The Fokker-Planck formalism for closed bosonic strings

JPS meeting at Tohoku University

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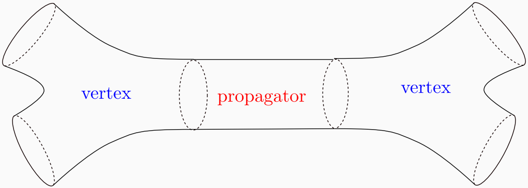
String Field Theory (SFT)

- Second-quantized theory of strings.
- In order to construct an SFT, we need a rule to decompose worldsheets (Riemann surfaces) into propagators and vertices.



- In other words, we need to find fundamental building blocks from which all the Riemann surfaces are generated.
- In general, we need infnitely many such blocks and the SFT action tends to be very complicated.

$$S = \phi K \phi + \phi^3 + \cdots$$

SFT

$$S = \phi K \phi + \phi^3 + \cdots$$

- SFT people nowadays are not afraid of such an action.
  - Zwiebach, ... developed a sophisticated machinery to deal with such an action.
  - It seems that the action for superstring field theory should be like that to deal with the spurious singularity problem. (Sen, ...)
- It is very difficult to solve the equation of motion for S.
- There are two known rules (light-cone, Witten) for which the action becomes very simple.

$$S = \phi K \phi + \phi^3$$

  - SFT for bosonic strings were constructed and exact solutions for eom. were discovered.
  - These rules are not good for superstrings.
- It may be helpful to find out yet another rule to decompose Riemann surfaces such that the SFT becomes simple.

This talk

- We construct an SFT using the pants decomposition of Riemann surfaces.
- In order to do so, we employ the Fokker-Planck (FP) formalism.
- We get an SFT for bosonic strings with FP action which consists of kinetic term and three string vertices.

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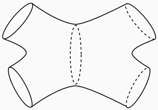
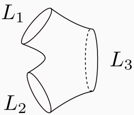
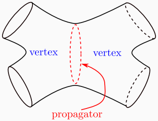
1. Pants decomposition
2. Mirzakhani's idea
3. The Fokker-Planck formalism
4. Conclusions

1. Pants decomposition



a pair of pants





- A Riemann surface with $2g - 2 + n > 0$ admits a hyperbolic metric such that the boundaries are geodesics.
- It can be decomposed into pairs of pants whose boundaries are geodesics.
- This fact implies that we may be able to construct an SFT with

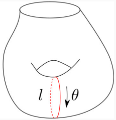
$$S = \phi K \phi + \phi^3$$

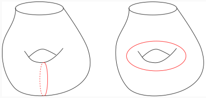
An SFT based on the pants decomposition?

$$S = \phi K \phi + \phi^3$$

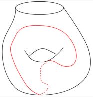
- This action does not work. (D'Hoker-Gross)
  - One-loop one point amplitudes diverge because the pants decomposition is not unique. There are infinitely many pants decompositions related by modular transformations.
  - Most of the amplitudes diverge in the same way.

$$A = \int \frac{l dl d\theta}{2\pi} \langle BV \rangle$$





$\cdots$

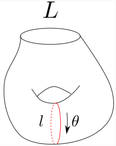


$$A = \infty \times$$

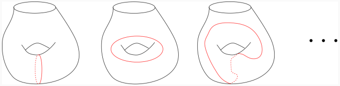
2. Mirzakhani's idea

- Mathematicians were interested in calculating the volume of the moduli space of Riemann sufaces of genus g and with n boundaries.

$$V_{1,1}(L) = \int_{\mathcal{F}} \frac{l dl d\theta}{2\pi}$$



- Integrating over $0 < l < \infty$, the integral diverges.
  - The pants decomposition is not unique. There are infinitely many pants decompositions related by modular transformations.



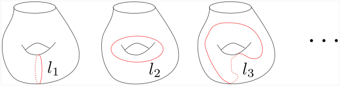
$$\int_0^\infty \int_0^{2\pi} \frac{l dl d\theta}{2\pi} = \infty \times$$

- We should integrate over the fundamental domain $\mathcal{F}$, which is very complicated in general.

McShane identity $(g = n = 1, L = 0)$

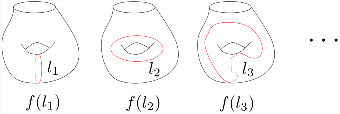
- McShane identity: for $f(l) = \frac{2}{1+e^{l}}$

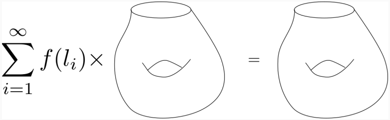
$$1 = f(l_1) + f(l_2) + f(l_3) + \cdots$$



- $V_{1,1}$ can be calculated using this identity (Mirzakhani)

$$V_{1,1}(0) = \int_{\mathcal{F}} \frac{l dl d\theta}{2\pi} = \int \frac{dl d\theta l}{2\pi} f(l) = \frac{\pi^2}{6}$$

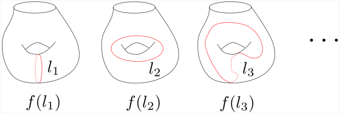


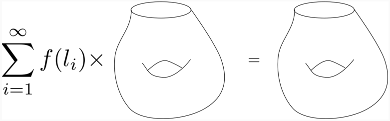


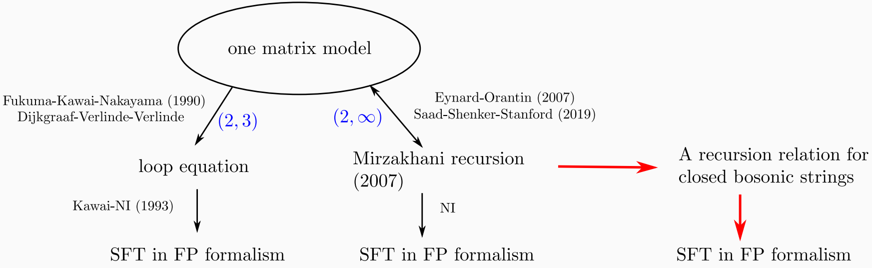
- Mirzakhani generalized the McShane identity and construct a recursion relation for $V_{g,n}$.

3. The Fokker-Planck formalism

- We would like to construct an SFT based on the pants decomposition using the Mirzakhani's idea.







The Fokker-Planck formalism

- Euclidean field theory

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = \frac{\int [d\phi] e^{-S[\phi]}\phi(x_1)\cdots\phi(x_n)}{\int [d\phi] e^{-S[\phi]}}$$

- Fokker-Planck formalism

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = \lim_{\tau\to\infty} \langle 0| e^{-\tau \hat{H}_{\rm FP}} \hat{\phi}(x_1)\cdots\hat{\phi}(x_n)|0\rangle$$

$$[\hat{\pi}(x), \hat{\phi}(y)] = \delta(x-y)\,, [\hat{\pi}, \hat{\pi}] = [\hat{\phi}, \hat{\phi}] = 0$$

$$\langle 0|\hat{\phi}(x) = \hat{\pi}(x)|0\rangle = 0$$

$$\hat{H}_{\rm FP} = -\int dx \left(\hat{\pi}(x) + \frac{\delta S}{\delta \phi(x)}[\hat{\phi}]\right)\hat{\pi}(x)$$

- path integral

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = \frac{\int [d\pi d\phi] e^{-I_{\rm FP}}\phi(0, x_1)\cdots\phi(0, x_n)}{\int [d\pi d\phi] e^{-I_{\rm FP}}}$$

$$I_{\rm FP} = \int_0^\infty d\tau \left[-\int dx \pi \partial_\tau \phi + H_{\rm FP}\right]$$

SFT

- We have constructed an SFT for closed bosonic strings based on the pants decomposition via the Fokker-Planck formalism.

$$
\begin{aligned}
I_{\mathrm{FP}} &= \int_0^\infty d\tau \Big[-\int_0^\infty dL \langle R|\pi_\alpha(\tau,L)\rangle \frac{\partial}{\partial\tau}|\phi^\alpha(\tau,L)\rangle + H(\tau) \\
&\qquad + \int_0^\infty dL \left(\langle R|\mathcal{Q}^\alpha(\tau,L)\rangle|\lambda^{\mathcal{Q}}_\alpha(\tau,L)\rangle + \langle R|\mathcal{T}^\alpha(\tau,L)\rangle|\lambda^{\mathcal{T}}_\alpha(\tau,L)\rangle \right) \Big]
\end{aligned}
$$

$$
\begin{aligned}
H &= \int_0^\infty dL L \left[\langle R|\phi^\alpha(L)\rangle|\pi_\alpha(L)\rangle - \langle R|\pi_\alpha(L)\rangle|\pi_{-\alpha}(L)\rangle \right] \\
&\quad - g_{\mathrm{s}} \int dL_1 dL_2 dL_3 \langle T_{L_2 L_3 L_1}|B^1_{-\alpha_1} B^2_{\alpha_2} B^3_{\alpha_3}|\phi^{\alpha_1}(L_1)\rangle_1 |\pi_{\alpha_2}(L_2)\rangle_2 |\pi_{\alpha_3}(L_3)\rangle_3 \\
&\quad - \frac{1}{2} g_{\mathrm{s}} \int dL_1 dL_2 dL_3 \langle D_{L_3 L_1 L_2}|B^1_{-\alpha_1} B^2_{-\alpha_2} B^3_{\alpha_3}|\phi^{\alpha_1}(L_1)\rangle_1 |\phi^{\alpha_2}(L_2)\rangle_2 |\pi_{\alpha_3}(L_3)\rangle_3
\end{aligned}
$$

  - This action consists of kinetic terms and three string interaction terms.
  - One can calculate the amplitudes perturbatively starting from this action.
  - It is manifestly invariant under a nilpotent BRST transformation and we can define the physical states using it.
  - $S[\phi]$ diverges.

4. Conclusions

5. Conclusions

- We have constructed an SFT for closed bosonic strings based on the pants decomposition via the Fokker-Planck formalism.

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\begin{aligned}
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- SFT for superstrings?