GWW Phase Transition in Induced QCD on the Graph

Based on works with K. Ohta in Meiji Gakuin University

arXiv:2303.03692 (appearing in PRD)

arXiv:2309.*****

So Matsuura

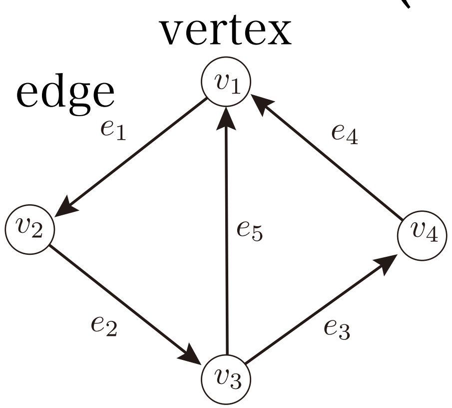
Department of Physics Hiyoshi, Keio University

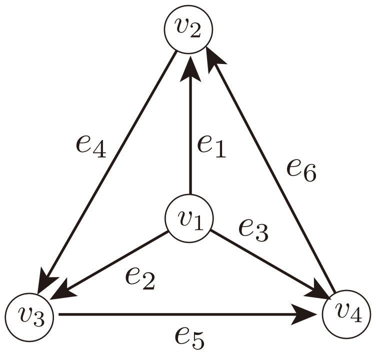
Plan of the talk

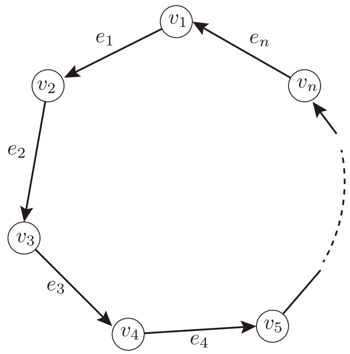
1. Graph zeta functions
2. Fundamental Kazakov-Migdal model on the Graph and graph zeta functions
3. Duality of the FKM model
4. Stability of the FKM model
5. GWW phase transitions in the FKM model
6. Numerical results
7. Conclusion and Future works

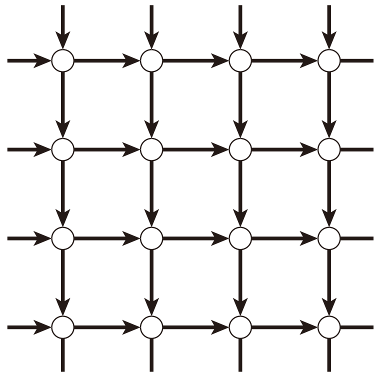
Graph zeta functions

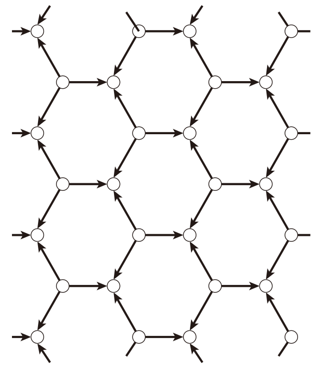
(directed) Graph and cycles



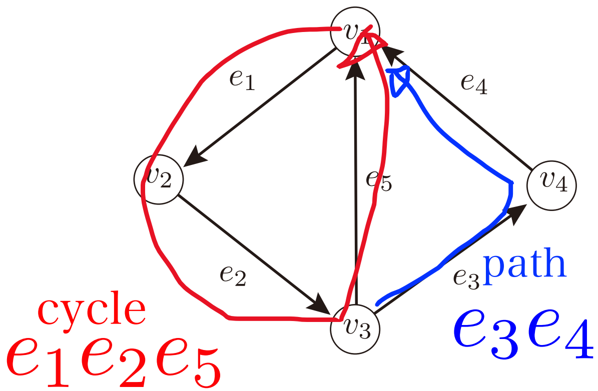


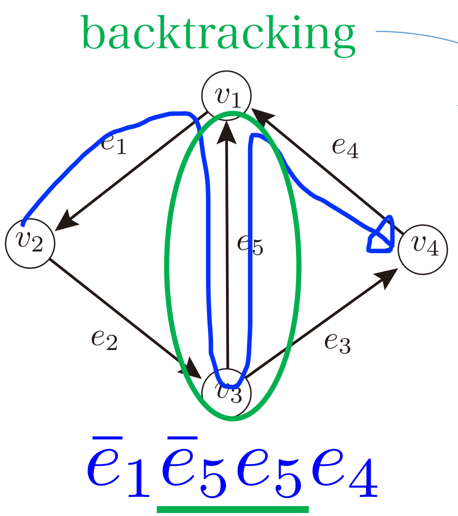




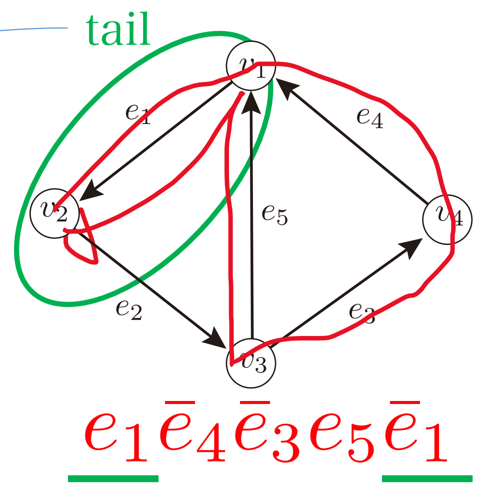


terminology





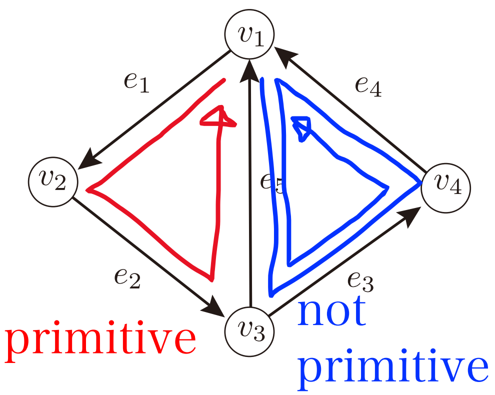
bump



Classification of cycles and Ihara zeta function

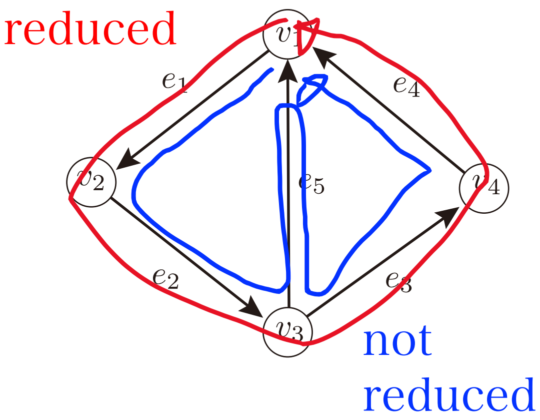
<u>primitive cycle</u>

$$C \neq B^n$$



<u>reduced cycle</u>

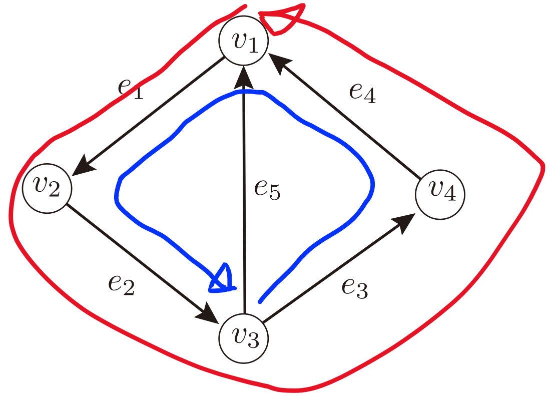
cycle without bumps



<u>equivalence cycle</u>

$$C_1 \sim C_2 \qquad \begin{array}{l} C_1 = e_1 \cdots e_l \\ C_2 = e'_1 \cdots e'_l \end{array}$$

$$\Leftrightarrow \ {}^{\exists}n \in \mathbb{N} \text{ s.t. } e'_i = e_{n+i} \text{ for } {}^{\forall}i$$



<u>Ihara zeta function</u>

$$\zeta_G(q) \equiv \prod_{[C]:\text{primitive reduced}} \frac{1}{1 - q^{|C|}}$$

cf) Riemann zeta function

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$$

$$\begin{array}{l} q \equiv e^{-s} \\ p_C \equiv e^{|C|} \end{array} \ \Rightarrow \ \zeta_G(q) = \prod_{[C]:PR} \frac{1}{1 - p_C^{-s}}$$

Determinant expression of Ihara zeta function

<u>Vertex representation</u>

Ihara zeta function is the inverse of a polynomial

Ihara 1966

$$\zeta_G(q) = (1-q^2)^{-(n_E - v_V)} \det\Big(I - qA + q^2(D-I)\Big)^{-1}$$

D: detree matrix

A: adjacency matrix

($n_V \times n_V$ *matrix*)

n_V: #(verices)

n_E: #(edges)

<u>Edge representation</u>

Equivalently,

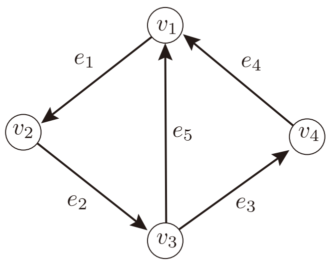
Bass 1992

$$\zeta_G(q) = \det(1-qW)^{-1}$$

<u>edge adjacency matrix</u> $\boldsymbol{e} = \{e, e^{-1} | e \in E\}$

$$W_{\boldsymbol{e}\boldsymbol{e}'} = \begin{cases} 1 & \text{if } t(\boldsymbol{e}) = s(\boldsymbol{e}') \text{ and } \boldsymbol{e}'^{-1} \neq \boldsymbol{e} \\ 0 & \text{others} \end{cases}$$

(例) Double Triangle



$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$W =$

	e_1	e_2	e_3	e_4	e_5	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$	$\bar{e}_4$	$\bar{e}_5$
e_1		1								
e_2			1		1					
e_3				1						
e_4	1									1
e_5	1								1	
$\bar{e}_1$									1	1
$\bar{e}_2$						1				
$\bar{e}_3$					1		1			
$\bar{e}_4$								1		
$\bar{e}_5$			1				1			

Bartholdi zeta function

Bartholdi 2000

$$\zeta_G(q,u) \equiv \prod_{[C]:\text{primitive}} \frac{1}{1 - q^{|C|} u^{b(C)}}$$

bumpの数

Bartholdi zeta function $\xrightarrow{u=0}$ Ihara zeta function

<u>Vertex representation</u>

$$\zeta_G(q,u) = \left(1-(1-u)^2 q^2\right)^{-(n_E - n_V)} \det\left(1 - qA + (1-u)q^2(D-(1-u)1)\right)^{-1},$$

<u>Edge representation</u>

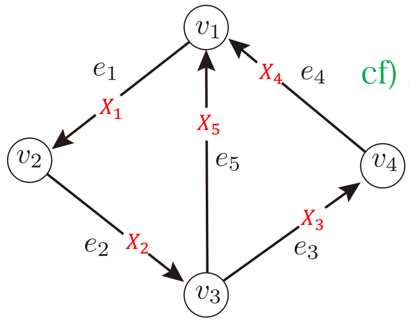
$$\zeta_G(q,u) = \det(1 - q(W + uJ))^{-1} \qquad J_{ee'} = \begin{cases} 1 & (\boldsymbol{e'}^{-1} = \boldsymbol{e}) \\ 0 & (\text{others}) \end{cases}$$

Matrix weighted Bartholdi zeta function

Ohta-S.M. 2022

cf) Mizuno, Sato 2003,2006

- regular matrix X_e (size K) on each edge e
- $X_{e^{-1}} = X_e^{-1}$
- $X_C \equiv X_{e_{i_1}} \cdots X_{e_{i_n}}$ for $C = e_{i_1} \cdots e_{i_n}$
- matrix weighted adjacency matrices



$$A(X)_{vv'} = \begin{cases} X_e & \langle v, v' \rangle = e \\ 0 & \text{others} \end{cases} \qquad (W_X)_{\boldsymbol{e}\boldsymbol{e}'} = \begin{cases} X_{\boldsymbol{e}} & \text{if } t(\boldsymbol{e}) = s(\boldsymbol{e}') \text{ and } \boldsymbol{e}'^{-1} \neq \boldsymbol{e} \\ 0 & \text{others} \end{cases}, \qquad (J_X)_{\boldsymbol{e}\boldsymbol{e}'} = \begin{cases} X_{\boldsymbol{e}} & \text{if } \boldsymbol{e}'^{-1} = \boldsymbol{e} \\ 0 & \text{others} \end{cases}.$$

<u>Matrix weighted Bartholdi zeta function</u>

$$\zeta_G(q,u;X) \equiv \prod_{C \in [\mathcal{P}]} \det\left(1_K - q^{|C|} u^{b(C)} X_C\right)^{-1},$$

$$= \left(1-(1-u)^2 q^2\right)^{-K(n_E - n_V)} \det\left(1_{K v_N} - q A_X + (1-u) q^2 (D - (1-u) 1_{K v_N})\right)^{-1}$$

$$= \det\left(1_{2K n_E} - q(W_X + u J_X)\right)^{-1}$$

Important properties of Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

(1) Euler product

$$\zeta(s) = \prod_{p} \frac{1}{1-p^{-s}}$$

(2) functional relation

completed zeta function : $\xi(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ $\Rightarrow$ $\xi(1-s) = \xi(s)$

(3) Riemann's hypothesis

non-trivial zeros of $\zeta(s)$will be only on $Re(s) = \frac{1}{2}$

Is Ihara zeta function a zeta function?

(1) Euler product?

$$\zeta_G(q) \equiv \prod_{[C]:PR} \frac{1}{1-q^{|C|}}$$

(2) functional relation?

- If the graph is (t+1)-regular, $\zeta_G(q) = (1-q^2)^{-(n_E-n_V)} \det\left((1+tq^2)\mathbf{1}_{n_V} - qA\right)^{-1}$
- Completed Ihara zeta：$\xi_G(q) \equiv (1-q^2)^{n_E-\frac{n_V}{2}}(1-t^2q^2)^{\frac{n_V}{2}}\zeta_G(q) = (1-q^2)^{\frac{n_V}{2}}(1-t^2q^2)^{\frac{n_V}{2}} \det\left((1+tq^2)\mathbf{1}_{n_V} - qA\right)$

$$\xi_G\left(\frac{1}{tq}\right) = (-1)^{n_V}\xi_G(q)$$

(3) Riemann's hypothesis?

If the graph is Ramanujan, non-trivial zeros of $\zeta_G(t^{-s})$ are only on $Re(s) = \frac{1}{2}$

(Ramanujan graph：(t+1)-regular and the eigenvalues of the adjacency matrix satisfies $\lambda^2 - 4t < 0$)

<u>proof</u>

Ihara zeta function of (t+1)-regular graph:

$\zeta_G(q) = (1-q^2)^{n_V-n_E} \det\left((1-tq^2)\mathbf{1}_{n_V} - qA\right) = (1-q^2)^{n_V-n_E}\prod_\lambda (tq^2 - \lambda q + 1)$ ➡ zeros：$q = \frac{\lambda \pm \sqrt{\lambda^2-4t}}{2t} \equiv t^{-s_\pm}$

If $\lambda^2 - 4t < 0$, since $s_- = s_+^*$, $t^{-s_+} \cdot t^{-s_-} = t^{-s_+-s_-} = t^{-2Re(s_+)} = t^{-1}$

Fundamental Kazakov-Migdal model on the Graph and graph zeta functions

FKM model

Arefeva 1993
Ohta-S.M. 2023

Fundamental Kazakov-Migdal (FKM) model on a general graph

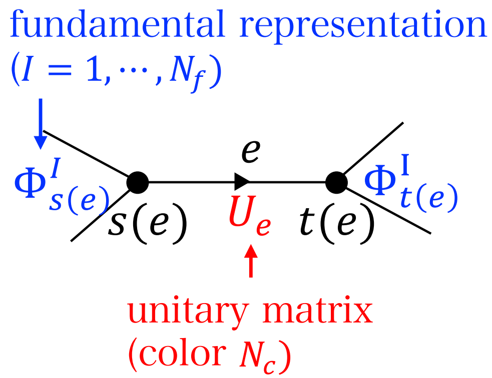
$$S = \sum_{v \in V} m_v^2 \Phi_v^{\dagger I} \Phi_{vI} - q \sum_{e \in E} \left(\Phi^{\dagger}_{s(e)}{}^{I} U_e \Phi_{t(e)I} + \Phi^{\dagger}_{t(e)}{}^{I} U_e^{\dagger} \Phi_{s(e)I} \right)$$

(cf) KM model on the graph

Kazakov-Migdal 1992
Ohta-S.M. 2022

$$S_{\mathrm{KM}} = \mathrm{Tr}\left\{ \frac{m_0^2}{2} \sum_{v \in V} \Phi_v^2 + q \sum_{e \in E} \left(\frac{r}{2} \left(\Phi_{s(e)}^2 + \Phi_{t(e)}^2 \right) - \Phi_{s(e)} U_e \Phi_{t(e)} U_e^{\dagger} \right) \right\}$$

)



Partition function as a Gaussian integral

$$Z = \prod_{I=1}^{N_f} \int \prod_{v \in V} d\Phi_{vI} d\Phi^{\dagger vI} \prod_{e \in E} dU_e \, e^{-\Phi^{\dagger vI} \Delta(U)_v{}^{v'} \Phi_{v'I}} \, ,$$

$$\Delta(U)_v{}^{v'} \equiv m_v^2 \delta_v{}^{v'} \mathbf{1}_{N_c} - q (A_U)_v{}^{v'} \qquad (A_U)_v{}^{v'} = \sum_{\boldsymbol{e} \in E_D} U_{\boldsymbol{e}} \, \delta_{\langle v, v' \rangle, \boldsymbol{e}}$$

Partition function as a graph zeta function

Recall

1) $\zeta_G(q,u;U) = \left(1-(1-u)^2q^2\right)^{-N_c(n_E-n_V)} \det\left(\mathbf{1}_{N_c n_V} - qA_U + (1-u)q^2(D-(1-u)\mathbf{1}_{N_c n_V})\right)^{-1}$

2) $Z_G = \prod_{I=1}^{N_f} \int \prod_{v\in V} d\Phi_{vI} d\Phi^{\dagger vI} \prod_{e\in E} dU_e \, e^{-\Phi^{\dagger vI}\Delta(U)_v{}^{v'}\Phi_{v'I}} \propto (\det \Delta(U))^{-N_f} \quad \Delta(U)_v{}^{v'} \equiv m_v^2 \delta_v{}^{v'} \mathbf{1}_{N_c} - q(A_U)_v{}^{v'}$

tunning the mass parameter

$$m_v^2 = 1 - q^2(1-u)^2 + q^2(1-u)\deg v$$

$$Z_G = \mathcal{N} \int \prod_{e\in E} dU_e \, \zeta_G(q,u;U)^{N_f}$$

$$\left(\mathcal{N} = (2\pi)^{N_f N_c n_V} \left(1-(1-u)^2q^2\right)^{N_f N_c (n_E - n_V)}\right)$$

FKM model is described by the unitary matrix weighted Graph zeta function

Effective action and relation to Wilson action

(For simplicity, $u = 0$ in the following)

$$Z_G = \mathcal{N} \int \prod_{e \in E} dU_e \, \zeta_G(q, u; U)^{N_f} \equiv \mathcal{N} \int \prod_{e \in E} dU_e \, e^{-\gamma S_{\mathrm{eff}}(U)} \quad (\gamma \equiv N_f / N_c)$$

$$\zeta_G(q; U) \equiv \prod_{C \in [\mathcal{P}_R]} \det\left(\mathbf{1}_{N_c} - q^{|C|} U_C\right)^{-1} = \exp\left(\sum_{C \in [\mathcal{P}_R]} \sum_{n=1}^{\infty} \frac{q^n}{n} \left(\mathrm{Tr}\, U_C^n + \mathrm{Tr}\, U_C^{\dagger\, n}\right)\right)$$

$$S_{\mathrm{eff}}(U) = -N_c \sum_{C \in [\Pi_+]} \sum_{n=1}^{\infty} \frac{q^n}{n} \left(\mathrm{Tr}\, U_C^n + \mathrm{Tr}\, U_C^{\dagger\, n}\right)$$

valid only for small $|q|$
(at most |q|<1)

$\gamma \equiv N_f / N_c \quad q \to 0, \quad \gamma \to \infty, \quad \lambda \equiv \frac{1}{\gamma q^l} \, : \, \text{fixed}$ (l : minimal length of the cycles)

$$S_{\mathrm{eff}}(U) \to -\frac{N_c}{\lambda} \sum_{C \in [\Pi_+^l]} \left(\mathrm{Tr}\, U_C + \mathrm{Tr}\, U_C^{\dagger}\right)$$

FKM model includes the usual lattice gauge theory with Wilson action

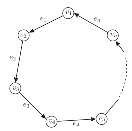
Duality of the FKM model

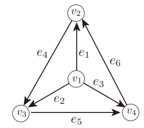
When the graph is regular

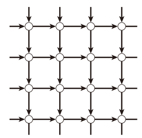
$$S_{\mathrm{eff}}(q;U) = -N_c \log \zeta_G(q;U)$$

<u>functional relation for d-regular graph</u> $(t \equiv d-1)$ Ohta-S.M. coming soon

$$\zeta_G(1/tq;U) = (tq^2)^{n_V N_c} \left(\frac{-tq^2(1-q^2)}{1-t^2q^2} \right)^{(n_E - n_C)N_c} \zeta_G(q;U)$$



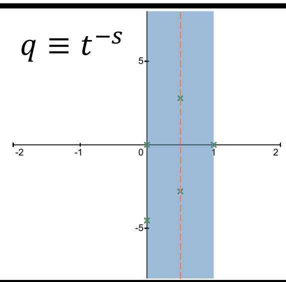


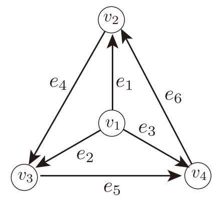


<u>cf)</u> Kotani-Sunada

When G is d-regular, $\zeta_G(q)$ has poles only in $\frac{1}{d-1} \leq Re(q) \leq 1$.
In particular, there is a simple pole at $q = \frac{1}{d-1}$.

$q \equiv t^{-s}$





The FKM model on a regular graph is symmetric under the dual transformation,

$$q \leftrightarrow \frac{1}{tq} \quad \text{or } s \leftrightarrow 1-s \quad (q \equiv t^{-s})$$

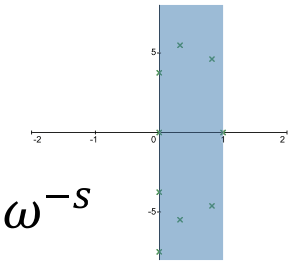
When the graph is irregular

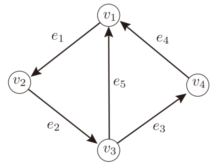
<u>fact</u> Kotani-Sunada

- All the poles of $\zeta_G(q)$ are in $\frac{1}{\omega} \leq Re(q) \leq 1$ ($1/\omega$: maximal radius of convergence)
- $\frac{1}{t_{max}} \leq \frac{1}{\omega} \leq \frac{1}{t_{min}}$
- There is a simple pole at $q = 1/\omega$

➡ $\omega \leftrightarrow t$

natural parametrization : $q = \omega^{-s}$





<u>"functional relation" for irregular graph</u> Ohta-S.M. coming soon

$$\zeta_G(1/\omega q; U) \propto \det\left(1 - \omega q\left(\tilde{Q}^{-1} W_U - (1 - \tilde{Q}^{-1}) J_U\right)\right)^{-1}$$

$$\left(\tilde{Q} \equiv \mathrm{diag}_e(\deg s(e) - 1)\right)$$

matrix Bartholdi zeta function with (unfamiliar) weights

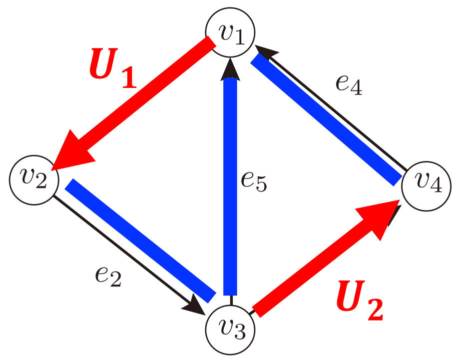
The FKM model on an irregular graph has also a dual expression in $q > 1$

Stability of the FKM model

Gauge fixing and degrees of freedom

$$S_{\text{eff}}(U) = -N_c \log\left(\zeta_G(q;U)\right) = -N_c \log\det\left(1 - qW_U\right)$$

<u>degrees of freedom after gauge fixing</u>



- We can set $U_e = 1$ on a spanning tree
- The number of the remaining edge = **rank r**
- The remaining unitary matrices = **independent plaquette variables**

$U_1, \cdots, U_r$: fundamental cycles

$$S_{\text{eff}}(U) = -N_C \sum_{[C]} \sum_{n=1}^{\infty} \frac{q^{n|C|}}{n} \left(\operatorname{Tr} U_C^n + \operatorname{Tr} U_C^{-n}\right) \qquad (U_C = U_{a_1} \cdots U_{a_l})$$

Saddle points

$$\delta S_{\text{eff}}(U) = -i \sum_{a=1}^{r} \sum_{C \in R_a} q^{|C|} \operatorname{Tr}\left(\delta A_a \left(U_C - U_C^\dagger\right)\right) = 0$$

reduced cycles including C_a

- $U_C = U_C^\dagger$ for all reduced cycles
- C can be a fundamental cycle: $\Rightarrow$ $U_a = U_a^\dagger$
- For $C = C_a C_b$: $U_a U_b = (U_a U_b)^\dagger = U_b^\dagger U_a^\dagger = U_b U_a$

U_a are diagonalizable simultaneously.

$\Downarrow$

$$U_a = \operatorname{diag}(\pm 1, \cdots, \pm 1)$$

Vacuum and the stability

$$S(U)\Big|_{\mathrm{FP}} = -N_c \sum_C \sum_{n=1}^{\infty} \frac{q^{n|C|}}{n} \left(N_C^+ + N_C^- (-1)^n\right)$$

$\Rightarrow$ vacuum : $U_a = \mathbf{1}_{N_c}$

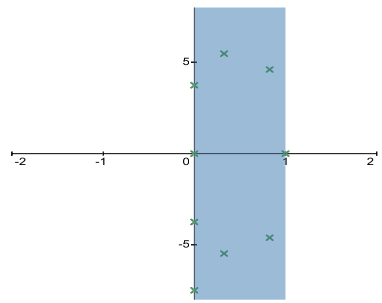
The stability of the vacuum

$$S_{\mathrm{eff}}(U) = -N_c^2 \log \zeta_G(q) - \sum_{a,b=1}^{r} \mathrm{Tr}\left(\delta A_a \delta A_b\right) \left(\mathcal{M}_G\right)_{ab} + \mathcal{O}(\delta A^3)$$

Proposal

Spec($\mathcal{M}_G$) includes negative values in $0 < s < 1$

$\mathcal{M}_G$ is positive definite otherwise



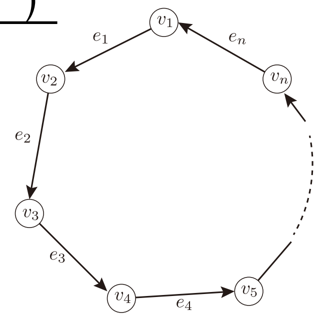
GWW phase transitions

GWW phase transition in the cycle graph

<u>After gauge fixing : $U_2 = \cdots = U_n = 1$ $(U_1 \equiv U, \alpha \equiv q^n)$</u>

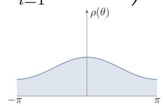
$$Z_{C_n} = \left(\frac{2\pi}{q^n}\right)^{N_f N_c n_V} \alpha^{N_c N_f} \int dU\, e^{N_f \sum_{m=1}^{\infty} \frac{\alpha^m}{m}\left(\mathrm{Tr}\, U^m + \mathrm{Tr}\, U^{-m}\right)}$$

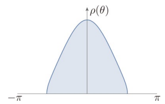
$$= \mathcal{N} \int_{-\pi}^{\pi} \prod_{i=1}^{N_c} d\theta_i\, e^{\sum_{j\neq k} \log\left|\sin\frac{\theta_j-\theta_k}{2}\right| - N_f \sum_i \log\left(1-2\alpha\cos\theta_i+\alpha^2\right)}$$



<u>Eigenvalue density in large N_c</u> $\left(\rho(\theta) \equiv \frac{1}{N_c}\sum_{i=1}^{N_c}\delta(\theta-\theta_i)\right)$

$$\rho(\theta) = \begin{cases} \frac{1}{2\pi}\left(1 + 2\gamma\frac{\alpha\cos\theta - \alpha^2}{1-2\alpha\cos\theta+\alpha^2}\right), & (\theta_0 = \pi) \\ \frac{2(\gamma-1)\alpha}{\pi}\frac{\cos\frac{\theta}{2}}{1-2\alpha\cos\theta+\alpha^2}\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}, & (\theta_0 < \pi) \end{cases}$$





$$\left(\sin^2\frac{\theta_0}{2} = \frac{(1-\alpha)^2}{4\alpha}\frac{2\gamma-1}{(\gamma-1)^2}\right)$$

<u>Free energy</u>

$$F_{C_n} \equiv -\lim_{N_c\to\infty}\frac{1}{N_c^2}\log Z_{C_n}$$

$$= -\int_0^1 dx dy \log\left|\sin\frac{\theta(x)-\theta(y)}{2}\right| + \gamma\int_0^1 dx\, \log(1-2\alpha\cos\theta(x)+\alpha^2)$$

$$= \begin{cases} F_{C_n}^- \equiv \gamma^2\log\left(1-\alpha^2\right) & (0 < \alpha \leq \alpha^*) \\ F_{C_n}^+ \equiv (2\gamma-1)\log(1-\alpha) + \frac{1}{2}\log\alpha + f(\gamma) & (\alpha^* < \alpha \leq 1) \end{cases} \quad \left(\alpha^* = \frac{1}{2\gamma-1}\right)$$

3^{rd} order GWW phase transition (appears only for $\gamma > 1$)

<u>Wilson limit</u>

$$q \to 0, \quad \gamma \to \infty, \quad \lambda \equiv \frac{1}{\gamma q^l} \ : \ \text{fixed}$$

$$\rho(\theta) \to \begin{cases} \frac{1}{2\pi}\left(1 + \frac{2}{\lambda}\cos\theta\right), & (\theta_0 = \pi) \\ \frac{2}{\pi\lambda}\cos\frac{\theta}{2}\sqrt{\frac{\lambda}{2} - \sin^2\frac{\theta}{2}}, & (\theta_0 < \pi) \end{cases}$$

$$F_{C_n} \to \begin{cases} F_{C_n}^- \equiv -\frac{1}{\lambda^2} & (0 < \alpha \leq \alpha^*) \\ F_{C_n}^+ \equiv -\frac{2}{\lambda} - \frac{1}{2}\log\frac{\lambda}{2} + \frac{3}{4} & (\alpha^* < \alpha < 1) \end{cases},$$

exactly reproduces
the result of GWW model

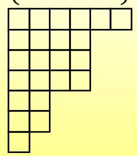
Partition function for a general graph in large N

large N decomposition of Wilson loops ($qN_f \ll N_c$)

$$\int \prod_{e\in E} dU_e \left(\prod_{C\in[\Pi_+]} \Upsilon_{\lambda_C}(U_C)\Upsilon_{\mu_C}(U_C^\dagger) \right) = \prod_{C\in[\Pi_+]} \left(\int \prod_{e\in E} dU_e \, \Upsilon_{\lambda_C}(U_C)\Upsilon_{\mu_C}(U_C^\dagger) \right) + \mathcal{O}(1/N_c)$$

$$\left(\Upsilon_\lambda(U) \equiv \prod_{i=1}^{k} \mathrm{Tr}\left(U^{l_i}\right)^{m_i} \quad \lambda = (l_1^{m_1}, l_2^{m_2}, \cdots) \right)$$

$(6^1 4^3 2^2 1^1)$



$$Z_G \to (2\pi)^{N_f N_c n_V} \left(1-q^2\right)^{N_f N_c (n_E - n_V)} \prod_{C\in[\Pi_+]} \frac{1}{\left(1-q^{2|C|}\right)^{N_f^2}}$$

saddle point approximation around $U_C = 1$ ($N_c \ll N_f$) (r : rank of the graph)

$$Z_G \to \mathcal{N} (2\pi)^{N_f N_c n_V} \left(1-q^2\right)^{N_f N_c (n_E - n_V)} N_f^{\frac{r}{2}N_c^2} \prod_{C\in[\Pi_+]} \frac{1}{(1-q^{|C|})^{2N_f N_c}} (\det \mathcal{M}_G)^{-\frac{N_c^2}{2}}$$

Free energy in the both limits

different analytic expressions in the both region

$$F_G \equiv -\frac{1}{N_c^2} \log Z_G = \begin{cases} -\frac{\gamma^2}{2} \log \zeta_G(q^2)\,, & (q \ll 1) \\ -\gamma \log \zeta_G(q) + \frac{1}{2} \mathrm{Tr} \log \mathcal{M}_G + \frac{r}{2} \log \gamma + f(\gamma) & (q \lesssim 1,\ \gamma \gg 1) \end{cases}$$

Check in the cycle graph

【recall】

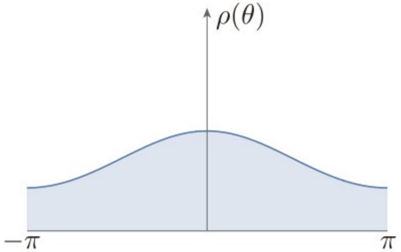
$$\zeta_{C_n}(q) = (1-q^n)^{-2} = (1-\alpha)^{-2}$$

$$\mathcal{M}_{C_n} = \frac{2\alpha}{(1-\alpha)^2}$$

【recall 2】 exact solution for C_n

$$F_{C_n} = \begin{cases} \gamma^2 \log\left(1-\alpha^2\right) & (0 < \alpha \le \alpha^*) \\ (2\gamma - 1)\log(1-\alpha) + \frac{1}{2}\log\alpha + f(\gamma) & (\alpha^* < \alpha \le 1) \end{cases}$$

$$F_{C_n} = \begin{cases} -\frac{\gamma^2}{2}\log\zeta_{C_n}(q^2)\,, \\ -\gamma\log\zeta_{C_n}(q) + \frac{1}{2}\,\mathrm{Tr}\log\mathcal{M}_{C_n} + \frac{r}{2}\log\gamma + f(\gamma) \end{cases} = \begin{cases} \gamma^2\log(1-\alpha^2) \\ (2\gamma-1)\log(1-\alpha) + \frac{1}{2}\log\alpha + f(\gamma) \end{cases}$$

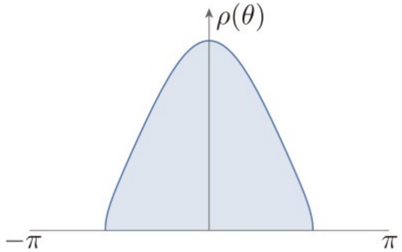


small q phase

$$\gamma q_c^n = \frac{1}{2}$$

GWW

phase transition

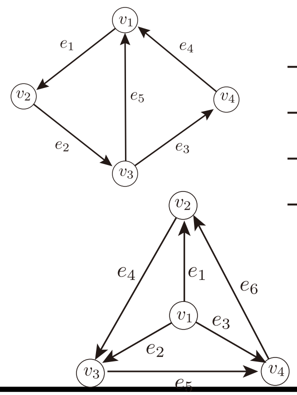


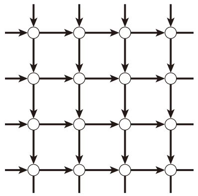
large q phase

Phase structure of the FKM model in $q<1$

when all fundamental cycles are symmetric:

The system should have two phases





U_1 . . . U_r

$$F_G = -\frac{\gamma^2}{2}\log\zeta_G(q^2)$$

ρ_a are all connected

U_1 . . . U_r

$$F_G = -\gamma\log\zeta_G(q) + \frac{1}{2}\mathrm{Tr}\log\mathcal{M}_G + \frac{r}{2}\log\gamma + f(\gamma)$$

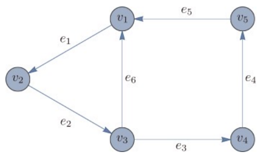
ρ_a are all disconnected

unstable

0 GWW $1/\omega$ 1 q

when the fundamental cycles are composed of different types of cycles:

The system should have intermediate phases



U_1 . . . U_r

$$F_G = -\frac{\gamma^2}{2}\log\zeta_G(q^2)$$

U_1 . . . U_r

. . .

U_1 . . . U_r

$$F_G = -\gamma\log\zeta_G(q) + \frac{1}{2}\mathrm{Tr}\log\mathcal{M}_G + \frac{r}{2}\log\gamma + f(\gamma)$$

$1/\omega$

unstable

0 . . . 1 q

Numerical results

Observables

``temperature''

$$Z_G = \int \prod_{e\in E} dU_e\, \zeta_G(q;U)^{N_f} = \int \prod_{e\in E} dU_e\, e^{-\gamma S_{\mathrm{eff}}(q;U)}$$

$$S_{\mathrm{eff}}(q;U) = -N_c \log \zeta_G(q;U)$$

<u>free energy</u>

$$F_G = \begin{cases} -\frac{\gamma^2}{2}\log\zeta_G(q^2)\,, & (0<q<q_c) \\ [\text{intermediate}] & (q_c<q<q_c') \\ -\gamma\log\zeta_G(q) + \frac{1}{2}\,\mathrm{Tr}\log\mathcal{M}_G + \frac{r}{2}\log\gamma + f(\gamma) & (q_c'<q<q_0) \end{cases}$$

<u>specific heat</u>

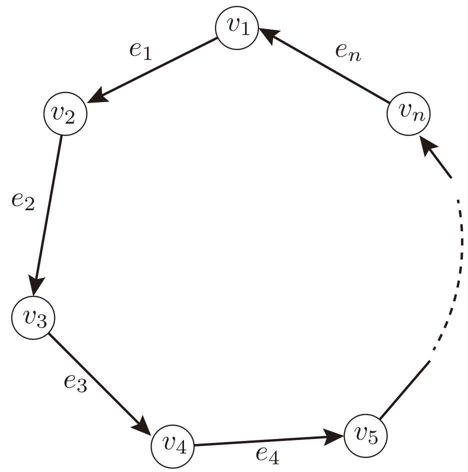
$$C_G = -\gamma^2\frac{\partial^2}{\partial\gamma^2}F_G = \frac{\gamma^2}{N_c^2}\left(\langle S_{\mathrm{eff}}^2\rangle - \langle S_{\mathrm{eff}}\rangle^2\right)$$

$$= \begin{cases} \gamma^2\log\zeta_G(q^2), & (0<q<q_c) \\ [\text{intermediate}] & (q_c<q<q_c') \\ \gamma^2 f''(\gamma) & (q_c'<q) \end{cases}$$

q-independent

Cycle graph

We know the exact solution



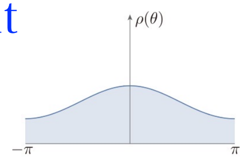
Specific heat ($\alpha = q^n$)

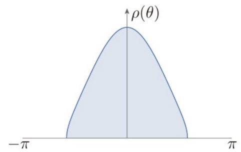
$$C_{C_n} = \begin{cases} -2\gamma^2 \log\left(1-\alpha^2\right) & (0 < \alpha < \alpha^*) \\ -2\gamma^2 \log\left(\frac{\gamma(\gamma-1)}{(\gamma-1/2)^2}\right) & (\alpha^* < \alpha < 1) \end{cases} \qquad \left(\alpha^* = \frac{1}{2\gamma - 1}\right)$$

q-independent

Eigenvalue density

$$\rho(\theta) = \begin{cases} \frac{1}{2\pi}\left(1 + 2\gamma\frac{\alpha\cos\theta - \alpha^2}{1 - 2\alpha\cos\theta + \alpha^2}\right), & (\theta_0 = \pi) \\ \frac{2(\gamma-1)\alpha}{\pi}\frac{\cos\frac{\theta}{2}}{1 - 2\alpha\cos\theta + \alpha^2}\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}, & (\theta_0 < \pi) \end{cases}$$

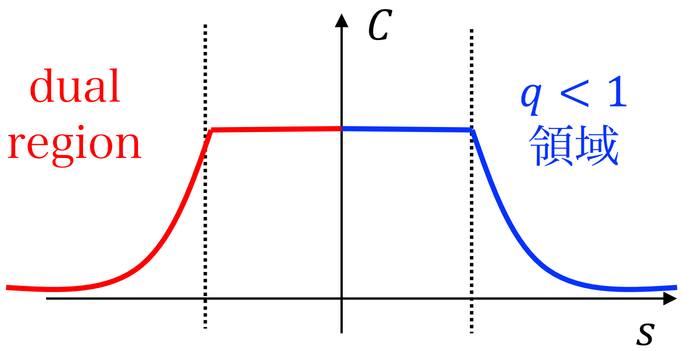




$$\left(\sin^2\frac{\theta_0}{2} = \frac{(1-\alpha)^2}{4\alpha}\frac{2\gamma-1}{(\gamma-1)^2}\right)$$

Duality

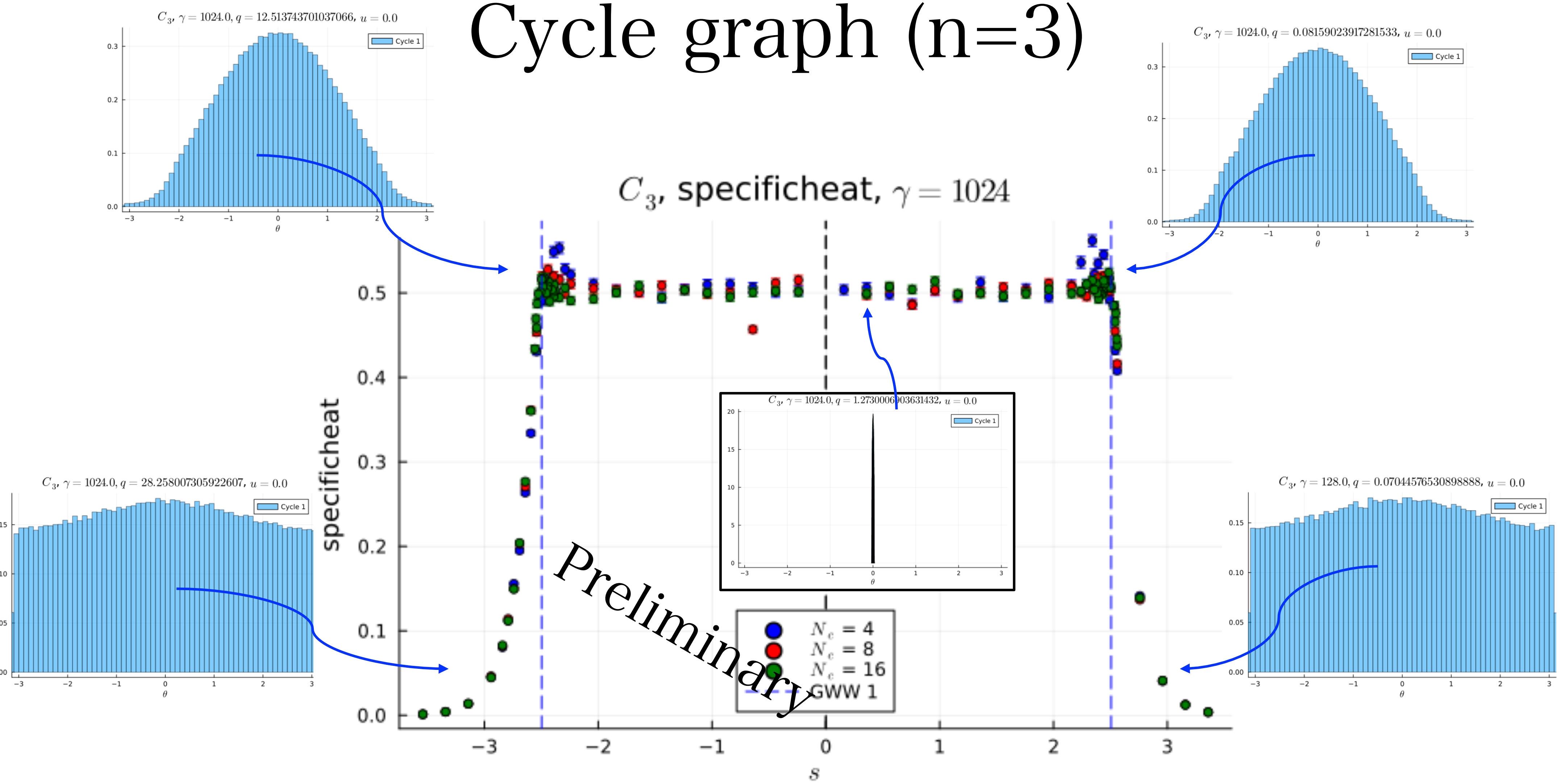
$$\zeta_{C_n}(1/q : U) \propto \zeta_{C_n}(q : U)$$



Parametrization

$$s \equiv -\log q \quad : 0 < q < 1 \text{ corresponds to } s > 0$$

Cycle graph (n=3)



General graphs

Specific heat ($\alpha = q^n$)

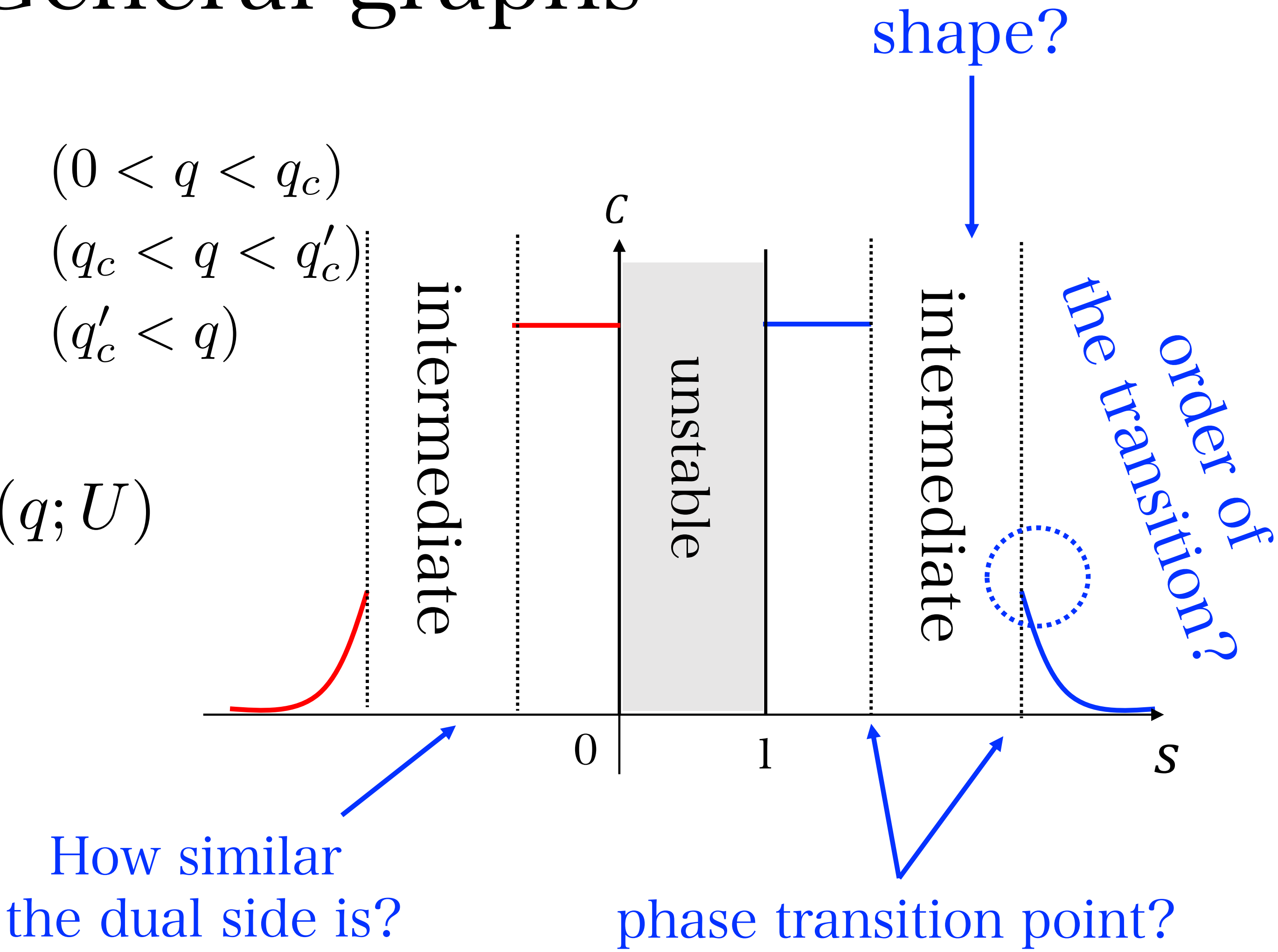
$$C_G = \begin{cases} \gamma^2 \log \zeta_G(q^2), & (0 < q < q_c) \\ [\text{intermediate}] & (q_c < q < q_c') \\ \gamma^2 f''(\gamma) & (q_c' < q) \end{cases}$$

Duality

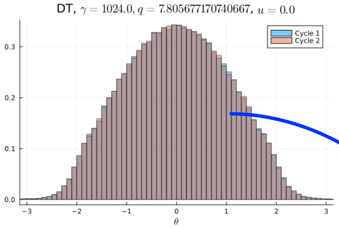
$$\zeta_{C_n}(1/\omega q; U) \propto \tilde{\zeta}_G(q; U)$$

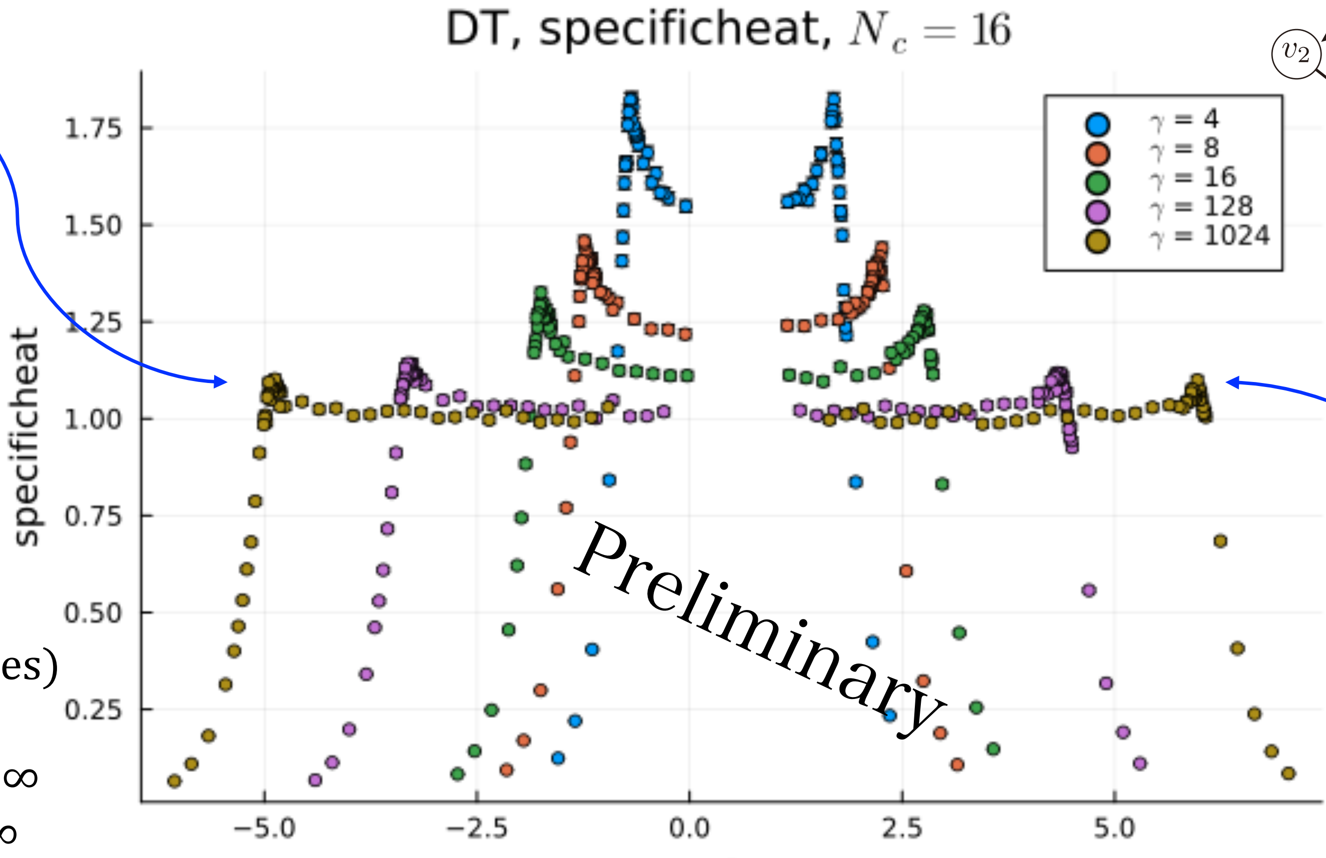
Parametrization

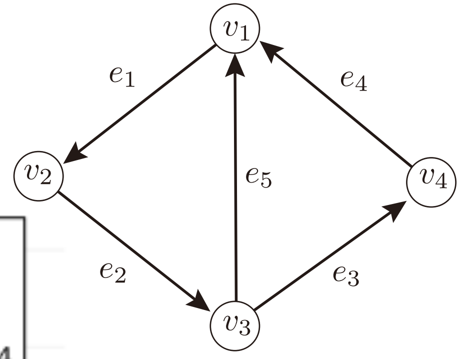
$$q = \omega^{-s}$$

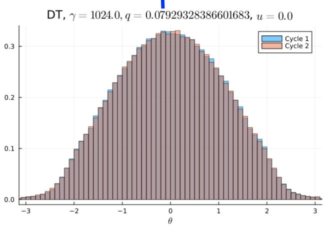


Double Triangle



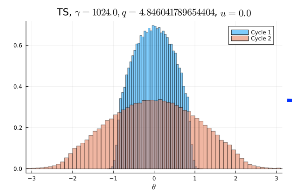


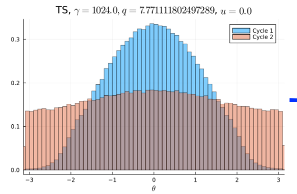


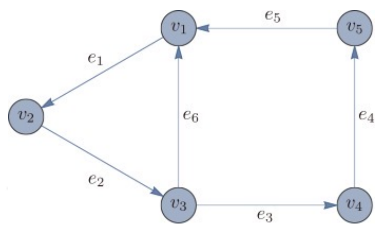


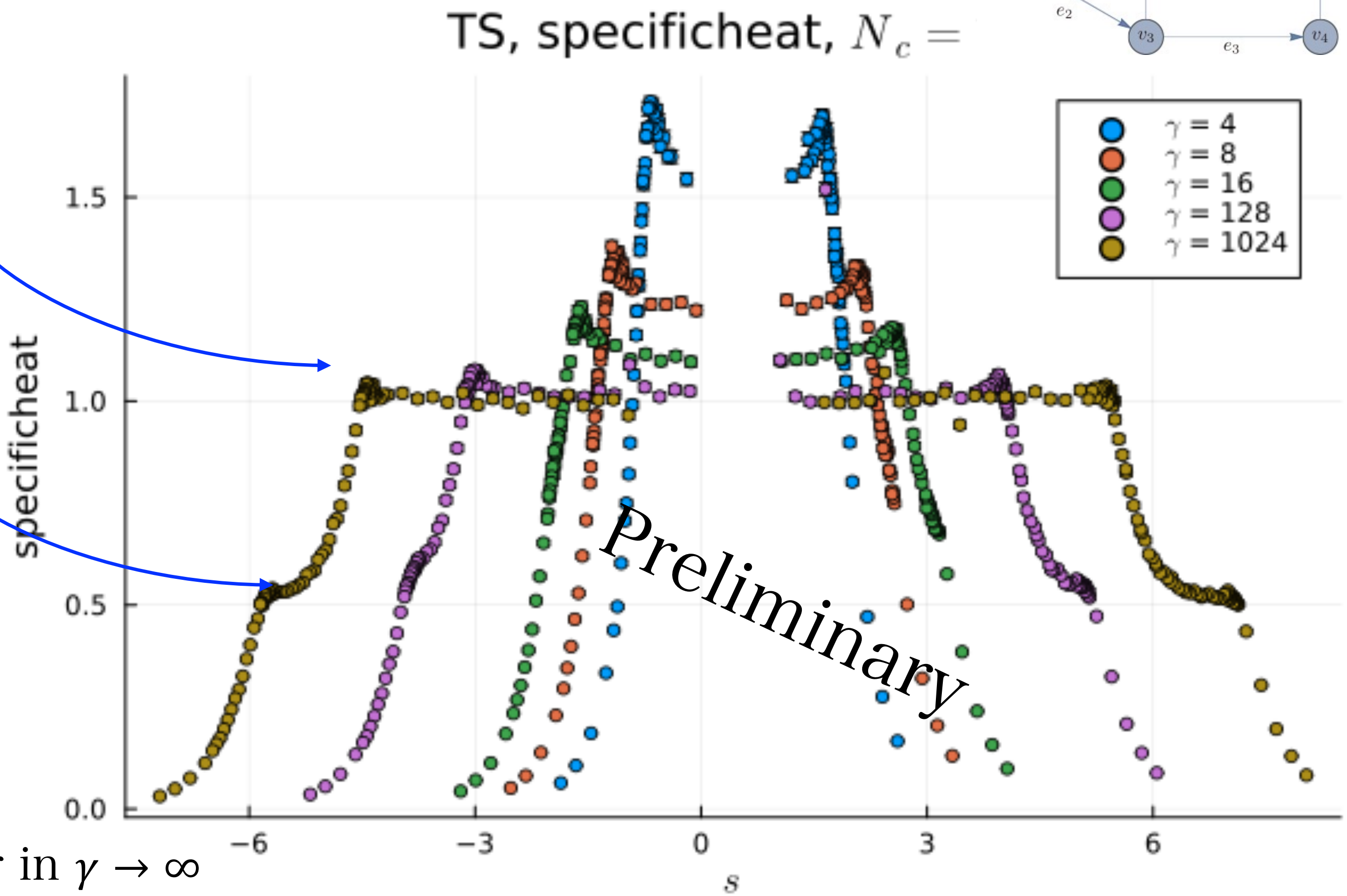
- RANK=2 (2 triangles)
- $q_c^3 \simeq 1/2\gamma$
- 2nd order for $\gamma < \infty$
- 3rd order in $\gamma \to \infty$
- slightly asymmetric (consistent to the dual description)

Triangle-Square



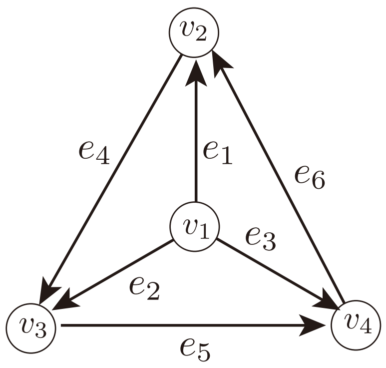


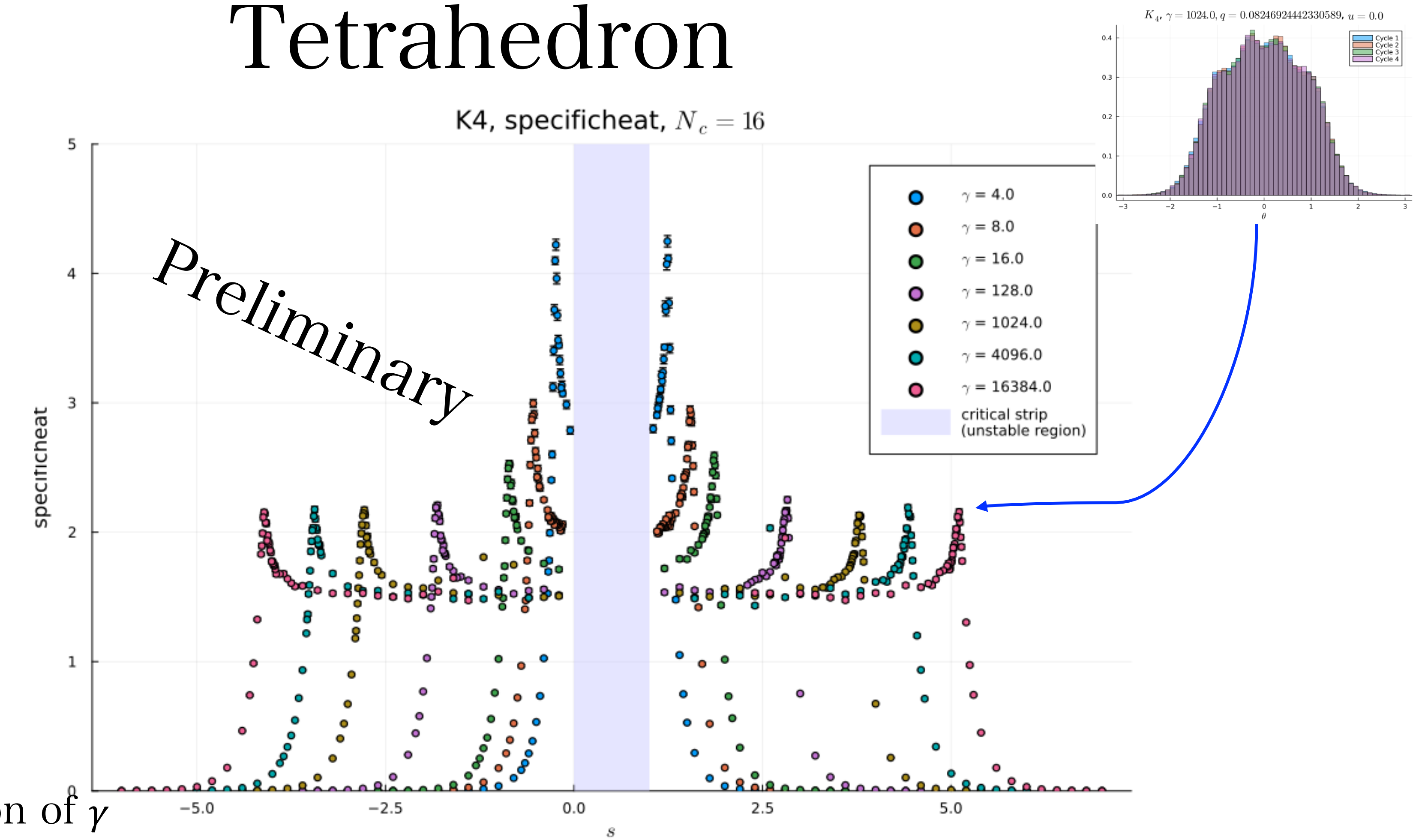




- RNAK=2 (triangle and square)
- there is an intermediate phase
- $q^3_{c1} \simeq 1/2\gamma$, $q^4_{c2} \simeq 1/2\gamma$
- c1: 3rd order for all γ
- c2: 2nd order for $\gamma < \infty$, 3rd order in $\gamma \to \infty$
- slightly asymmetric (consistent to the dual description

Tetrahedron





- RANK=3 (3 triangles)
- $q_c^3 \simeq 1/4\gamma$
- 2nd order for all region of γ
- This is due to the difference of the number of the fundamental cycles and the number of the cycles of the minimal length.
- symmetric around s=1/2 (consistent to the duality)

Conclusion

- We have constructed the FKM model on the graph.
- The effective action of the FKM model is written by unitary matrix weighted graph zeta function.
- The FKM model reduces to Wilson's lattice gauge theory when the graph is a lattice.
- The FKM model has a strong/week coupling duality because of the functional relation of the graph zeta function.
- The FKM model enjoys the GWW phase transition in large N_c
- The phase structure of the FKM model depends on the structure of the fundamental cycles of the graph.

Future works

- Analytical description of the intermediate phases?
- Continuum limit?
- dynamical fermions?
- Physical meaning of the Riemann's hypothesis of graph zeta function or Ramanujan graph?
- Can graph zeta function be an observable of SUSY gauge theory on the graph?
- Relation to other zeta functions?