

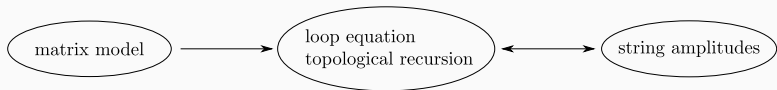
Fokker-Planck formalism and string theory

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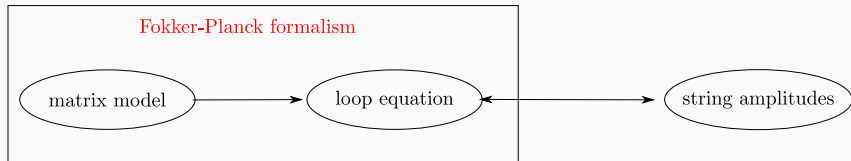


- noncritical strings

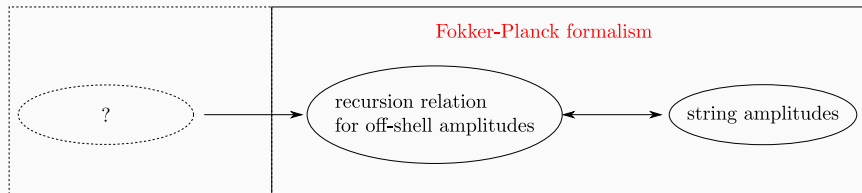


- One possible way to prove duality is to follow this path.
- It will be very nice if all these are on the same "OS".

- $(2, 3)$ minimal strings (NI-Kawai, Jevicki-Rodrigues)



- closed bosonic strings (PTEP 023B05(2023), PTEP 073B02(2024))



1. Fokker-Planck formalism for noncritical strings
2. Amplitudes of closed bosonic strings
3. Recursion relations
4. The FP formalism for closed bosonic strings
5. Strebel differentials
6. Conclusions

1. Fokker-Planck formalism for noncritical strings

- One matrix model

$$\int \frac{dM}{\text{Vol. (SU}(N))} e^{-N\text{Tr}V(M)}$$

- $M \rightarrow \Phi(l) \equiv \text{Tr} e^{lM},$

$$\begin{aligned} \int \frac{dM}{\text{Vol. (SU}(N))} e^{-N\text{Tr}V(M)} &= \int [d\Phi(l)] J[\Phi] e^{-N\text{Tr}V(M)} \\ &= \int [d\Phi(l)] e^{-S_{\text{eff.}}[\Phi]} \end{aligned}$$

- $J[\Phi]$ satisfies (Jevicki-Sakita)*

$$\int dl' \Phi(l+l') l' \frac{\delta \log J[\Phi]}{\delta \Phi(l')} = \int_0^l dl'' \Phi(l'') \Phi(l-l')$$

- $S_{\text{eff.}}[\Phi]$ is not well-defined in the continuum limit.

The Fokker-Planck formalism

- Euclidean field theory

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \frac{\int [d\phi] e^{-S[\phi]} \phi(x_1) \cdots \phi(x_n)}{\int [d\phi] e^{-S[\phi]}}$$

- Fokker-Planck formalism*

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \lim_{\tau \rightarrow \infty} \langle 0 | e^{-\tau \hat{H}_{\text{FP}}} \hat{\phi}(x_1) \cdots \hat{\phi}(x_n) | 0 \rangle$$

$$[\hat{\pi}(x), \hat{\phi}(y)] = \delta^D(x - y), [\hat{\pi}, \hat{\pi}] = [\hat{\phi}, \hat{\phi}] = 0$$

$$\langle 0 | \hat{\phi}(x) = \hat{\pi}(x) | 0 \rangle = 0, \quad \langle 0 | 0 \rangle = 1$$

$$\hat{H}_{\text{FP}} = - \int d^D x \left(\hat{\pi}(x) - \frac{\delta S}{\delta \phi(x)} [\hat{\phi}] \right) \hat{\pi}(x)$$

- One may consider FP Hamiltonian with kernel $K(x, x') > 0$;

$$\hat{H}_{\text{FP}} = - \int d^D x d^D x' \left(\hat{\pi}(x) - \frac{\delta S}{\delta \phi(x)} [\hat{\phi}] \right) K(x, x') \hat{\pi}(x')$$

- FP Hamiltonian

$$H_{\text{FP}}^M = \text{Tr} \frac{\partial}{\partial M} \left(\frac{\partial}{\partial M} + N \frac{\partial V(M)}{\partial M} \right)$$

$$H_{\text{FP}}^\Phi = J[\Phi] H_{\text{FP}}^M J^{-1}[\Phi]$$

- By taking the continuum limit $a \rightarrow 0$ with

$$\Phi\left(\frac{l}{a}\right) = e^{\frac{l}{a}\lambda_c} \left(\frac{1}{g_s} w(l) + \phi(l) \right) + \frac{Na}{2} V'(a\partial_l) \delta(l)$$

$$N = \frac{1}{g_s} a^{-\frac{5}{2}}, \quad H_{\text{FP}} = a^{-\frac{1}{2}} H_{\text{FP}}^\Phi$$

one obtains $\hat{H}_{\text{FP}}$ which describes **(2,3) minimal string theory**
(Jevicki-Rodrigues, NI-Kawai)

$$\begin{aligned} \hat{H}_{\text{FP}} = & -2 \int dldl' w(l) \hat{\phi}(l') \hat{\pi}(l+l')(l+l') - \int dldl' w(l+l') \hat{\pi}(l) l \hat{\pi}(l') l' \\ & - g_s \int dldl' \hat{\phi}(l+l') \hat{\pi}(l) l \hat{\pi}(l') l' - g_s \int dldl' \hat{\phi}(l) \hat{\phi}(l') \hat{\pi}(l+l')(l+l') \end{aligned}$$

- $\hat{H}_{\text{FP}}$ can be expressed as

$$\hat{H}_{\text{FP}} = g_s \int dl \hat{T}(l) \hat{\pi}(l) l$$

$$\begin{aligned} \hat{T}(l) = & -\frac{2}{g_s} \int_0^l dl' w(l') \hat{\phi}(l-l') - \frac{1}{g_s} \int_0^\infty dl' w(l+l') \hat{\pi}(l') l' \\ & - \int_0^l dl' \hat{\phi}(l') \hat{\phi}(l-l') - \int_0^\infty dl' \hat{\phi}(l+l') \hat{\pi}(l') l' \end{aligned}$$

- In this formalism, one can derive **the loop equation** given in the form

$$\lim_{\tau \rightarrow \infty} \langle 0 | e^{-\tau \hat{H}_{\text{FP}}} l \hat{T}(l) \hat{\phi}(x_1) \cdots \hat{\phi}(x_n) | 0 \rangle = 0$$

- $\hat{T}(l)$ satisfies the Virasoro algebra

$$[\hat{T}(l), \hat{T}(l')] = (l - l') \hat{T}(l + l')$$

- $\hat{H}_{\text{FP}}$ can be formally expressed as

$$\hat{H}_{\text{FP}} = - \int dl dl' \left(\hat{\pi}(l) - \frac{\delta S_{\text{eff.}}}{\delta \phi(l)} \left[\hat{\phi} \right] \right) \hat{K}(l, l') \hat{\pi}(l')$$

$$\hat{K}(l, l') = \left(w(l + l') + g_s \hat{\phi}(l + l') \right) ll'$$

- The continuum theory can be regarded as the one based on the dual theory (matrix model) action $S_{\text{eff.}}$.
- In the continuum limit, $S_{\text{eff.}}$ satisfies

$$- \int dl' \frac{\delta S_{\text{eff.}}}{\delta \phi(l')} K(l', l) = 2l \int_0^l dl' w(l') \phi(l - l') + g_s l \int_0^l dl'' \phi(l') \phi(l - l')$$

- $S_{\text{eff.}}$ can be calculated perturbatively but it is divergent and ill-defined.
- Although $S_{\text{eff.}}$ is ill-defined, the FP formalism is well-defined.

$$-\int dl' \frac{\delta S_{\text{eff.}}}{\delta \phi(l')} K(l', l) = 2l \int_0^l dl' w(l') \phi(l - l') + g_s l \int_0^l dl'' \phi(l') \phi(l - l')$$

- Solutions to $\delta S_{\text{eff.}} = 0$ can be studied by using this equation.
- In $N \rightarrow \infty$ limit, ZZ-brane corresponds to

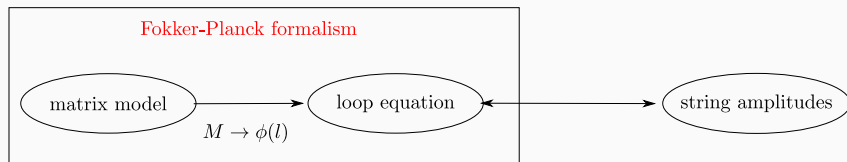
$$\phi(l) \propto e^{\frac{\kappa}{2}l}$$

which is a solution to $l \int_0^l dl' w(l') \phi(l - l') = 0$.

$$\tilde{w}(\zeta) \propto (\zeta - \frac{\kappa}{2}) \sqrt{\zeta + \kappa}$$

$$\tilde{w}(\zeta) \tilde{\phi}(\zeta) = 0$$

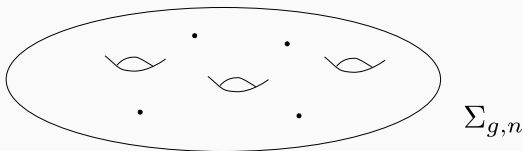
- (2, 3) minimal strings



- We would like to construct a “loop equation” and FP formalism for critical strings.
- In order to do so, we derive a recursion relation of the off-shell amplitudes which can be regarded as a loop equation.

2. Amplitudes of closed bosonic strings

Amplitudes of closed bosonic strings



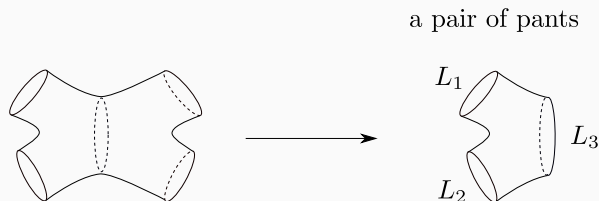
- Amplitudes of closed bosonic string theory

$$A_{g,n} = \int_{\mathcal{M}_{g,n}} \langle \prod_i \left(dt^i \mathcal{B}[\partial_{t^i}] \right) \prod_a V_a \rangle_{\Sigma_{g,n}}$$

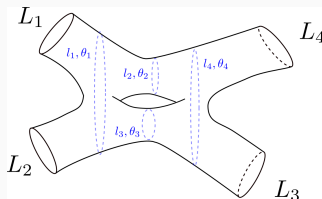
$\mathcal{M}_{g,n}$: moduli space of Riemann surfaces with
genus g and n punctures

- In order to get a recursion relation, we need a way to describe $\mathcal{M}_{g,n}$ concretely.
- $\mathcal{M}_{g,n}$ can be described by the length-twist (Fenchel-Nielsen) coordinates.

Pants decomposition



- A Riemann surface (genus g , n boundaries) with $2g - 2 + n > 0$ admits a unique hyperbolic metric such that the boundaries are geodesics.
- It can be decomposed into pairs of pants whose boundaries are geodesics.
- Namely, **all such Riemann surfaces can be realized by gluing the pairs of pants.**

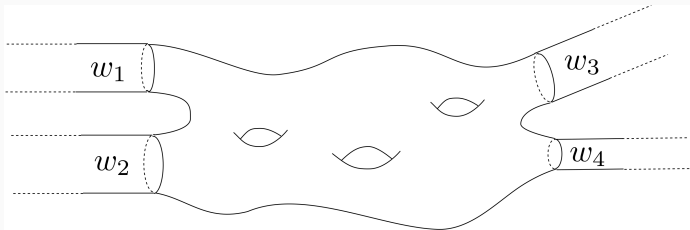


- The moduli space of Riemann surfaces (genus g , n boundaries) can be parametrized by the coordinates $(l_s; \theta_s)$ ($s = 1, \dots, 3g - 3 + n$).

$$\mathcal{M}_{g,n}(\mathbf{L}) = \mathcal{T}_{g,n}(\mathbf{L}) / \text{modular group}$$

$\mathcal{M}_{g,n}(\mathbf{L})$: the moduli space of hyperbolic surfaces with
genus g and n boundaries with lengths $\mathbf{L} = (L_1, \dots, L_n)$

$$\mathcal{T}_{g,n}(\mathbf{L}) \cong \{(l_s; \theta_s) | 0 < l_s < \infty, -\infty < \theta_s < \infty\}$$



- Given $\Sigma_{g,n}(\mathbf{L}) \in \mathcal{M}_{g,n}(\mathbf{L})$, one can get a punctured Riemann surface $\Sigma_{g,n} \in \mathcal{M}_{g,n}$ with local coordinates around punctures, by attaching semi-infinite cylinders to the boundaries (grafting).

$$\text{gr}_{\infty} : \Sigma_{g,n}(\mathbf{L}) \mapsto \Sigma_{g,n}$$

- $\mathcal{M}_{g,n}(\mathbf{L}) \cong \mathcal{M}_{g,n}$ for any $\mathbf{L}$. (Mondello)
- One can define off-shell amplitudes by using the hyperbolic surfaces with boundaries.

$$\int_{\mathcal{M}_{g,n}(\mathbf{L})} \prod_s [dl_s d\theta_s] \langle \prod_s (\mathcal{B}[\partial_{l_s}] \mathcal{B}[\partial_{\theta_s}]) \prod_a V_a \rangle_{\Sigma_{g,n}(\mathbf{L})}$$

3. Recursion relations

3. Recursion relations

$$\int_{\mathcal{M}_{g,n}(\mathbf{L})} \prod_s [dl_s d\theta_s] \langle \prod_s (\mathcal{B}[\partial_{l_s}] \mathcal{B}[\partial_{\theta_s}]) \prod_a V_a \rangle_{\Sigma_{g,n}(\mathbf{L})}$$

- Mathematicians were trying to calculate the volume of $\mathcal{M}_{g,n}(\mathbf{L})$.

$$V_{g,n}(\mathbf{L}) = \int_{\mathcal{M}_{g,n}(\mathbf{L})} \prod_s \left[\frac{l_s dl_s d\theta_s}{2\pi} \right] = \int_{\mathcal{F}} \prod_s \left[\frac{l_s dl_s d\theta_s}{2\pi} \right]$$

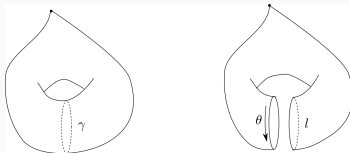
- Mirzakhani discovered a way to calculate the volumes without taking the fundamental domain.

$$\int_{\mathcal{F}} \prod_s \left[\frac{l_s dl_s d\theta_s}{2\pi} \right] \sim \int \prod_s \left[\frac{l_s dl_s d\theta_s}{2\pi} \right] F(l_s; \theta_s)$$

- Let us consider one of the simplest examples $V_{1,1}(0)$.



McShane identity ($g = n = 1, L = 0$)



- McShane identity (1998): for $f(l) = \frac{2}{1+e^l}$
$$1 = \sum_{\gamma} f(l)$$
- The integral $V_{1,1}(0) = \int_{\mathcal{F}} \frac{ldld\theta}{2\pi}$ can be unfolded by using this identity.

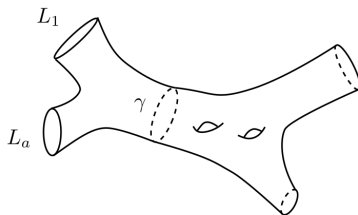
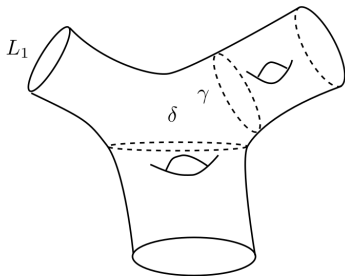
$$\begin{aligned} V_{1,1}(0) &= \int_{\mathcal{F}} \frac{ldld\theta}{2\pi} = \int_{\mathcal{M}_{1,1}(0)} \sum_{\gamma} f(l) \frac{ldld\theta}{2\pi} = \int_{\mathcal{M}_{1,1}^{\gamma}(0)} \frac{ldld\theta}{2\pi} f(l) \\ &= \int_0^{\infty} dl \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{2l}{1+e^l} = \frac{\pi^2}{6} \end{aligned}$$

$$\begin{aligned} \mathcal{M}_{1,1}^{\gamma}(0) &= \{(\Sigma_{1,1}, \gamma) \mid \Sigma_{1,1} \in \mathcal{M}_{1,1}(0), \gamma \subset \Sigma_{1,1}\} \\ &\cong \{(l, \theta) \mid 0 < l < \infty, 0 \leq \theta < 2\pi\} \end{aligned}$$

generalized McShane identity

- For general g, n with $2g - 2 + n > 0$.

$$L_1 = \sum_{\{\gamma, \delta\} \in \mathcal{C}_1} D_{L_1 l_\gamma l_\delta} + \sum_{a=2}^n \sum_{\gamma \in \mathcal{C}_a} (T_{L_1 L_a l_\gamma} + D_{L_1 L_a l_\gamma})$$



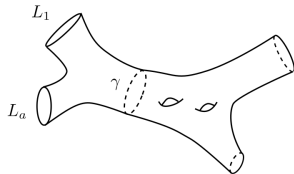
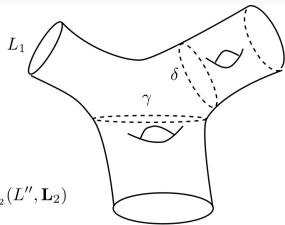
$$D_{LL'L''} = 2 \left(\log(e^{\frac{L}{2}} + e^{\frac{L'+L''}{2}}) - \log(e^{-\frac{L}{2}} + e^{\frac{L'+L''}{2}}) \right)$$

$$T_{LL'L''} = \log \frac{\cosh \frac{L''}{2} + \cosh \frac{L+L'}{2}}{\cosh \frac{L''}{2} + \cosh \frac{L-L'}{2}}$$

Mirzakhani recursion relation

- Multiplying the generalized McShane identity by $\int_{\mathcal{F}} \prod_s \left[\frac{l_s dl_s d\theta_s}{2\pi} \right]$, one can partially unfold the integral and get

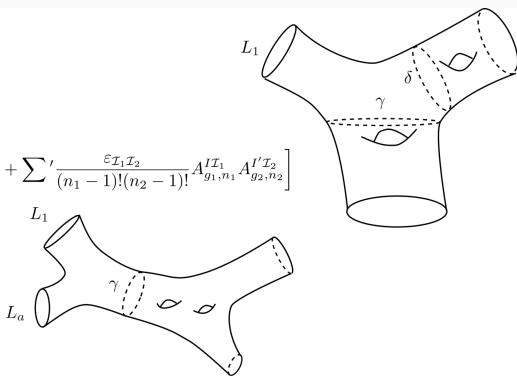
$$\begin{aligned}
 LV_{g,n+1}(L, \mathbf{L}) &= \frac{1}{2} \int_0^\infty dL' L' \int_0^\infty dL'' L'' D_{LL' L''} V_{g-1, n+2}(L', L'', \mathbf{L}) \\
 &+ \frac{1}{2} \int_0^\infty dL' L' \int_0^\infty dL'' L'' D_{LL' L''} \sum_{\text{stable}} V_{g_1, n_1}(L', \mathbf{L}_1) V_{g_2, n_2}(L'', \mathbf{L}_2) \\
 &+ \sum_{a=1}^n \int_0^\infty dL' L' (T_{L_1 L_a L'} + D_{L_1 L_a L'}) V_{g, n}(L, \mathbf{L} \setminus L_a)
 \end{aligned}$$



The recursion relation of off-shell amplitudes

- An off-shell amplitude is given by an integral over $\mathcal{M}_{g,n}(\mathbf{L})$.
- Using the generalized McShane identity, we can partially unfold the integration and get **a recursion relation for off-shell amplitudes**.

$$\begin{aligned}
 L_1 A_{g,n}^{I_1 \cdots I_n} &= L_1 G^{I_1 I_2} \delta_{g,0} \delta_{n,2} \\
 &+ \frac{1}{2} D^{I_1 J' J} G_{JI} G_{J'I'} \left[A_{g-1,n+1}^{I'I_2 \cdots I_n} + \sum' \frac{\varepsilon_{I_1 I_2}}{(n_1 - 1)!(n_2 - 1)!} A_{g_1,n_1}^{I I_1} A_{g_2,n_2}^{I' I_2} \right] \\
 &+ \sum_{a=2}^n \varepsilon_a T^{I_1 I_a J} G_{JI} A_{g,n-1}^{I_2 \cdots \hat{I}_a \cdots I_n}
 \end{aligned}$$



The recursion relation

$$\begin{aligned}
 L_1 A_{g,n}^{I_1 \cdots I_n} &= L_1 G^{I_1 I_2} \delta_{g,0} \delta_{n,2} \\
 &+ \frac{1}{2} D^{I_1 J' J} G_{JI} G_{J'I'} \left[A_{g-1,n+1}^{II' I_2 \cdots I_n} + \sum' \frac{\varepsilon_{\mathcal{I}_1 \mathcal{I}_2}}{(n_1 - 1)!(n_2 - 1)!} A_{g_1, n_1}^{I \mathcal{I}_1} A_{g_2, n_2}^{I' \mathcal{I}_2} \right] \\
 &+ \sum_{a=2}^n \varepsilon_a T^{I_1 I_a J} G_{JI} A_{g,n-1}^{II_2 \cdots \hat{I}_a \cdots I_n}
 \end{aligned}$$

$$A_{g,n}^{I_1 \cdots I_n} = \int_{\mathcal{F}} \prod_s [dl_s d\theta_s] \langle \prod_s [\mathcal{B}[\partial_{l_s}] \mathcal{B}[\partial_{\theta_s}]] B_{\alpha_1} \cdots B_{\alpha_n} V_{i_1} \cdots V_{i_n} \rangle_{\Sigma_{g,n,L}}$$

$$B_{\alpha_a} \equiv \begin{cases} 1 & \alpha_a = + \\ (b_0^{(a)} - \bar{b}_0^{(a)}) b_{S_a} (\partial_{L_a}) \int_0^{2\pi} \frac{d\theta_a}{2\pi} e^{i\theta_a (L_0^{(a)} - \bar{L}_0^{(a)})} & \alpha_a = - \end{cases}$$

$$T^{I_1 I_2 I_3} \equiv T_{L_1 L_2 L_3} \langle B_{\alpha_1} B_{\alpha_2} B_{\alpha_3} V^{i_1} V^{i_2} V^{i_3} \rangle$$

$$D^{I_1 I_2 I_3} \equiv D_{L_1 L_2 L_3} \langle B_{\alpha_1} B_{\alpha_2} B_{\alpha_3} V^{i_1} V^{i_2} V^{i_3} \rangle$$

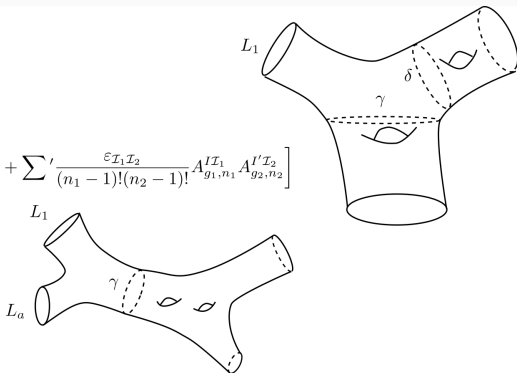
$$G_{I_1 I_2} \equiv \langle \varphi_{i_1}^c | \varphi_{i_2}^c \rangle (-1)^{n_{\varphi_{i_2}}} \delta(L_1 - L_2) \delta_{\alpha_1, -\alpha_2},$$

- The external lines are labeled by $I = (i, L, \alpha)$, where $\alpha = \pm$ distinguish the b insertions.*
- One can obtain all the amplitudes by solving this recursion relation.

4. The FP formalism for closed bosonic strings

Recursion relation

$$\begin{aligned}
 L_1 A_{g,n}^{I_1 \cdots I_n} = & L_1 G^{I_1 I_2} \delta_{g,0} \delta_{n,2} \\
 & + \frac{1}{2} D^{I_1 J' J} G_{JI} G_{J'I'} \left[A_{g-1,n+1}^{II' I_2 \cdots I_n} + \sum' \frac{\varepsilon_{I_1 I_2}}{(n_1 - 1)!(n_2 - 1)!} A_{g_1, n_1}^{II_1} A_{g_2, n_2}^{I' I_2} \right] \\
 & + \sum_{a=2}^n \varepsilon_a T^{I_1 I_a J} G_{JI} A_{g,n-1}^{II_2 \cdots \hat{I}_a \cdots I_n}
 \end{aligned}$$



- Mirzakhani recursion can be considered as a “loop equation”.
(Eynard-Orantin, Saad-Shenker Stanford)
- This recursion may also be considered as a “loop equation”.
- One can develop the FP formalism.

The FP formalism for closed bosonic strings

- We develop the FP formalism

$$\langle\langle \phi^{I_1} \dots \phi^{I_n} \rangle\rangle = \lim_{\tau \rightarrow \infty} \langle\langle 0 | e^{-\tau \hat{H}_{\text{FP}}} \hat{\phi}^{I_1} \dots \hat{\phi}^{I_n} | 0 \rangle\rangle$$

$$\langle\langle \phi^{I_1} \dots \phi^{I_n} \rangle\rangle^c = \sum_{g=0}^{\infty} g_s^{2g-2+n} A_{g,n}^{I_1 \dots I_n}$$

$$[\hat{\pi}_I, \hat{\phi}^K] = \delta_I^K$$

$$[\hat{\pi}_I, \hat{\pi}_K] = [\hat{\phi}^I, \hat{\phi}^K] = 0$$

$$\langle\langle 0 | \hat{\phi}^I = \hat{\pi}_I | 0 \rangle\rangle = 0, \quad \langle\langle 0 | 0 \rangle\rangle = 1$$

- $\hat{H}_{\text{FP}}$ can be constructed from the recursion relation

$$\begin{aligned} \hat{H}_{\text{FP}} &= \hat{T}^I \hat{\pi}_I \\ &= -L \hat{\pi}_I \hat{\pi}_{I'} G^{I'I} + L \hat{\phi}^I \hat{\pi}_I \\ &\quad - \frac{1}{2} g_s D^{II'I''} G_{I''K''} G_{I'K'} \hat{\phi}^{K''} \hat{\phi}^{K'} \hat{\pi}_I \\ &\quad - g_s T^{II'I''} G_{I''K''} \hat{\phi}^{K''} \hat{\pi}_{I'} \hat{\pi}_I \end{aligned}$$

The FP formalism for closed bosonic strings

$$\langle\langle \phi^{I_1} \dots \phi^{I_n} \rangle\rangle = \lim_{\tau \rightarrow \infty} \langle\langle 0 | e^{-\tau \hat{H}_{\text{FP}}} \hat{\phi}^{I_1} \dots \hat{\phi}^{I_n} | 0 \rangle\rangle$$

$$\begin{aligned} \hat{H}_{\text{FP}} = & -L \hat{\pi}_I \hat{\pi}_{I'} G^{I' I} + L \hat{\phi}^I \hat{\pi}_I \\ & -\frac{1}{2} g_s D^{II' I''} G_{I'' K''} G_{I' K'} \hat{\phi}^{K''} \hat{\phi}^{K'} \hat{\pi}_I \\ & -g_s T^{II' I''} G_{I'' K''} \hat{\phi}^{K''} \hat{\pi}_{I'} \hat{\pi}_I \end{aligned}$$

- By construction, the results coincide with perturbative amplitudes of closed bosonic string theory.
- $\hat{H}_{\text{FP}}$ consists of kinetic terms and three string interactions.
- $\hat{H}_{\text{FP}}$ is not invariant under the BRST symmetry on the worldsheet. One can construct a BRST invariant formulation* by introducing auxiliary fields.

- Formally $\hat{H}_{\text{FP}}$ can be recast into

$$\hat{H}_{\text{FP}} = - \left(\hat{\pi}_I - \frac{\delta S_{\text{eff.}}}{\delta \phi^I} \left[\hat{\phi} \right] \right) \hat{K}^{II'} \pi_{I'}$$

$$\begin{aligned} K^{I'I} &= LG^{II'} + g_s T^{II'J} G_{JJ'} \phi^{J'} (-1)^{|I'| |J|} \\ \frac{\delta S_{\text{eff.}}}{\delta \phi^{I'}} K^{I'I} &= L\phi^I - \frac{g_s}{2} D^{II'I''} G_{I''J''} G_{I'J'} \phi^{J''} \phi^{J'} \\ &\quad + g_s T^{II'I''} G_{I'I''} \end{aligned}$$

- One can calculate $S_{\text{eff.}}$ perturbatively. It turns out to be divergent and ill defined.*
- $S_{\text{eff.}}$ should be identified with the action of a fictitious dual theory.

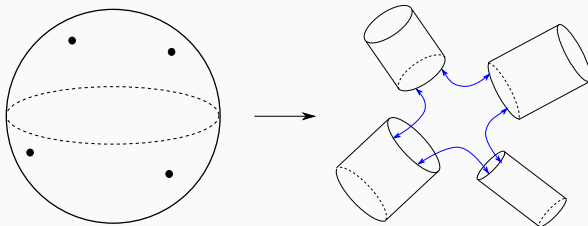
5. Strebel differentials

Strebel's theorem

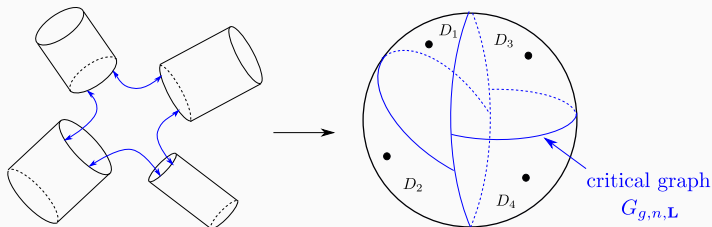
- Given a punctured Riemann surface ($2g - 2 + n > 0$) and positive numbers $L_1, \dots, L_n$, there exists the unique quadratic differential $\phi(z)dz^2$ (**Strebel differential**) such that
 - for $z \sim z_a$, $\phi(z)dz^2 \sim -\left(\frac{L_a}{2\pi}\right)^2 \frac{dz^2}{(z-z_a)^2}$
 - holomorphic for $z \neq z_a$
 - with the metric

$$ds^2 = |\phi(z)| dz d\bar{z} = dw d\bar{w}$$
$$w = \int^z dz' \sqrt{\phi(z')}$$

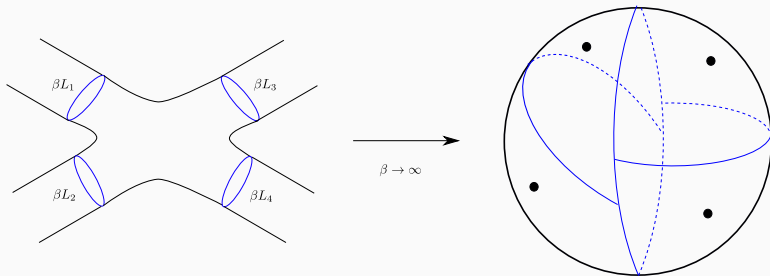
the surface looks like



Strebel differentials



- To any punctured Riemann surface, one can associate a graph called critical graph.
- Moduli spaces of punctured Riemann surfaces can be parametrized by the lengths of the edges of the critical graphs (**combinatorial moduli space** $\mathcal{M}_{g,n}^{\text{comb.}}(\mathbf{L})$).



- The combinatorial description can be considered as a limit of the one using the hyperbolic surfaces. (Mondello, Do)

$$\mathcal{M}_{g,n}^{\text{comb.}}(\mathbf{L}) = \lim_{\beta \rightarrow \infty} \mathcal{M}_{g,n}(\beta \mathbf{L}) \cong \mathcal{M}_{g,n}$$

- The combinatorial version of Mirzakhani recursion was given by Andersen et al., Bennett et al..

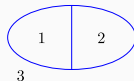
- For the combinatorial description of the moduli space, we get

$$\begin{aligned}\hat{H}_{FP} = & -L\hat{\pi}_I\hat{\pi}_{I'}G^{I'I} + L\hat{\phi}^I\hat{\pi}_I \\ & -\frac{1}{2}g_sV^{II'I''}G_{I''K''}G_{I'K'}\hat{\phi}^{K''}\hat{\phi}^{K'}\hat{\pi}_I \\ & -g_sW^{II'I''}G_{I''K''}\hat{\phi}^{K''}\hat{\pi}_{I'}\hat{\pi}_I,\end{aligned}$$

propagator

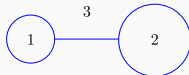


vertices



θ type

$$|L_1 - L_2| < L_3 < L_1 + L_2$$



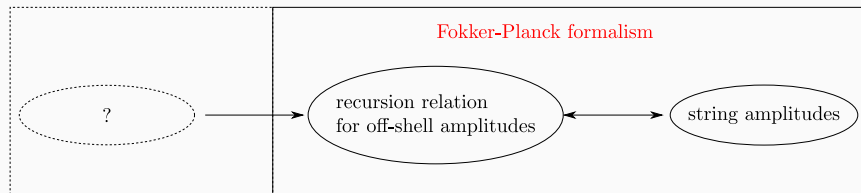
dumbbell type

$$L_1 + L_2 < L_3$$

- $S_{\text{eff.}}$ can be defined in the same way as the hyperbolic case.

6. Conclusions

5. Conclusions



- The technique here can be applied to construct classical solutions of SFT .
Firat-Valdes-Meller, Kim
- The combinatorial version may be useful in studying gauge/string duality
(Gopakumar, Razamat, ...)
- Superstrings?

$$J[\Phi] = \int \frac{dM}{\text{Vol. (SU}(N))} \prod_l \delta \left(\Phi(l) - \text{Tre}^{lM} \right)$$

Using

$$\text{Tr} \frac{\partial \text{Tre}^{lM}}{\partial M} \frac{\partial \text{Tre}^{l'M}}{\partial M} = ll' \text{Tre}^{(l+l')M},$$

$$\begin{aligned} & \int dl' \Phi(l+l') ll' \frac{\delta \log J[\Phi]}{\delta \Phi(l')} \\ &= \frac{l}{J} \int dl' \int \frac{dM}{\text{Vol. (SU}(N))} \text{Tr} \frac{\partial \text{Tre}^{lM}}{\partial M} \frac{\partial \Phi(l')}{\partial M} \frac{\delta}{\delta \Phi(l')} \left[\prod_l \delta \left(\Phi(l) - \text{Tre}^{lM} \right) \right] \\ &= -\frac{l}{J} \int \frac{dM}{\text{Vol. (SU}(N))} \text{Tr} \frac{\partial \text{Tre}^{lM}}{\partial M} \frac{\partial}{\partial M} \prod_l \delta \left(\Phi(l) - \text{Tre}^{lM} \right) \\ &= \frac{l}{J} \int \frac{dM}{\text{Vol. (SU}(N))} \text{Tr} \frac{\partial^2 \text{Tre}^{lM}}{\partial M^2} \prod_l \delta \left(\Phi(l) - \text{Tre}^{lM} \right) \\ &= l \int_0^l dl' \Phi(l') \Phi(l-l') \end{aligned}$$

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \lim_{\tau \rightarrow \infty} \langle 0 | e^{-\tau \hat{H}_{\text{FP}}} \hat{\phi}(x_1) \cdots \hat{\phi}(x_n) | 0 \rangle$$

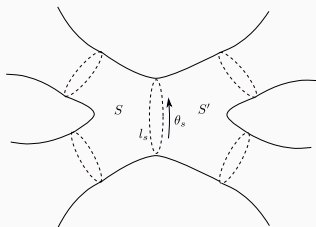
- Probability density

$$P[\phi, \tau] \equiv \langle 0 | e^{-\tau \hat{H}_{\text{FP}}} \prod_x \delta(\hat{\phi}(x) - \phi(x)) | 0 \rangle$$

$$\begin{aligned} \partial_\tau P[\phi, \tau] &= -H_{\text{FP}} \left[\phi(x), \frac{\delta}{\delta \phi(x)} \right] P[\phi, \tau] \\ H_{\text{FP}} \left[\phi(x), \frac{\delta}{\delta \phi(x)} \right] &= - \int d^D x \frac{\delta}{\delta \phi(x)} \left(\frac{\delta}{\delta \phi(x)} + \frac{\delta S}{\delta \phi(x)} \right) \end{aligned}$$

- Since $e^{\frac{1}{2}S} H_{\text{FP}} e^{-\frac{1}{2}S} \geq 0$, $\lim_{\tau \rightarrow \infty} P[\phi, \tau] = \frac{e^{-S}}{\int [d\phi] e^{-S}}$,

$$\begin{aligned} \lim_{\tau \rightarrow \infty} \langle 0 | e^{-\tau \hat{H}_{\text{FP}}} \hat{\phi}(x_1) \cdots \hat{\phi}(x_n) | 0 \rangle &= \lim_{\tau \rightarrow \infty} \int [d\phi] P[\phi, \tau] \phi(x_1) \cdots \phi(x_n) \\ &= \frac{\int [d\phi] e^{-S[\phi]} \phi(x_1) \cdots \phi(x_n)}{\int [d\phi] e^{-S[\phi]}} \end{aligned}$$



$$\mathcal{B}[\partial_{l_s}] = \mathcal{B}_S[\partial_{l_s}] + \mathcal{B}_{S'}[\partial_{l_s}]$$

$$\mathcal{B}[\partial_{\theta_s}] = -2\pi i b_0^-$$

$$\begin{array}{c}
 \begin{array}{c} S \quad l_s \quad S' \\ \text{---} \quad \text{---} \quad \text{---} \end{array} \\
 \mathcal{B}[\partial_{\theta_s}] (\mathcal{B}_S[\partial_{l_s}] + \mathcal{B}_{S'}[\partial_{l_s}])
 \end{array}
 =
 \begin{array}{c}
 \text{---} \quad \text{---} \\
 \text{---} \quad \text{---}
 \end{array}
 \times
 \begin{array}{c}
 \text{---} \quad \text{---} \\
 \text{---} \quad \text{---}
 \end{array}
 +
 \begin{array}{c}
 \text{---} \quad \text{---} \\
 \text{---} \quad \text{---}
 \end{array}
 \times
 \begin{array}{c}
 \text{---} \quad \text{---} \\
 \text{---} \quad \text{---}
 \end{array}$$

- String states with these insertions are labeled by $\alpha = -$, and those with no insertions are labeled by $\alpha = +$.

BRST symmetry on the worldsheet

- We need the worldsheet **BRST symmetry** to define the physical states with positive norm.

$$Q|\text{phys.}\rangle = 0$$

$$|\rangle \sim |\rangle + Q|\rangle'$$

- In order to discuss this symmetry, we change the notation

$$|\phi^\alpha(L)\rangle \equiv \sum_i \hat{\phi}^I |\varphi_i^c\rangle$$

$$|\pi_\alpha(L)\rangle \equiv \sum_i |\varphi_i\rangle \hat{\pi}_I$$

$$\begin{aligned} \hat{H} = & \int_0^\infty dL L [\langle R|\phi^\alpha(L)\rangle |\pi_\alpha(L)\rangle - \langle R|\pi_\alpha(L)\rangle |\pi_{-\alpha}(L)\rangle] \\ & - g_s \int dL_1 dL_2 dL_3 \langle T_{L_2 L_3 L_1} | B_{-\alpha_1}^1 B_{\alpha_2}^2 B_{\alpha_3}^3 | \phi^{\alpha_1}(L_1)\rangle_1 |\pi_{\alpha_2}(L_2)\rangle_2 |\pi_{\alpha_3}(L_3)\rangle_3 \\ & - \frac{1}{2} g_s \int dL_1 dL_2 dL_3 \langle D_{L_3 L_1 L_2} | B_{-\alpha_1}^1 B_{-\alpha_2}^2 B_{\alpha_3}^3 | \phi^{\alpha_1}(L_1)\rangle_1 |\phi^{\alpha_2}(L_2)\rangle_2 |\pi_{\alpha_3}(L_3)\rangle_3 \end{aligned}$$

- The BRST transformation

$$\begin{aligned} \delta_\epsilon |\phi^+(L)\rangle &= \epsilon P_- Q |\phi^+(L)\rangle & \delta_\epsilon |\pi_+(L)\rangle &= \epsilon Q |\pi_+(L)\rangle - \epsilon b_0^- P \partial_L |\pi_- \\ \delta_\epsilon |\phi^-(L)\rangle &= \epsilon Q |\phi^-(L)\rangle - \epsilon b_0^- P \partial_L |\phi^+(L)\rangle & \delta_\epsilon |\pi_-(L)\rangle &= \epsilon P_- Q |\pi_-(L)\rangle \end{aligned}$$

$\hat{H}$ is not BRST invariant

- $\hat{H}$ is not BRST invariant.
 - If it were, FP formalism would be modular invariant
 - Let $\hat{Q}$ be the generator of the BRST transformation

$$\delta\hat{H} = [\hat{Q}, \hat{H}] = \int_0^\infty dL \left(\langle R | \mathcal{Q}^\alpha(L) \rangle | \pi_\alpha(L) \rangle + \langle R | \mathcal{T}^\alpha(L) \rangle [\hat{Q}, | \pi_\alpha(L) \rangle] \right)$$

$$\hat{H} = \int_0^\infty dL \langle R | \mathcal{T}^\alpha(L) \rangle | \pi_\alpha(L) \rangle$$

$$| \mathcal{Q}^\alpha(L) \rangle \equiv [\hat{Q}, | \mathcal{T}^\alpha(L) \rangle]$$

- **The amplitudes are invariant**, because $| \mathcal{Q}^\alpha(L) \rangle, | \mathcal{T}^\alpha(L) \rangle$ are “null quantities” satisfying

$$\left[\lim_{\tau \rightarrow \infty} \langle\langle 0 | e^{-\tau \hat{H}} \right] | \mathcal{T}^\alpha(L) \rangle = 0$$

$$\left[\lim_{\tau \rightarrow \infty} \langle\langle 0 | e^{-\tau \hat{H}} \right] | \mathcal{Q}^\alpha(L) \rangle = 0$$

BRST invariant formulation*

- We can modify the Hamiltonian by introducing the auxiliary fields $|\lambda_\alpha^{\mathcal{Q}}(L)\rangle, |\lambda_\alpha^{\mathcal{T}}(L)\rangle$ so that **it becomes BRST invariant and still yields the correct amplitudes.**

$$\hat{H} \rightarrow \hat{H} + \int_0^\infty dL \left(\langle R | \mathcal{Q}^\alpha(L) \rangle |\lambda_\alpha^{\mathcal{Q}}(L)\rangle + \langle R | \mathcal{T}^\alpha(L) \rangle |\lambda_\alpha^{\mathcal{T}}(L)\rangle \right)$$

$$\delta \hat{H} = \int_0^\infty dL \left(\langle R | \mathcal{Q}^\alpha(L) \rangle |\pi_\alpha(L)\rangle + \langle R | \mathcal{T}^\alpha(L) \rangle [\hat{Q}, |\pi_\alpha(L)\rangle] \right)$$

- The action

$$I_{\text{FP}} = \int_0^\infty d\tau \left[- \int_0^\infty dL \langle R | \pi_\alpha(\tau, L) \rangle \frac{\partial}{\partial \tau} |\phi^\alpha(\tau, L)\rangle + H(\tau) + \int_0^\infty dL \left(\langle R | \mathcal{Q}^\alpha(\tau, L) \rangle |\lambda_\alpha^{\mathcal{Q}}(\tau, L)\rangle + \langle R | \mathcal{T}^\alpha(\tau, L) \rangle |\lambda_\alpha^{\mathcal{T}}(\tau, L)\rangle \right) \right]$$

- This action is invariant under the BRST transformation.
- It consists of kinetic terms and three string interaction terms.

$$S_{\text{eff.}} = \frac{1}{2} G_{IJ} \phi^I \phi^J - \frac{g_s}{6} A_{0,3}^{II'I''} G_{IJ} G_{I'J'} G_{I''J''} \phi^{J''} \phi^{J'} \phi^J \\ + \frac{g_s}{L} T^{II'I''} G_{I'I''} G_{IJ} \phi^J + \mathcal{O}(g_s^2)$$

$$S_{\text{eff.}} = \underbrace{\phi^2 + g_s \phi^3}_{\text{D'Hoker-Gross}} + \underbrace{g_s \phi + \dots}_{(\infty - 1) \times \text{diagram}}$$

D'Hoker-Gross

$(\infty - 1) \times$



- $S_{\text{eff.}}$ is divergent and ill defined.
 - The 1 loop 1 point amplitude becomes finite.
 - $S_{\text{eff.}}$ includes infinitely many divergent counterterms.

- In $N \rightarrow \infty$ limit,

$$\Phi(l) = \int d\lambda e^{l\lambda} \rho(\lambda)$$

$$S_{\text{eff.}} = \int d\lambda V(\lambda) \rho(\lambda) - \frac{1}{2} \int d\lambda d\lambda' \rho(\lambda) \rho(\lambda') \log |\lambda - \lambda'|$$

- disk amplitude

$$W(l) = \int d\lambda e^{l\lambda} \rho_0(\lambda)$$

$$V'(\lambda) = \text{P} \int d\lambda' \frac{\rho_0(\lambda')}{\lambda - \lambda'}$$

- ZZ-brane: $\rho(\lambda) = \rho_0(\lambda) + C\delta(\lambda - x)$

$$S_{\text{eff.}}(x) \propto \int d\lambda \rho_0(\lambda) \log |x - \lambda|$$

$$S'_{\text{eff.}}(x) \propto \int d\lambda \frac{\rho_0(\lambda)}{x - \lambda} = 0$$