The analysis of the bosonic Lorentzian IKKT matrix model at large D

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1/D expansion

Replace the number of bosonic matrices with D: $10 \to D$

[Hotta-Nishimura-Tsuchiya('98)]
(Euclidean model without mass term)

$$h_{ab} \sim A_\mu^a A_\mu^b, \quad A_\mu = \sum_{a=1}^{N^2-1} A_\mu^a t^a$$

$$Z = \int dA e^{i(A^4 + \gamma A^2)}$$

$$= \int dh \int dA e^{i(h^2 + hA^2 + \gamma A^2)}$$

$$= \int dh e^{ih^2 - \frac{D}{2}\mathrm{logdet}K}$$

$$= \int d\tilde{h} e^{-D S_{\mathrm{eff}}[\tilde{h}]}$$

$$\tilde{h}_{ab} = \frac{N}{\sqrt{D}} h_{ab}$$

$$K \sim \frac{1}{N}\tilde{h} + \tilde{\gamma}$$

$$\gamma = \tilde{\gamma}\sqrt{D}$$

The saddle points dominates the path integral at Large D $\quad \frac{\partial S_{\mathrm{eff}}}{\partial \tilde{h}_{ab}} = \tilde{h}_{ab} + iK_{ab}^{-1} = 0$

Saddle point for N=2

The saddle point Eq. $\tilde{h}_{ab} + iK^{-1}_{ab} = 0$

For N=2, there are three saddle points in all.

$$\tilde{h}_{ab} = v^{(\pm)}\mathbf{1}, \quad \tilde{h} = \tilde{\gamma}\,\mathrm{diag}\left(1, 1, \frac{i}{\tilde{\gamma}^2}\right) \qquad v^{(\pm)} = \frac{\tilde{\gamma} \pm \sqrt{\tilde{\gamma}^2 + 8i}}{4}$$

The relevant saddle points

$\gamma < 0$	$\tilde{h}_{ab} = v^{(+)}\mathbf{1}$
$\gamma > 0$	$\tilde{h}_{ab} = v^{(-)}\mathbf{1}, \quad \tilde{h}_{ab} = v^{(+)}\mathbf{1}, \quad \tilde{h} = \tilde{\gamma}\,\mathrm{diag}\left(1, 1, \frac{i}{\tilde{\gamma}^2}\right)$

Identification of each saddle with classical solution ($\gamma > 0$)

remaining symmetries

$$\tilde{h}_{ab} = v^{(-)}\mathbf{1}, \quad \tilde{h}_{ab} = v^{(+)}\mathbf{1}, \quad \tilde{h} = \tilde{\gamma}\,\mathrm{diag}\left(1, 1, \frac{i}{\tilde{\gamma}^2}\right)$$

$$\mathrm{SU}(2) \qquad \mathrm{SU}(2) \qquad \mathrm{U}(1)$$

classical solutions [talk by W. Piensuk]

trivial solution

$$A_\mu = 0$$

(unbroken)

$$\mathrm{SO}(9,1) \times \mathrm{SU}(2)$$

Pauli solution

$$A_\mu = \sqrt{\frac{\gamma}{2}}\sigma_\mu \quad \mu = 1, 2, 3$$

diagonal subgroup of

$$\mathrm{SO}(3) \times \mathrm{SU}(2)$$

squashed Pauli solution

$$A_\mu = \sqrt{\gamma}\sigma_\mu \quad \mu = 1, 2$$

diagonal subgroup of

$$\mathrm{SO}(2) \times \mathrm{U}(1)$$

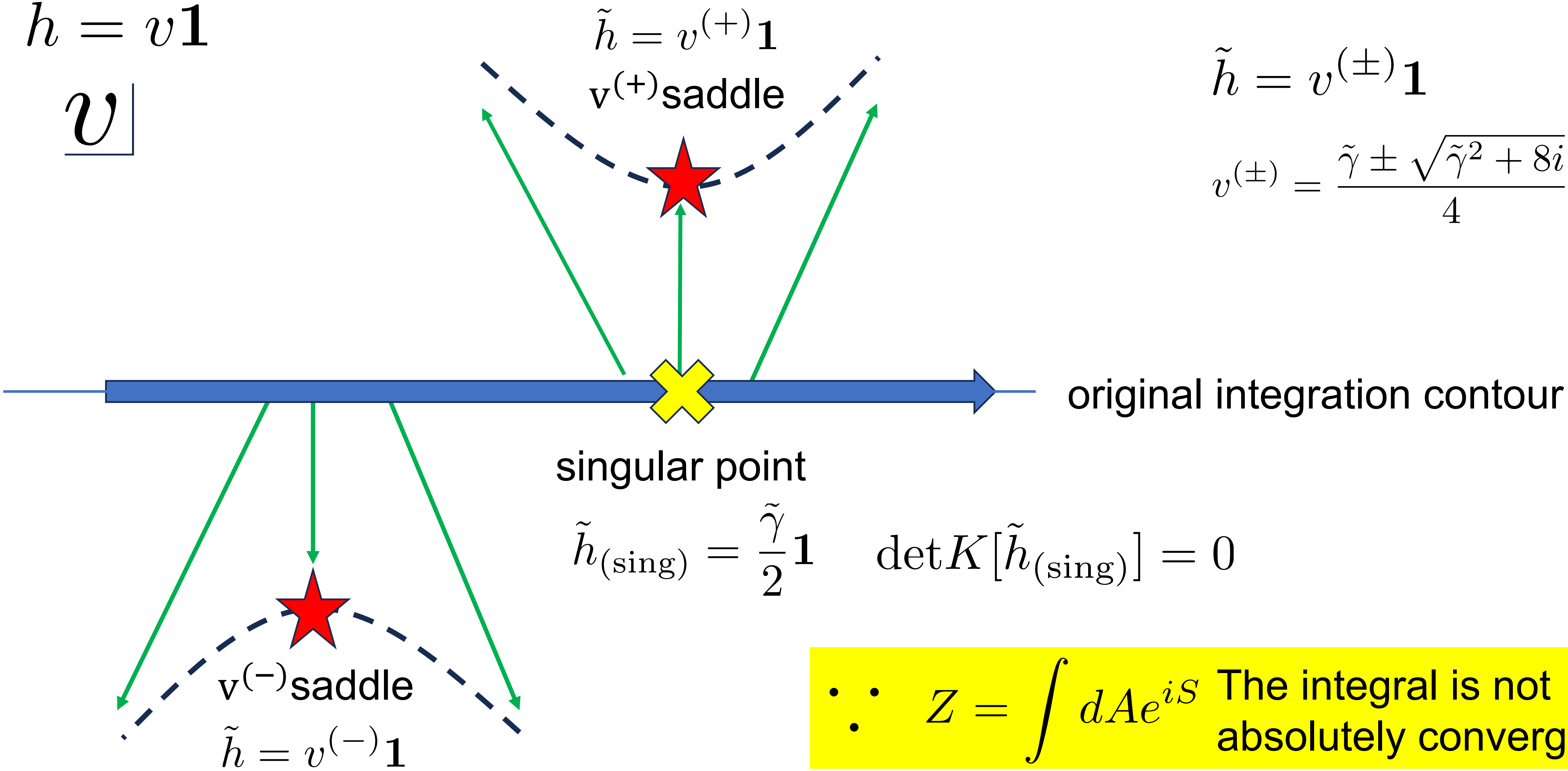
Singularity on the real axis

$$\tilde{h} = v\mathbf{1}$$

$$v$$

$$\tilde{h} = v^{(\pm)}\mathbf{1}$$

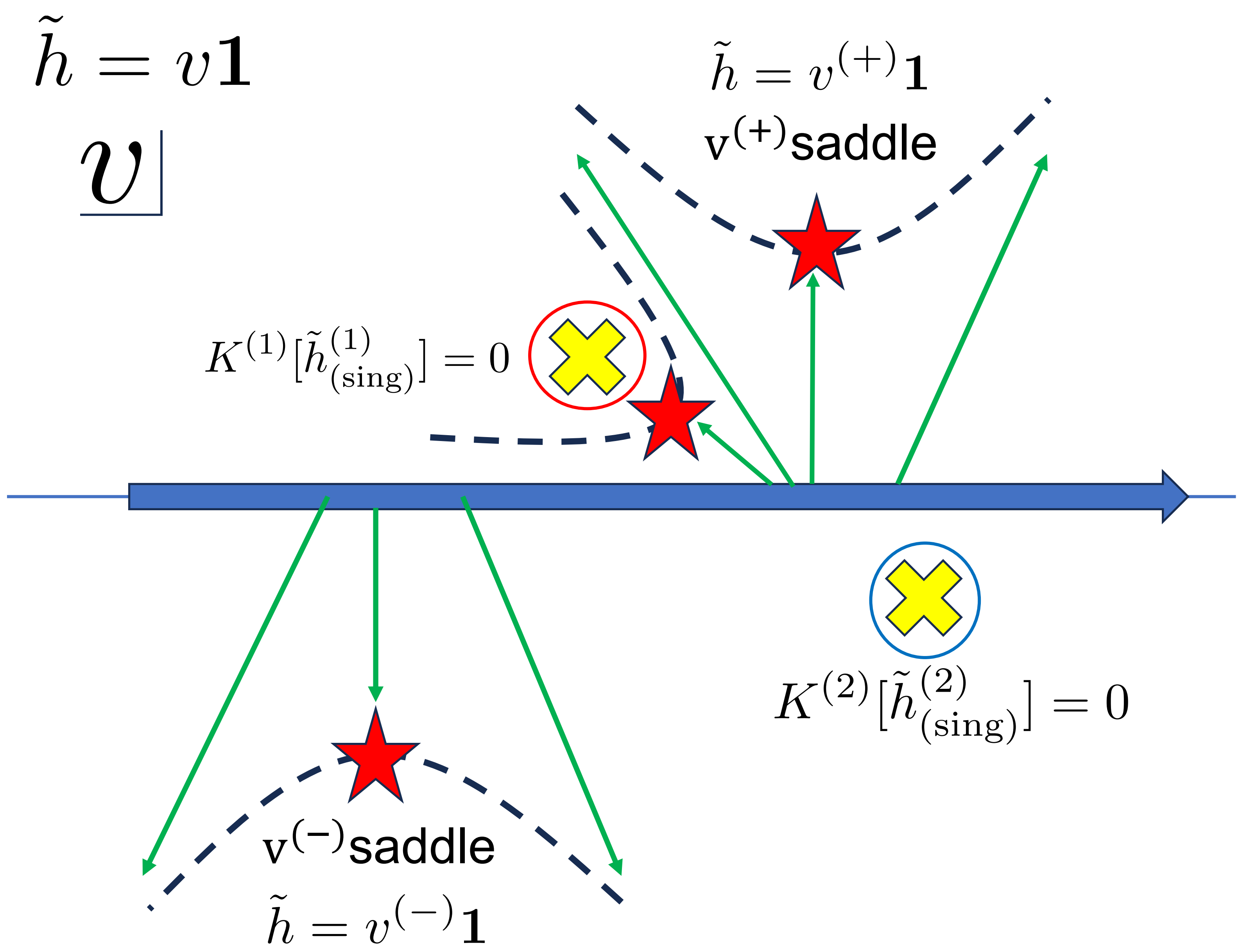
$$v^{(\pm)} = \frac{\tilde{\gamma} \pm \sqrt{\tilde{\gamma}^2 + 8i}}{4}$$



$$\tilde{h}_{(\mathrm{sing})} = \frac{\tilde{\gamma}}{2}\mathbf{1} \quad \det K[\tilde{h}_{(\mathrm{sing})}] = 0$$

$\because$ $Z = \int dA e^{iS}$ The integral is not absolutely convergent

The case with convergence factor ε



$$S_m = \frac{1}{2}N\gamma\left\{e^{i\varepsilon}\mathrm{tr}(A_0)^2 - e^{-i\varepsilon}\mathrm{tr}(A_i)^2\right\}$$

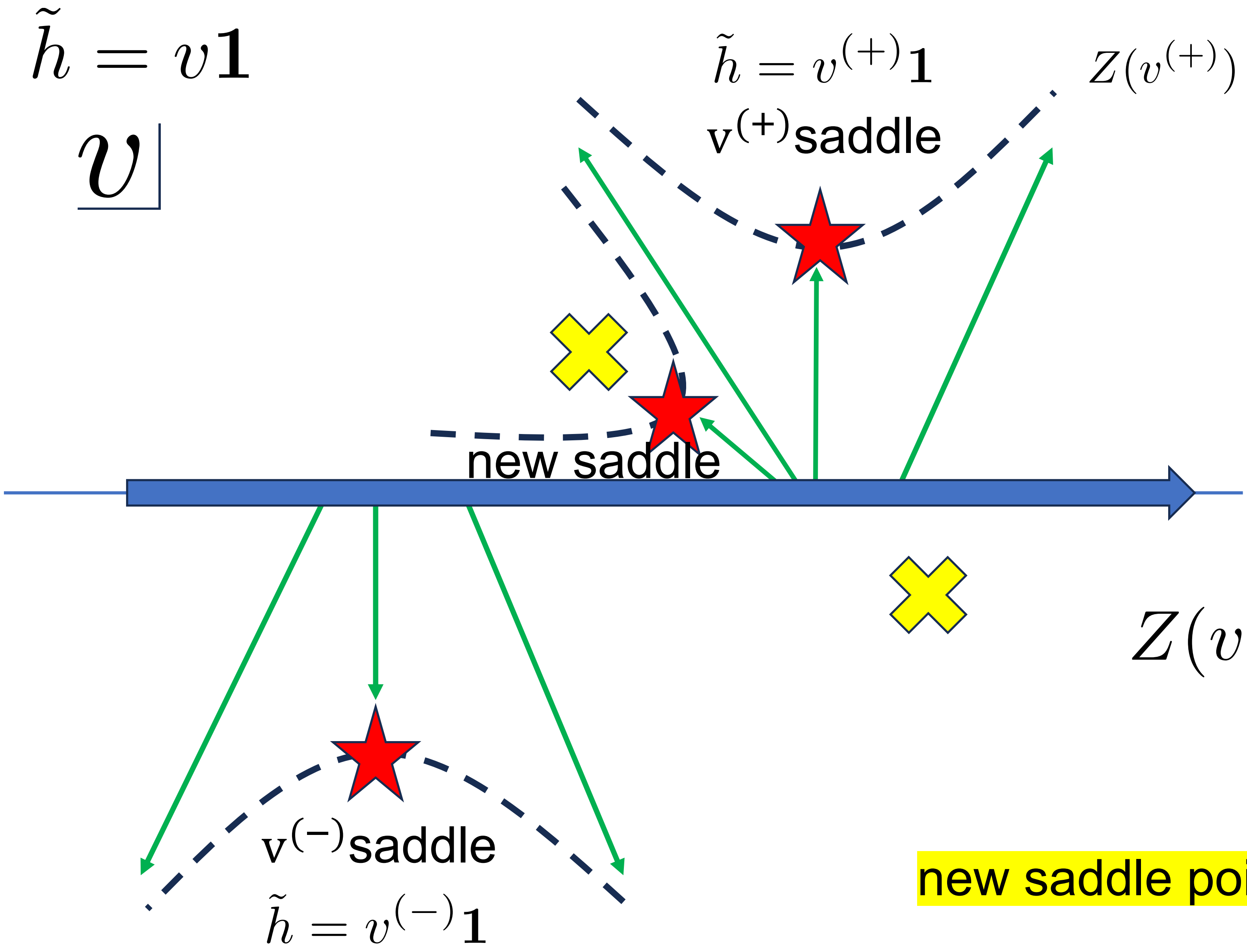
$$Z(\tilde{h}) \sim \det K^{(1)}[\tilde{h}]^{\frac{1}{2}} \cdot \det K^{(2)}[\tilde{h}]^{\frac{D-1}{2}}$$

ε splits the singular point and shifts them

The v$^{(+)}$ saddle becomes relevant and the partition function becomes finite at $\varepsilon \to 0$

$$Z(v^{(+)}) \sim e^{-\frac{3}{8}iD\tilde{\gamma}^2}\left(\frac{\tilde{\gamma}}{2}\right)^{\frac{3}{2}D}$$

The new saddle point



$$Z(v^{(+)}) \sim e^{-\frac{3}{8}iD\tilde{\gamma}^2}\left(\frac{\tilde{\gamma}}{2}\right)^{\frac{3}{2}D}$$

new saddle point appears around singular point

$$Z(v^{\mathrm{new}}) \sim e^{-\frac{3}{8}iD\tilde{\gamma}^2}(\tilde{\gamma}\varepsilon)^{-\frac{3}{2}D}$$

diverges at $\varepsilon \to 0$

new saddle point dominates in the $\varepsilon \to 0$ limit.

Transition at finite ε

$$\begin{cases} Z(v^{\text{new}}) \sim e^{-\frac{3}{8}iD\tilde{\gamma}^2} \left(\tilde{\gamma}\varepsilon\right)^{-\frac{3}{2}D} \\ Z(v^{(+)}) \sim e^{-\frac{3}{8}iD\tilde{\gamma}^2} \left(\frac{\tilde{\gamma}}{2}\right)^{\frac{3}{2}D} \end{cases}$$

$\Rightarrow$ $|Z(v^{\text{new}})| > |Z(v^{(+)})|$

for $\tilde{\gamma} < \tilde{\gamma}_c = \sqrt{\frac{2}{\varepsilon}}$

(large D)

$$\frac{1}{\sqrt{D}} \langle \mathrm{tr}(A_\mu A^\mu) \rangle = \frac{i}{D} \frac{\partial}{\partial \tilde{\gamma}} \ln Z = \begin{cases} \frac{3}{4}\tilde{\gamma} - i\frac{3}{2\tilde{\gamma}} & \text{(for the new saddle)} \\ \frac{3}{4}\tilde{\gamma} + i\frac{3}{2\tilde{\gamma}} & \text{(for the } v^{(+)} \text{ saddle)} \end{cases}$$

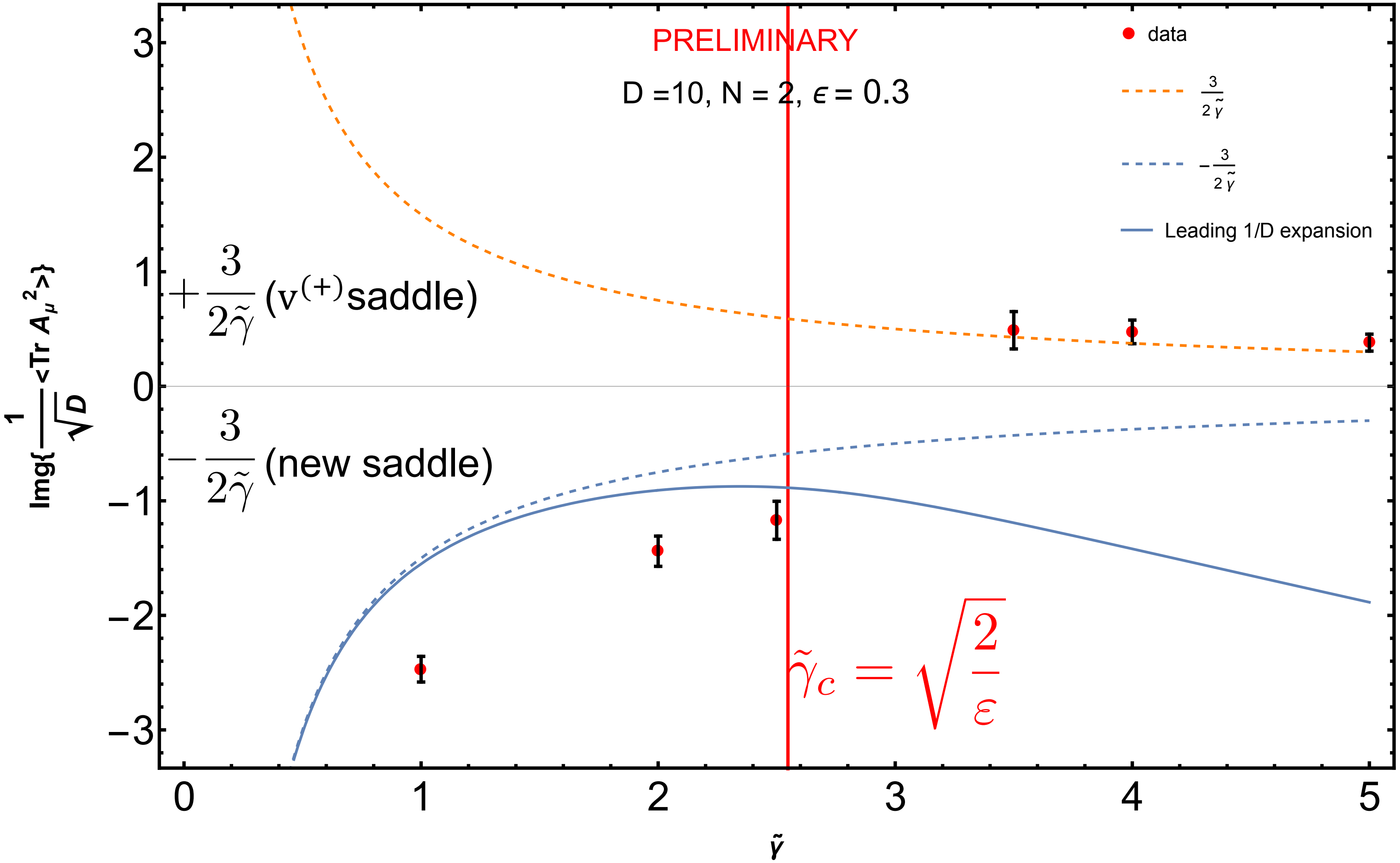
$$\mathrm{Im}\, \langle \mathrm{tr}(A_\mu A^\mu) \rangle = -\frac{3}{2\tilde{\gamma}} \to +\frac{3}{2\tilde{\gamma}} \quad \text{at} \quad \tilde{\gamma}_c = \sqrt{\frac{2}{\varepsilon}}$$

$$\frac{1}{\sqrt{D}} \langle \mathrm{tr}(A^2) \rangle_{\text{Pauli}} \sim \frac{3}{4}\tilde{\gamma}$$

(large γ)

Numerical result for the Pauli thimble

$$\mathrm{Im}\frac{1}{\sqrt{D}}\langle \mathrm{tr}(A_\mu A^\mu)\rangle$$



dominant saddle

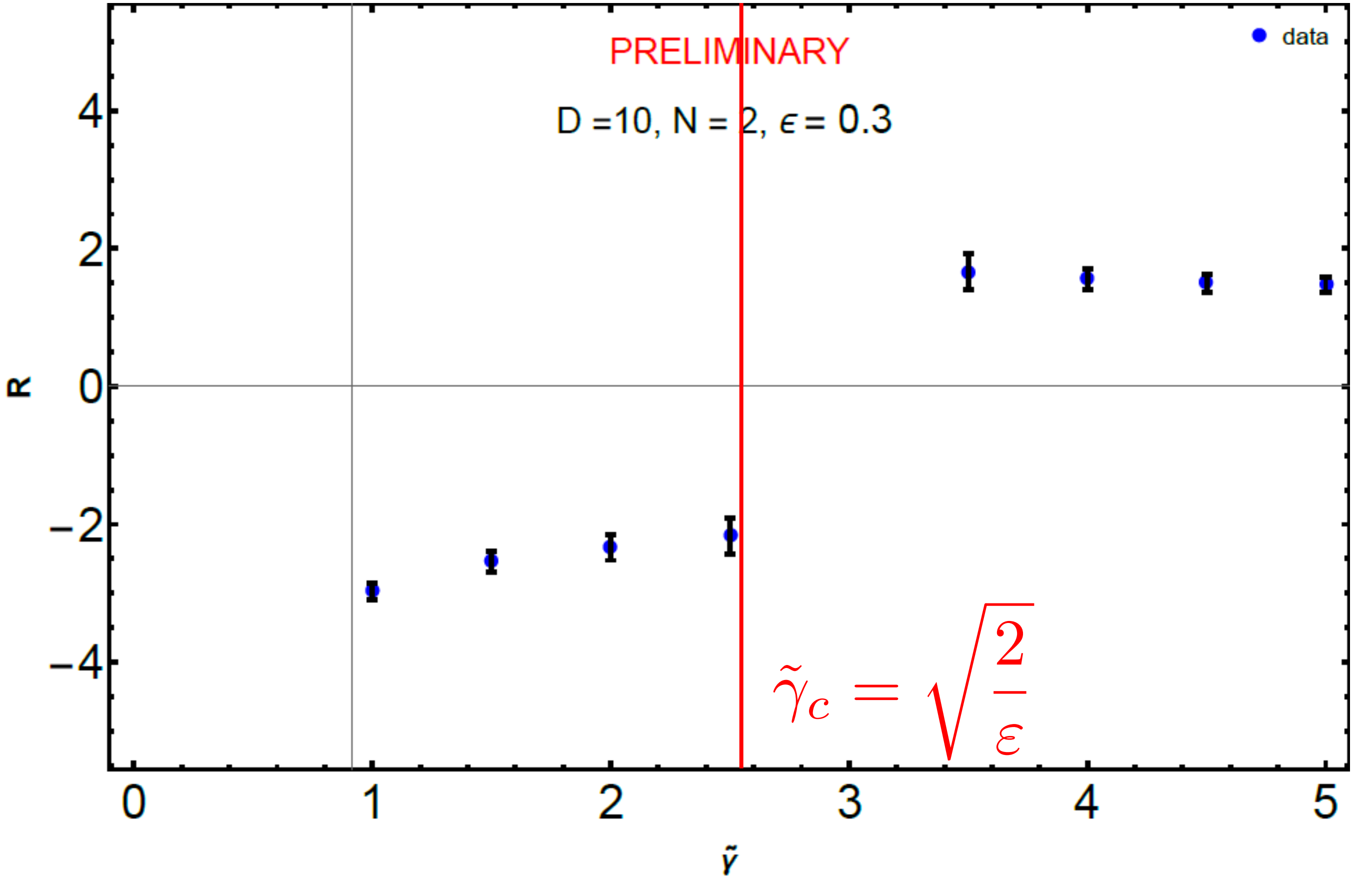
$\tilde{\gamma} < \tilde{\gamma}_c$	new saddle
$\tilde{\gamma} > \tilde{\gamma}_c$	$v^{(+)}$saddle

The transition was confirmed by thimble calculations.

time-like config. vs space-like config.

$$R = -\mathrm{tr}(A_0^\dagger A_0) + \mathrm{tr}(A_i^\dagger A_i)$$

$$\begin{cases} R < 0 & \text{time-like config.} \\ R > 0 & \text{space-like config.} \end{cases}$$



The dominant config. in the thimble associated with the Pauli solution

$\tilde{\gamma} < \tilde{\gamma}_c$	time-like
$\tilde{\gamma} > \tilde{\gamma}_c$	space-like

Divergence for time-like config.

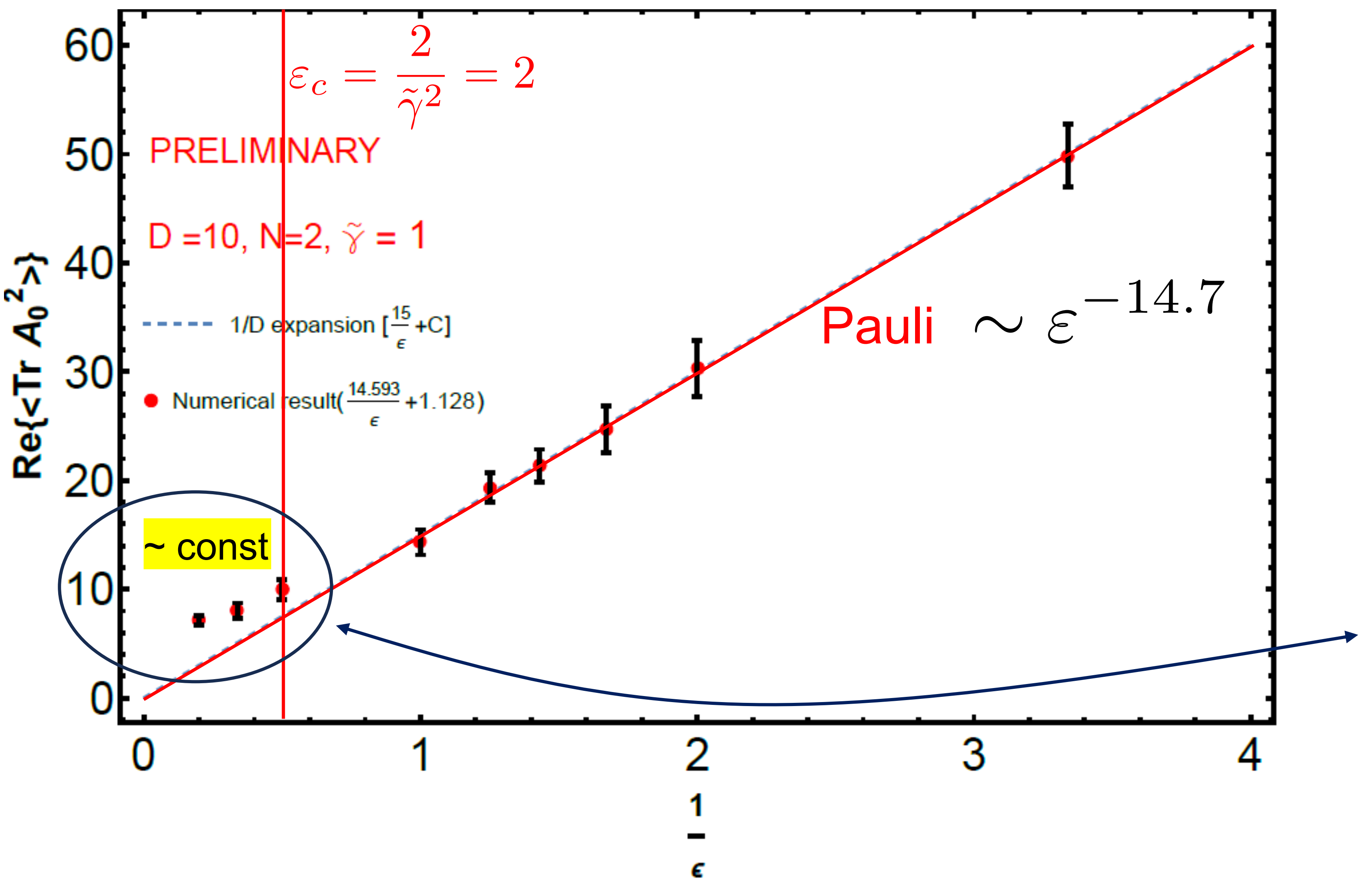
dominant saddle		dominant config.
new saddle	$\tilde{\gamma} < \tilde{\gamma}_c$	time-like
$\mathrm{v}^{(+)}$saddle	$\tilde{\gamma} > \tilde{\gamma}_c$	space-like

$$\begin{cases} Z(v^{\mathrm{new}}) \sim e^{-\frac{3}{8}iD\tilde{\gamma}^2} (\tilde{\gamma}\varepsilon)^{-\frac{3}{2}D} & \text{divergent at } \varepsilon \to 0 \\ Z(v^{(+)}) \sim e^{-\frac{3}{8}iD\tilde{\gamma}^2} \left(\frac{\tilde{\gamma}}{2}\right)^{\frac{3}{2}D} & \text{finite at } \varepsilon \to 0 \end{cases}$$

The divergence is caused by time-like config.

No divergence for space-like configs.

$\mathrm{Re}\left\langle \mathrm{tr}(A_0)^2 \right\rangle$



$$\left\langle \mathrm{tr}(A_0)^2 \right\rangle \sim -\frac{\partial}{\partial \varepsilon} \log Z$$

$$\left\langle \mathrm{tr}(A_0)^2 \right\rangle \sim \frac{c}{\varepsilon}$$

$$Z \sim \varepsilon^{-c}$$

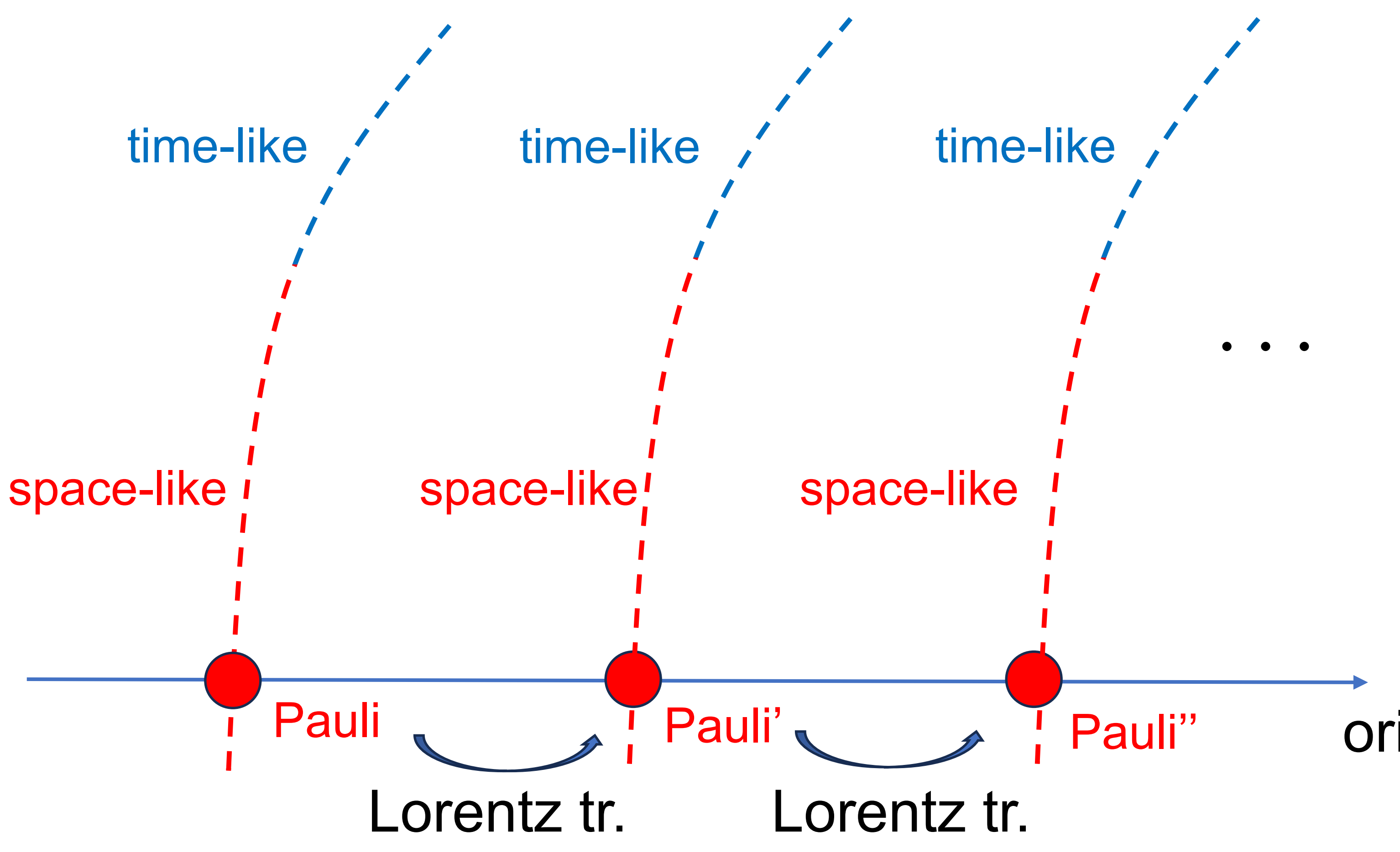
$$Z(v^{(+)}) \sim e^{-\frac{3}{8} i D \tilde{\gamma}^2} \left(\frac{\tilde{\gamma}}{2} \right)^{\frac{3}{2} D}$$

No ε dependence for $v^{(+)}$saddle

$$\left\langle \mathrm{tr}(A_0)^2 \right\rangle \sim f(\tilde{\gamma})$$

← 1/D expansion(on going)

Interpretation of the transition and divergence



non – compactness of Lorentz sym. cause divergence

non - compactness of Lorentz sym. does NOT cause divergence

original integration contour

Summary

- The new saddle appears by introducing ε and associated partition function diverges at $\varepsilon \to 0$
- This new saddle corresponds to the time-like configuration for the Pauli solution

⇒ The partition function for the Pauli thimble diverges due to the non - compact Lorentz transformations of time-like configurations.

- This property appears only in the model which has Lorentz symmetry (SO(D) model defined by $A_D = iA_0$ does not have this property)

Future prospects

- SUSY case

1/D expansion cannot be applied to the SUSY model (with fermion)

→ Numerical simulation by Lefschetz thimble method (on going for N=2)

- Large N

Increase the matrix size N to check the emergence of space-time

(The computational cost of the generalized Lefschetz thimble

method grows with N as $O(N^6)$. But we may still do N=4,8,16,…)