

Recent Developments in Grosse-Wulkenhaar Model from an Integrability Perspective

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Talk Plan

- 1 History and Overview
- 2 Origin from N.C. field theory
- 3 Set Up of N.C. Φ^3 model
- 4 Ward-Takahashi Ids & Schwinger-Dyson Eqs.) **Skip**
5. Large N, L limit
6. Finite N Φ^3 model
7. Finite N Φ^3 - Φ^4 mixed matrix model.
8. Φ^4 model \Rightarrow Harmonic Oscillator System

§1 History

► 90s' Matrix model

⊙ 2D gravity \Leftrightarrow random matrix

Brezin - Kazakov, Gross - Migdal, etc.

Kontsevich model (Witten conjecture)

$$Z[J] = \int \mathcal{D}\Phi \exp(-\text{tr}(\Lambda\Phi + \Phi^3))$$

↑
Fukuma-Kawai
- Nakayama

Φ : Hermite matrix, $\Lambda = \text{diag}(\Lambda_1, \Lambda_2, \dots)$

Makeenko - Semenoff solve this in $N \rightarrow \infty$

▶ 2000's

N.C. field Theory \Rightarrow Matrix model.

Grosse-Steinadcer

\mathbb{F}^3 model (Kontsevich model) ^{basically} Renormalizable

Grosse-Wulkenhaar

\mathbb{F}^4 models in 2, 4 dim are Renormalizable

\mathbb{F}^4 model is solvable

(SD-eq is recursively determined.)

▷ Overview.

① We will see Φ^3 model closely as Review.

- We will see every multi pts - function can be obtained explicitly.

(Not only $N \rightarrow \infty$, but also finite N)

② Φ^4 is still difficult.

(Φ^3 - Φ^4 mixed model will be introduced. ^{New}
(Integrable) _{2304.10369})

Newest

③ Φ^4 model (finite N) is

**N-body Harmonic Oscillator system
without interaction!! (integrable!!)**

§2. ~ The Origin from N.C. field Theory ~

\mathbb{R}_θ^2 : Moyal plane $[A, B] := AB - BA$

$$[x^1, x^2] = i \theta \Leftrightarrow [z, \bar{z}] = 2\theta$$

N.C. parameter

- Annihilation $\left(a := \frac{z}{\sqrt{2\theta}} \right)$ Creation $\left(a^\dagger := \frac{\bar{z}}{\sqrt{2\theta}} \right)$ op.

$$\Rightarrow [a, a^\dagger] = 1, \quad [a, a] = [a^\dagger, a^\dagger] = 0$$

- $\frac{\partial}{\partial z} = -\frac{1}{\sqrt{2\theta}} [a^\dagger,]$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{\sqrt{2\theta}} [a,]$

• Fock sp. $[a, a^\dagger] = 1$, $[a, a] = [a^\dagger, a^\dagger] = 0$

$|0\rangle : a|0\rangle = 0$, $|n\rangle := \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle$

Number op. $N := a^\dagger a$ $N|n\rangle = n|n\rangle$

$\langle n| = \text{dual of } |n\rangle$

$\langle n|m\rangle = \delta_{nm}$

• Scalar field $\Phi = \sum \Phi_{nm} |m\rangle \langle n|$

$\int d^2x \rightarrow \theta^2 \text{Tr}$

Action

$$\begin{aligned} S_1 &= \int d^4x -\phi \left(\frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \right) \phi \\ &= \frac{\theta^2}{2\theta} \text{Tr} \phi [a^\dagger, [a, \phi]] \quad N = a^\dagger a \\ &= \theta \text{Tr} (\phi N \phi - \theta a^\dagger \phi a \phi) \end{aligned}$$

Removing this term by a counter Lagrangian
Renormalizable model is obtained.

$$S_m = \theta \text{Tr} \frac{\mu^2}{2} \phi^2$$

μ : const. (mass)

Interaction

$$S_{\text{int}} = \Theta \frac{\lambda}{3} \text{Tr} \phi^3$$

λ : const
coupling const

$$S = S_1 + S_m + S_{\text{int}} + \text{tadpole}$$

$$= \Theta \text{Tr} \left(\phi E \phi - \underbrace{A \phi} + \frac{\lambda}{3} \phi^3 \right)$$

$$\text{where } E_{nm} = \left(\frac{1}{2} \mu^2 + \eta \right) \delta_{nm}$$

A : const

§3 Set Up of NC \mathbb{F}^3 (2-dim case for simplicity)

Hermitian Matrix $\Phi = \Phi^\dagger \in M_N(\mathbb{C})$

Action $S = L \text{Tr} (E \Phi^2 - A \Phi) + V(\Phi)$

$V(\Phi) = L \frac{\lambda}{3} \text{Tr} \Phi^3$ (Kontsevich model)

$E = (\underbrace{E_m}_{\delta_{mn}}) = \begin{pmatrix} E_1 & & 0 \\ 0 & E_2 & \\ & & \ddots \end{pmatrix}$ ← Not $U(N)$ Sym but $U(1)^N$ Sym

$E_m = \mu^2 \left(\frac{1}{2} + e \left(\frac{m}{\mu^2 L} \right) \right)$

$\theta \rightarrow$
 $A, \lambda, L, \mu : \text{const.}$ \uparrow $e(0)=0, C^\infty\text{-fun}$

$$S = L \left(\sum_{n,m} \frac{1}{2} \Phi_{nm} \Phi_{mn} H_{nm} - A \sum_{m=0}^N \Phi_{mm} + \frac{\lambda}{3} \sum_{k,l,m} \Phi_{kl} \Phi_{lm} \Phi_{mk} \right)$$

$$H_{mn} := E_m + E_n = \mu^2 \left(1 + e^{\left(\frac{m}{\mu^2 L}\right)} + e^{\left(\frac{n}{\mu^2 L}\right)} \right)$$

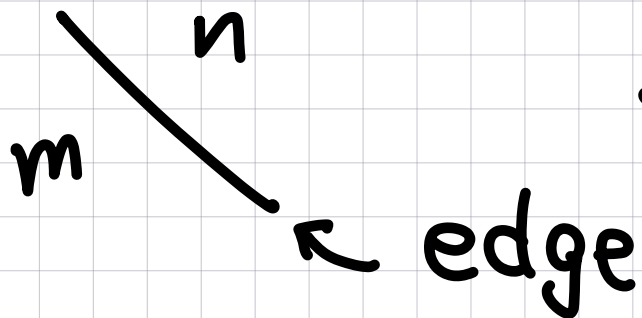
$$Z[J] := \int \mathcal{D}\Phi e^{-S + L \text{Tr}(J\Phi)}$$

$$= K \exp\left(-V\left(\frac{1}{L} \frac{\partial}{\partial J}\right)\right) Z_{\text{free}}[J]$$

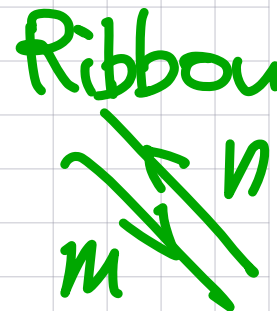
$$Z_{\text{free}} = e^{\sum \frac{L}{2} (\delta_{nm} A + J_{nm}) H_{nm}^{-1} (\delta_{nm} A + J_{nm})}$$

Remark) Correspondence with Graphs.

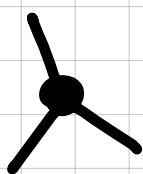
propagator:



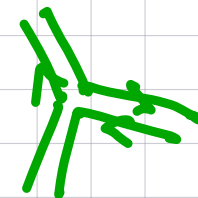
$$\sim \frac{1}{H_{mn}}$$



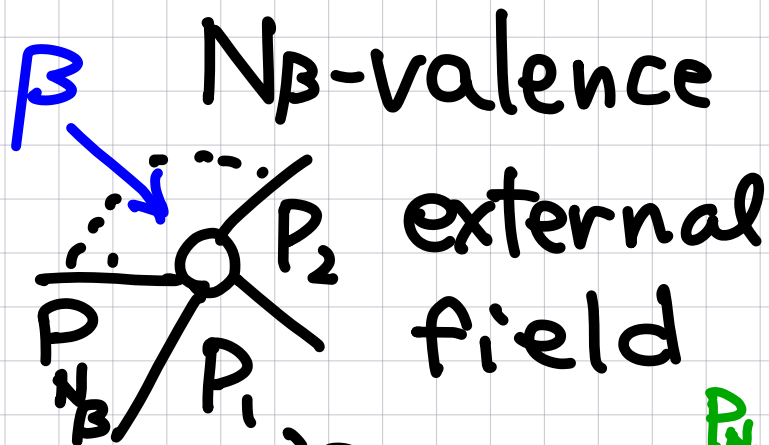
Black vertex:



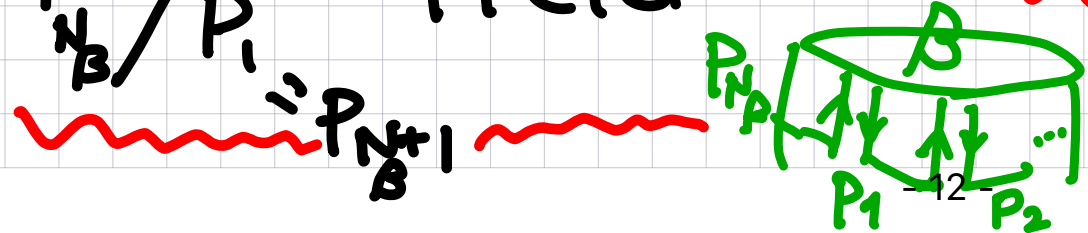
$$\sim \lambda \text{tr} \Phi^3$$



White vertex:



$$\sim \prod_{i=1}^{N_\beta} J_{P_i P_{i+1}}$$



$$(N_\beta + 1 \equiv 1)$$

$$\log \frac{\sum [J]}{\sum [0]} = \underbrace{(N_1 + \dots + N_B)\text{-point fun}}_{\text{}} \sum_{B=1}^{\infty} \sum_{1 \leq N_1 \leq \dots \leq N_B} \sum_{\substack{N \\ P_i^j = 0}} L^{2-B} \frac{G(|P_1^1 \dots P_{N_1}^1| \dots |P_1^B \dots P_{N_B}^B|)}{S(N_1, \dots, N_B)} \prod_{\beta=1}^B \frac{J_{P_1^\beta \dots P_{N_\beta}^\beta}}{N_\beta}$$

= Generating fun of connected graphs
(Green fun.)

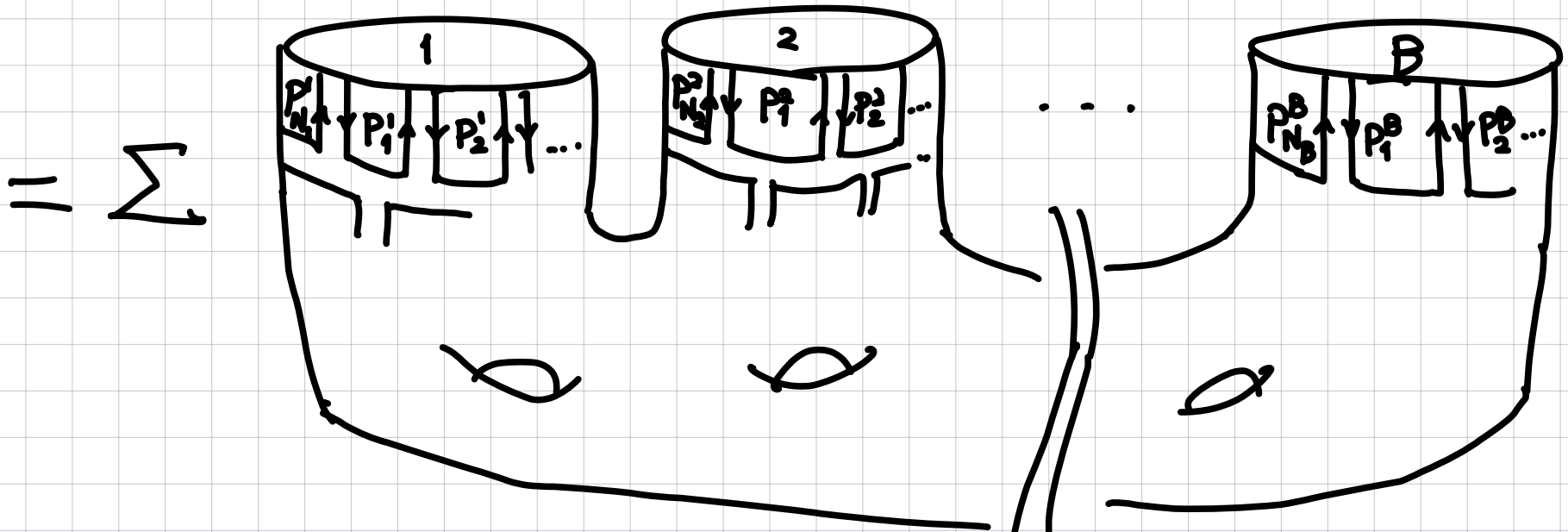
$$S(N_1, \dots, N_B) := \prod_{i=1}^B N_i! \quad \text{statistical factor}$$

for $(N_1, \dots, N_B) = (\underbrace{N_1', \dots, N_1'}_{N_1}, \dots, \underbrace{N_s', \dots, N_s'}_{N_s})$

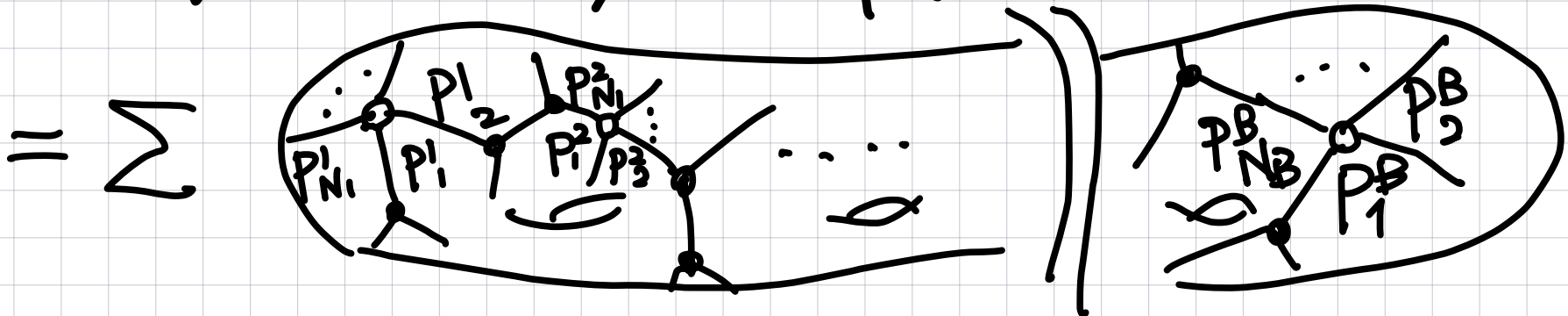
L^{2-B} : We choose this factor to obtain all Npt function as finite at Large (L, N) lim

Feynman Graph

$$G |P_1^1 P_2^1 \dots P_{N_1}^1 | P_1^2 P_2^2 \dots P_{N_2}^2 | \dots | P_1^B P_2^B \dots P_{N_B}^B |$$



O: white vertex is a puncture.



$\sim \mathcal{N} := \sum_{i=1}^B N_i$ - external ribbon case \sim

Let us count the power of L in connected Feynman diagram

$$\langle \Phi_{a_1 a_2} \cdots \Phi_{a_{N_1} a_1} \Phi_{a_1 a_2} \cdots \Phi_{a_{N_2} a_2} \cdots \Phi_{a_1^B a_2^B} \cdots \Phi_{a_{N_B}^B a_1^B} \rangle_c$$

$$= \frac{1}{L^{\mathcal{N}}} \frac{\partial}{\partial J_{a_1 a_2}} \cdots \frac{\partial}{\partial J_{a_1^B a_2^B}} \log Z[J] \Big|_{J=0}$$

(For simplicity, a_j^i ($i=1, \dots, B, j=1, \dots, N_i$) pairwise different)

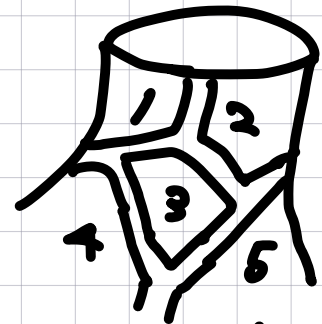
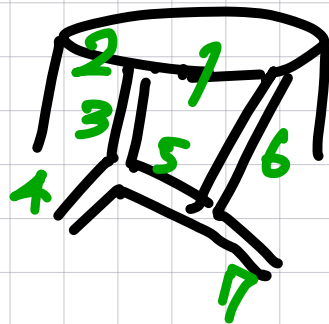
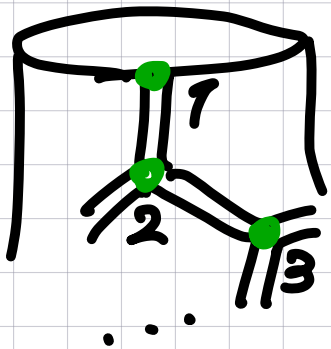
k_3 : # of $\frac{1}{3} \lambda \text{tr} \Phi^3$ interaction

graph $\sim L^{k_3 - \frac{3k_3 + \mathcal{N}}{2}}$

propagator $\left(\frac{1}{L}\right)^{\frac{3k_3 + \mathcal{N}}{2}}$
 vertex L^{k_3}

Euler $\chi = 2 - 2g - B$ (g : genus, B : boundaries)

$$= \underbrace{(R_3 + N)}_{\text{Vertex}} - \underbrace{\left(\frac{3R_3 + N}{2} + N\right)}_{\text{Edge}} + \underbrace{(N + \Sigma)}_{\text{Face}} \quad \leftarrow \text{loop}$$



$$\langle \dots \rangle_c \sim L^{R_3 - \frac{3R_3 + N}{2}} = L^{\chi - N - \Sigma} = L^{2 - 2g - B - N - \Sigma}$$

$$\langle \dots \rangle_c = L^{2 - N - B} \underbrace{G |a'_1 \dots a'_{N_1}| \dots |a^B_1 \dots a^B_{N_B}|}_{}$$

This $G_{|\dots|}$ is given by the sum over all Feynman graph with fixed $B, \{N_1, \dots, N_B\}$

↳ When we expand it by q .

$$G(a_1 \dots a_{N_1} | \dots | a_1^B \dots a_{N_B}^B) = \sum_{q=0}^{\infty} L^{-2q} G^{(q)}(a_1 \dots a_{N_1} | \dots | a_1^B \dots a_{N_B}^B)$$

(See for ex "Laplacian to compute Intersection Numbers on $\overline{M}_{g,n}$ and Correlation Functions in NCQFT" Hock - Grosse - Wulkenhaar)

SKIP

§4. ~ W.T. Id & SDEs. ~

~ Ward Takahashi like Id. ~

$$\Phi \rightarrow \Phi' = \Phi + [u, \Phi] \leftarrow Z[J] \text{ is inv.}$$

WT-Id.

$$\sum_m \frac{\partial^2 Z[J]}{\partial J_{am} \partial J_{mb}} = \sum_m \frac{L}{E_a - E_b} \left(J_{ma} \frac{\partial}{\partial J_{mb}} - J_{bm} \frac{\partial}{\partial J_{am}} \right) Z[J]$$

2pt functions are reduced to
1pt functions by the WT-Id.

~ Schwinger-Dyson Eqs. ~

▷ 1pt function

$$G_{|a|} := \frac{1}{L} \frac{\partial \log Z[J]}{\partial J_a} \Big|_{J=0}$$

$$= H_{aa}^{-1} (A - \lambda G_{|a|}^2 - \frac{\lambda}{L} \sum_{m=0}^N G_{|a| m} - \frac{\lambda}{L^2} G_{|a| a}) \quad \text{--- ①}$$

▷ 2pt fun.

$$G_{|a| b|} := \frac{1}{L} \frac{\partial^2 \log Z[J]}{\partial J_a \partial J_b} \Big|_{J=0}$$

Using W-T id.
2pt → 1pt

$$= H_{ab}^{-1} \left(1 + \lambda \frac{G_{|a|} - G_{|b|}}{E_a - E_b} \right) \quad \text{--- ②}$$

▷ Renormalization Condition

$$G_{101} = 0 \iff A = \frac{\lambda}{L} \sum_{m=0}^N G_{10m_1} + \frac{\lambda}{L^2} G_{10101}$$

⇓ Remove A in ① by using this condition

$$G_{101} = H_{aa}^{-1} \left\{ -\lambda G_{101}^2 - \frac{\lambda}{L} \sum_m (H_{am}^{-1} - H_{0m}^{-1}) - \frac{\lambda}{L^2} (G_{10101} \bar{G}_{10101}) - \frac{\lambda^2}{L} \sum_m \left(H_{am}^{-1} \frac{(G_{101} - G_{1m1})}{E_a - E_m} - H_{0m}^{-1} \frac{G_{1m1}}{E_m - E_0} \right) \right\} \dots \textcircled{1}'$$

⇓ using $\frac{W_{101}}{2\lambda} := G_{101} + \frac{H_{aa}}{2\lambda} = G_{101} + \frac{E_a}{\lambda}$

①' ② are simplified.

Schwinger - Dyson Eqs. for 1pt, 2pt fun.

$$\bullet W_{|a|}^2 = 4E_a^2 - \frac{4\lambda^2}{L^2} (G_{|a|a|} - G_{|0|0|})$$

$$- \frac{2\lambda^2}{L} \sum_{m=0}^{\infty} \left(\frac{W_{|a|} - W_{|m|}}{E_a^2 - E_m^2} - \frac{W_{|m|} - W_{|0|}}{E_m^2 - E_0^2} \right)$$

$$\bullet G_{|a|b|} = \frac{1}{2} \frac{W_{|a|} - W_{|b|}}{E_a^2 - E_b^2}$$

Skip

§5 ~ Large (N, L) -lim & Solutions ~

matrix size & N.C. parameter

$N, L \rightarrow \infty$ with fixing $\frac{N}{L} = \mu^2 \Lambda^2$

$$\lim_{L \rightarrow \infty} \frac{1}{L} \sum_{m=0}^N f\left(\frac{m}{L}\right) = \mu^2 \Lambda^2 \int_0^1 f(\mu^2 \Lambda^2 x) dx$$
$$= \mu^2 \int_0^{\Lambda^2} f(\mu^2 x) dx$$

$$\mu^2 W(x) = \lim_{L \rightarrow \infty} W|_{L\mu^2 x}$$

$$G_1(x) = \lim_{L \rightarrow \infty} G_1|_{L\mu^2 x} \text{ etc.}$$

$$X := (2E(x) + 1)^2$$

Using these
expression

↓ Schwinger-Dyson eq for 1pt fun. →

$$W^2(x) + \int_1^{\Xi} dY \rho(Y) \frac{W(x) - W(Y)}{x - Y} = X + \int_1^{\Xi} dY \rho(Y) \frac{W(1) - W(Y)}{1 - Y}$$

const.

where $\rho(Y) = \frac{2\tilde{\lambda}^2}{\sqrt{Y} e' \left(e^{-1} \left(\frac{\sqrt{Y}-1}{2} \right) \right)}$, $\Xi = (1 + 2e(\tilde{\lambda}^2))^2$

$$\tilde{\lambda} = \frac{\lambda \mu}{2}$$

↑ Makeenko-Semenoff solved similar type

Solution →

$$W(x) := \sqrt{x+c} + \frac{1}{2} \int_1^{\Xi} dz \frac{\rho(z)}{(\sqrt{x+c} + \sqrt{z+c}) \sqrt{z+c}}$$

ex). N.C. scalar ϕ^3 field theory

$$e(x) = x, \quad X = (2x+1)^2, \quad \rho(Y) = \frac{2\tilde{\lambda}^2}{\sqrt{Y}}$$

$$W(X) = \sqrt{X+c} + \frac{2\tilde{\lambda}^2}{\sqrt{X}} \log \left(\frac{(\sqrt{X}+1)(\sqrt{X+c} + \sqrt{X})}{\sqrt{X+c}X + \sqrt{X+c}} \right)$$

$$G(x) = \frac{\sqrt{(2x+1)^2+c} - (2x+1)}{2\tilde{\lambda}} + \frac{\tilde{\lambda}}{2x+1} \log \left(\frac{(2x+2)(\sqrt{(2x+1)^2+c} + 2x+1)}{(2x+1)\sqrt{1+c} + \sqrt{(2x+1)^2+c}} \right)$$

~ $(N_1 + \dots + N_B)$ -pt function ~

$$G(a_1^1 \dots a_{N_1}^1 \dots | a_1^B \dots a_{N_B}^B) = L^{B-2} \frac{\partial^{N_1}}{\partial J_{a_1^1 \dots a_{N_1}^1}} \dots \frac{\partial^{N_B}}{\partial J_{a_1^B \dots a_{N_B}^B}} \log \frac{Z[J]}{Z[0]}$$

where $\frac{\partial^N}{\partial J_{a_1 \dots a_N}} = \frac{\partial}{\partial J_{a_1 a_2}} \frac{\partial}{\partial J_{a_2 a_3}} \dots \frac{\partial}{\partial J_{a_{N-1} a_N}} \frac{\partial}{\partial J_{a_1 a_1}}$ Feynman graph

$N, L \rightarrow \infty$

(Using W-T Id.)



Point

$$G(x_1^1, \dots, x_{N_1}^1 | \dots | x_1^B, \dots, x_{N_B}^B)$$

$$= \lambda^{N_1 + \dots + N_B - B} \sum_{k_1=1}^{N_1} \dots \sum_{k_B=1}^{N_B} G(x_{k_1}^1 | \dots | x_{k_B}^B) \prod_{\beta=1}^B \prod_{\substack{\alpha=1 \\ \alpha \neq k_\beta}}^{N_\beta} \frac{4}{x_{k_\beta}^\beta - x_\alpha^\beta}$$

If we obtain this, then every $(N_1 + \dots + N_B)$ -pt function is solved!!

$$G(a^1 \dots a^B) = L^{B-2} \frac{\partial}{\partial J_{a_1 a_1}} \dots \frac{\partial}{\partial J_{a_B a_B}} \log \frac{Z[J]}{Z[0]} \Big|_{J=0}$$

\downarrow SD-eg. for this } Similar process
 $L, N \rightarrow \infty$ lim as previous discussions

S-D eg

$$\begin{aligned}
 & W(x^1) G(x^1 | X^{\{2, \dots, B\}}) + \frac{1}{2} \int_1^{\infty} d\tau \rho(\tau) \frac{G(x^1 | X^{\{2, \dots, B\}}) - G(\tau | X^{\{2, \dots, B\}})}{(x - \tau)} \\
 & = -\tilde{\lambda} \sum_{\beta=2}^B G(x^1, x^\beta, x^\beta | X^{\{2, \dots, B\}}) - \tilde{\lambda} \sum_{\substack{J \subset \{2, \dots, B\} \\ 1 \leq |J| \leq B-2}} G(x^1 | X^J) G(x^1 | X^{\{2, \dots, B\} \setminus J})
 \end{aligned}$$

where $G(x | Y^J) = G(x | Y^{j_1} | Y^{j_2} | \dots | Y^{j_p})$ for $J = \{j_1, \dots, j_p\}$

▷ Solution for (1+1)-pt fun.

↓ S-D eq.

$$W(x)G(x|\gamma) = -\tilde{\lambda} G(x, \gamma, \gamma) - \frac{1}{2} \int_1^{\infty} dz \rho(z) \frac{G(x|\gamma) - G(z|\gamma)}{x-z}$$

Solution

$$4\tilde{\lambda}^2$$

$$G(x|\gamma) = \frac{4\tilde{\lambda}^2}{\sqrt{x+c} \sqrt{\gamma+c} (\sqrt{x+c} + \sqrt{\gamma+c})^2}$$

▷ Solution for $B \geq 3$

$$G(x^1 \dots | x^B) = (-2\tilde{\lambda})^{3B-4} \left(\frac{d}{dt} \right)^{B-3} \left(\frac{\left(\frac{1}{\sqrt{x^1+c-2t}} \right)^3 \dots \left(\frac{1}{\sqrt{x^B+c-2t}} \right)^3}{\left(1 - \int_1^{\infty} d\tau \rho(\tau) \frac{1}{\sqrt{\tau+c} \sqrt{\tau+c-2t} (\sqrt{\tau+c} + \sqrt{\tau+c-2t})} \right)^{B-2}} \right) \Big|_{t=0}$$

Every N-pt function is solved exactly!

§6 Finite Φ^3 -model.

$$S = V \operatorname{tr} \left(E \Phi^2 + \kappa \Phi + \frac{\lambda}{3} \Phi^3 \right) - V \operatorname{tr} J \Phi$$

$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \text{diag}(E_1, E_2, \dots, E_N) & \text{const} & \text{external field} & J^\dagger = J \\ E_i > 0 & & & \uparrow \end{array}$

Remark.

When we put $\kappa = 0$ & $J = 0$,

this model becomes Kontsevich model.

\Rightarrow

$Z(E)$ corresponds to τ function

\uparrow of KdV hierarchy.

▷ Partition function for finite N

$$Z = \int d\Phi \exp(-iV \text{tr}(E\Phi^2 + K\Phi + \frac{\lambda}{3}\Phi^3)) \exp(iV t_2 J\Phi)$$

$$\left\{ \begin{array}{l} \Phi = X - E \end{array} \right.$$

$$= \exp(-iV \text{tr}(\frac{2}{3\lambda}E^3 - \frac{K}{\lambda}E + \frac{1}{\lambda}JE))$$

$$\int dX \exp(-i\frac{\lambda V}{3} \text{tr} X^3) \exp(i\lambda V K \text{tr}\{(M-I+K)UXU^{-1}\})$$

↓, where $M = \frac{J^2}{K^2}$, $K = \frac{J}{\lambda K}$

I Z (HC) integral

$$\int_{U(n)} \exp t \text{tr} A U B U^{-1} dU = C \frac{\det_{1 \leq i, j \leq n} \exp(t \lambda_i(A) \lambda_j(B))}{t^{\frac{n^2-n}{2}} \Delta(\lambda(A)) \Delta(\lambda(B))}$$

eigen value of A & B

Vandermonde det on $\lambda_i(A)$ & $\lambda_i(B)$

$$Z[J] = C \exp\left(-i \frac{V}{\lambda^2} \text{tr} \left(\frac{2}{3} E^3 - \lambda \kappa E + \lambda J E \right)\right) \frac{\det(\Phi_i(S_j))}{\prod_{1 \leq t < u \leq N} (S_u - S_t)}$$

where S_i is an eigenvalue of $\frac{E^2}{\lambda \kappa} - I + \frac{J}{\kappa}$

$$\Phi_R(z) = 2\pi i \left(\frac{i}{\lambda V}\right)^{k-1} \left(\frac{d}{dy}\right)^{R-1} A_i[y] \Big|_{y = -\frac{V \kappa z}{(\lambda V)^{\frac{1}{2}}}}$$

↑
Airy fun.

or using

$$A_N(y_1, \dots, y_N) = \prod (\partial_{y_i} - \partial_{y_j}) A_i(y_1) \dots A_i(y_N) = \det(A_i^{(j-1)}(y_i))$$

$$y_i = -\frac{V \kappa}{(\lambda V)^{\frac{1}{2}}} S_i$$

$$Z[J] = C' e^{\frac{-iV}{\lambda} \text{tr}(JE)} \frac{A_N(y_1, \dots, y_N)}{\prod_{1 \leq t < u \leq N} (S_u - S_t)}$$

Thm.

$$G|a_1^i \dots a_{N_1}^i| \dots |a_1^B \dots a_{N_B}^B|$$

Every N pt - function is described by $G|a_1^i| \dots |a_1^i|$ type.

$$= \lambda^{N-B} \sum_{k_1=1}^{N_1} \dots \sum_{k_B=1}^{N_B} G|a_{k_1}^i| \dots |a_{k_B}^B| \prod_{\substack{l_1=1 \\ l_1 \neq k_1}}^{N_1} \frac{1}{E_{a_{l_1}^i}^2 - E_{a_{k_1}^i}^2} \dots \prod_{\substack{l_B=1 \\ l_B \neq k_B}}^{N_B} \frac{1}{E_{a_{l_B}^B}^2 - E_{a_{k_B}^B}^2}$$

\Downarrow

We should do cal. of $G|a_1^i| \dots |a_1^i|$ type n pt - function.

$$G|a_1^i| \dots |a_1^i| = (iV)^{n-2} \frac{\partial^n}{\partial J_{a_1} \dots \partial J_{a_n}} \log Z[J] \Big|_{J=0}$$

\Downarrow

diagonal J is enough.

$$\text{"eigenvalue of } \frac{E_i^2}{\lambda k} - I + \frac{J}{k} \text{"} = \frac{E_i^2}{\lambda k} - 1 + \frac{J_i}{k} \quad i=1 \dots N$$

$$\Rightarrow G|a\rangle = -\frac{E_a}{\lambda} - \frac{\lambda}{i\nu} \sum_{i \neq a} \frac{1}{E_a^2 - E_i^2} + \frac{i}{(\lambda\nu)^{\frac{1}{2}}} \partial_a \log A_N(z)$$

$$z_i = -\frac{\nu E_i^2}{(\lambda\nu)^{\frac{1}{2}} \lambda} + \frac{\nu \kappa}{(\lambda\nu)^{\frac{1}{2}}}$$

Similarly

$$G|a_1 a_2 \dots a_n\rangle = (i\nu)^{n-1} C \left(\frac{\nu^{\frac{2}{3}n}}{\lambda^{\frac{1}{3}n}} \right) \partial_{a_1} \dots \partial_{a_n} \log A_N(z_1, \dots, z_N)$$

($n \geq 3$)

We can obtain every N -point function from above.

§7 $\Phi^3 - \Phi^4$ mixed matrix model (finite N)

Φ^4 (Grosse-Wulkenhaar) model is still not enough to be calculated, and its properties are not unveiled.

Integrable model $\Phi^3 - \Phi^4$ mixed model might help to reveal properties of Φ^4 model. ↔ Itykson Zuber 92

$$S[\Phi] = V \text{tr} \left(E \Phi^2 + \frac{1}{2} M \Phi M \Phi + \sqrt{\lambda} M \Phi^3 + \frac{\lambda}{4} \Phi^4 \right)$$

const. $\begin{pmatrix} E_1 & & & \\ & E_2 & & \\ & & \ddots & \\ 0 & & & \ddots \end{pmatrix}$
↑
coupling const.

$$M^2 = E, \quad M = \begin{pmatrix} \sqrt{E_1} & & & \\ & \sqrt{E_2} & & \\ & & \ddots & \\ 0 & & & \ddots \end{pmatrix}$$

constant matrix

$$Z[J] = \int D\Phi \, e^{-S} e^{V \text{tr} J \Phi}$$

▷ Feynman Rules

$$S_{\text{free}} = V \text{tr} \left(E \bar{\Phi}^2 + \frac{1}{2} M \Phi M \Phi \right)$$

$$Z_{\text{free}}[J] := \int D\Phi e^{-S + V \text{tr} J \bar{\Phi}}$$

$$= C \exp \left(\frac{V}{2} \sum_{n,m=1}^N J_{mn} \frac{1}{E_n + E_m + \sqrt{E_n E_m}} J_{nm} \right)$$

↑
const. $Z_{\text{free}}[0]$


• Propagator

$$\begin{array}{ccc} a & \longrightarrow & d \\ b & \longleftarrow & c \end{array} := \langle \Phi_{ba} \bar{\Phi}_{dc} \rangle = \frac{1}{V} \frac{\delta_{ad} \delta_{bc}}{E_a + E_b + \sqrt{E_a E_b}}$$

$$Z[J] = \int D\Phi e^{-S_{\text{free}}} e^{-S_{\text{int}}} e^{V \text{tr} J \Phi}$$

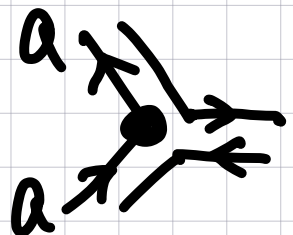
$$S_{\text{int}} = V \text{tr} \left(\sqrt{\lambda} M \Phi^3 + \frac{\lambda}{4} \Phi^4 \right)$$

- 4pt Interaction



$$= -\frac{V\lambda}{4} \quad (\text{without statistical factors})$$

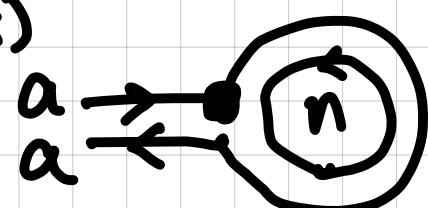
- 3pt Interaction



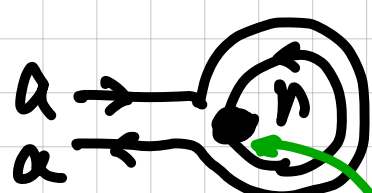
$$= -V\sqrt{\lambda} E_a \quad (\text{without statistical factors})$$

- loop $\sum_{i=1}^n M_i$

Ex)

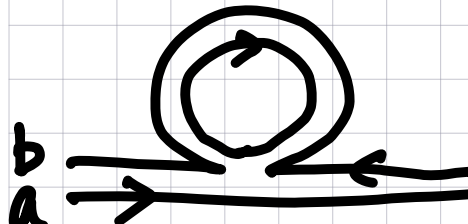


$$= \frac{1}{v 3 E_a} \times (-V \sqrt{\lambda E_a}) \sum_{n=1}^N \frac{1}{v (E_a + E_n + \sqrt{E_a E_n})}$$



$$= \frac{1}{v 3 E_a} \times \sum_{n=1}^N \frac{-V \sqrt{\lambda E_n}}{v (E_a + E_n + \sqrt{E_a E_n})}$$

on the loop

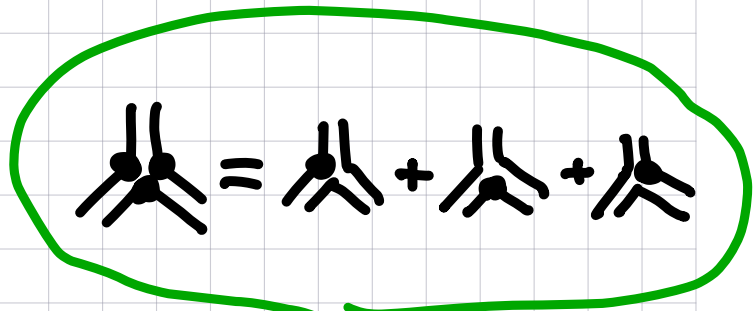


$$= -\frac{\lambda}{4V^2} \frac{1}{(E_a + E_b + \sqrt{E_a E_b})^2}$$

$$\times \sum_{n=1}^N \frac{1}{E_b + E_n + \sqrt{E_b E_n}}$$

$\Delta E_x.$

$$G|a|b| = \frac{\partial^2 \log Z[J]}{\partial J_{aa} \partial J_{bb}} \Big|_{J=0}$$



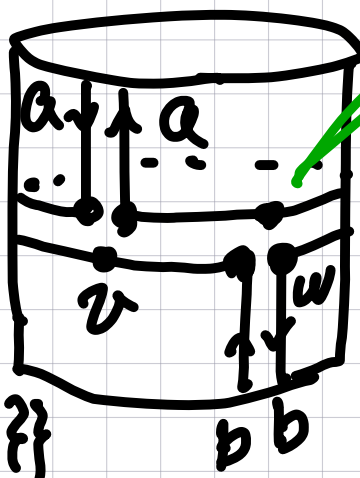
$$= 4V^2$$



$$+ V^2 \sum_{\omega} \sum_{\nu} \sum_{\pi}$$

$\{b, b, a\}$

$\{a, a, b\}$



$$=$$

$$\frac{\lambda}{9 E_a E_b (E_a + E_b + \sqrt{E_a E_b})}$$

$$+ \sum_{\omega} \sum_{\nu} \frac{\lambda \sqrt{E_\nu} \sqrt{E_\omega}}{9 E_a E_b (E_a + E_b + \sqrt{E_a E_b})^2} + O(\lambda^2)$$

▷ Exact calculation.

Using IZ integral

$$Z[J] = C \frac{e^{-\frac{V}{\lambda} \text{tr}(JM)} P_N(s_1, s_2, \dots, s_N)}{\prod_{1 \leq t < u \leq N} (s_u - s_t)}$$

where $P_N(s_1, \dots, s_N) := \det(P^{(i-1)}(s_i)) = \prod_{1 \leq i < j \leq N} (2s_i - 2s_j) P(s_1) \cdots P(s_N)$,

$$P(z) := \int_{-\infty}^{\infty} dx e^{-\frac{\lambda V}{4} x^4 + Vxz} \quad (\text{like Pearcey integral})$$

s_t : eigenvalues of $\frac{1}{\sqrt{\lambda}} M^3 + J$

$$\text{ex). } G|a|b| = \frac{\partial^2}{\partial J_{aa} \partial J_{bb}} \log Z[J] \Big|_{J=0}$$


We can choose $J = \text{diag}(J_1, \dots, J_N)$

$$\Rightarrow S_t = \frac{1}{\sqrt{\lambda}} E_t^{2\frac{1}{\lambda}} + J_{tt}$$

$$G|a|b| = \frac{\partial_a \partial_b P_N(z_1, \dots, z_N)}{P_N(z_1, \dots, z_N)} - \frac{\partial_a P_N(z_1, \dots, z_N)}{P_N(z_1, \dots, z_N)} \frac{\partial_b P_N(z_1, \dots, z_N)}{P_N(z_1, \dots, z_N)}$$

$$- \frac{\lambda}{(E_a \sqrt{E_a} - E_b \sqrt{E_b})^2}$$

↓ If we use saddle point approximation then we get the previous result.

 Φ^4 model \Rightarrow Harmonic Oscillator System

$$S = N \operatorname{tr} \left(E \bar{\Phi}^2 + \frac{\eta}{4} \Phi^4 \right) \quad \eta \in \mathbb{R}_{>0}$$

$$Z = \int d\bar{\Phi} e^{-S} \quad E = \operatorname{diag}(E_1, E_2, \dots, E_N)$$

$E_i > 0, E_i \neq E_j$

Main Thm.

$$\Psi(E, \eta) := e^{-\frac{N}{2\eta} \sum_i E_i^2} \Delta(E) Z, \quad \uparrow$$

where $\Delta(E) = \prod_{k < l} (E_l - E_k)$

$\Rightarrow \mathcal{H} \Psi = 0$, Schrödinger eq

$$\mathcal{H} = -\frac{\eta}{N} \sum_{i=1}^N \left(\frac{\partial}{\partial E_i} \right)^2 + \frac{N}{\eta} \sum_{i=1}^N (E_i)^2 : \text{Hamiltonian}$$

N-body (N-dim) Harmonic Oscillator System

Proof). $E = \begin{pmatrix} E_1 & & 0 \\ & E_2 & \\ 0 & & \ddots \\ & & & E_N \end{pmatrix} = U H U^\dagger$ $H = (H_{ij})$

Hermitian non-diagonal

$$S = N \text{tr} (H \Phi^2 + \frac{\lambda}{4} \Phi^4), \quad Z = \int d\Phi e^{-S}$$

$$\text{S-D eq} \int d\Phi \frac{\partial}{\partial \Phi_{ij}} (\Phi_{ij} e^{-S}) = 0$$

$$\Leftrightarrow Z - N \sum_k (H_{ki} \langle \Phi_{ij} \Phi_{jk} \rangle + H_{jk} \langle \Phi_{ki} \Phi_{ij} \rangle)$$

$$- N \sum_{k,l} \langle \bar{\Phi}_{jk} \bar{\Phi}_{kl} \bar{\Phi}_{li} \Phi_{ij} \rangle = 0$$

, where $\langle O \rangle = \int d\Phi O e^{-S}$

$$\frac{\partial Z}{\partial H_{ij}} = -N \sum_k \langle \bar{\Phi}_{jk} \Phi_{ki} \rangle, \quad \frac{\partial^2 Z}{\partial H_{ij} \partial H_{mn}} = N^2 \sum_{k,l} \langle \bar{\Phi}_{jk} \bar{\Phi}_{kl} \bar{\Phi}_{ln} \Phi_{im} \rangle$$

Diff eg.

$$\left(N^2 + 2 \sum_{i,k} H_{ki} \frac{\partial}{\partial H_{ki}} - \frac{2}{N} \sum_{i,k} \left(\frac{\partial}{\partial H_{ki}} \frac{\partial}{\partial H_{ik}} \right) \right) \mathcal{Z} = 0$$

• Change of variables

$$\mathbb{R}^{N^2} \rightarrow \mathbb{R}^N \times U(N)$$

$$H_{ij} \mapsto E_i + \text{others}$$

\mathcal{Z} depends on only E_i .

• Laplacian: $\sum_{i,j} \frac{\partial}{\partial H_{ji}} \frac{\partial}{\partial H_{ij}} = \sum_i \left(\frac{\partial}{\partial E_i} \right)^2 + \sum_{i \neq j} \frac{1}{E_i - E_j} \left(\frac{\partial}{\partial E_i} - \frac{\partial}{\partial E_j} \right)$

• $\sum_{i,j} H_{i,j} \frac{\partial}{\partial H_{i,j}} = \sum_k E_k \frac{\partial}{\partial E_k}$

$$\left\{ \frac{2}{N} \sum_{i=1}^N \left(\frac{\partial}{\partial E_i} \right)^2 + \frac{2}{N} \sum_{i \neq j} \frac{1}{E_i - E_j} \left(\frac{\partial}{\partial E_i} - \frac{\partial}{\partial E_j} \right) - 2 \sum_k E_k \frac{\partial}{\partial E_k} - N^2 \right\} \mathcal{Z} = 0$$

$\therefore \hookrightarrow$ SD

Lem. Diagonalization →

N-Body harmonic oscillator Hamiltonian

$$\mathcal{H} := -\frac{\hbar^2}{2m} \sum_{i=1}^N \left(\frac{\partial}{\partial E_i} \right)^2 + \frac{1}{2} \sum_{i=1}^N (E_i)^2, \quad ,$$

$$\downarrow e^{-\frac{N}{2\hbar} \sum_i E_i^2} \Delta(E) \mathcal{L}_{SD} \Delta(E)^{-1} e^{\frac{N}{2\hbar} \sum_i E_i^2} = -\mathcal{H}_{HO}$$

Thm. →

$$\bar{\Psi}(E, \hbar) := e^{-\frac{N}{2\hbar} \sum_i E_i^2} \Delta(E) \mathbb{Z}(E, \hbar)$$

$$\Rightarrow \mathcal{H} \bar{\Psi}(E, \hbar) = 0$$

$\bar{\Psi}$ is a 0-energy solution of HO system //

▷ Concrete expression

↙ IZ integral

$$Z(E, \eta) = \frac{C}{\Delta(E)} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N dx_i e^{-N \left(\frac{\eta}{4} x_i^4 + E_i x_i^2 \right)} \right) \left(\prod_{k < l} \frac{x_k - x_l}{x_k + x_l} \right)$$

↕

$$\bar{\Psi}(E, \eta) = C \int_{\mathbb{R}^N} \left(\prod_{i=1}^N dx_i e^{-N \left(\frac{\eta}{4} x_i^4 + E_i x_i^2 + \frac{1}{2\eta} E_i^2 \right)} \right) \left(\prod_{k < l} \frac{x_k - x_l}{x_k + x_l} \right)$$

Rem. $N=1$ case

H.O \rightarrow Weber eq $y''(u) = u^2 y(u)$ ($u = \sqrt{\frac{\eta}{2}} E$) \rightarrow

$$\bar{\Psi}(u) = e^{-\frac{u^2}{2}} \int_{-\infty}^{\infty} dx e^{-\sqrt{\frac{\eta}{2}} u x^2 - \frac{\eta}{4} x^4} = \frac{1}{\sqrt{\frac{\eta}{2}}} \sqrt{u} K_{\frac{1}{4}} \left(\frac{u^2}{2} \right)$$

↓

modified Bessel fun of 2nd kind

Summary

o Scalar field theories on Moyal plane

⇒ 1-Matrix model?

renormalized ϕ^3 → Grosse-Steinacker model Kontsevich type KdV τ
" ϕ^4 → Grosse-Wulkenhaar model

Every $G(a_1^1 \dots a_{N,1}^1 \dots | a_1^D \dots a_{N,D}^D)$ for finite N

or $G(x_1^1 \dots x_{N,1}^1 \dots | x_1^D \dots x_{N,D}^D)$ for $N \rightarrow \infty$ ($\frac{1}{N}$ fixed)

is calculated exactly.

Note: The relation between $G(a_1^1 \dots a_{N,1}^1 \dots | a_1^D \dots a_{N,D}^D)$
& $G(x_1^1 \dots x_{N,1}^1 \dots | x_1^D \dots x_{N,D}^D)$ is unknown.

◦ Φ^3 - Φ^4 hybrid model?

Φ^4 Grosse-Wulkenhaar model ← More difficult than Φ^3 model

Φ^3 - Φ^4 hybrid model is more easy than Φ^4 .

↳ Indeed it is an integrable model.

(Itikson-Zuber 92. generalized Kontsevich, W3-alg. higher K_p .)

For finite N ,

- perturbation theory
- exact solutions

Note: Renormalization, $N \rightarrow \infty$, etc have not been studied yet.

◦ Φ^4 Grosse-Wulkenhaar model



N-body Harmonic Oscillator system

without interaction !!

(Quantum solvable System)