Recursion Relations for $(2, p)$ Minimal String

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Contents

1. Introduction

Main focus

FZZT brane partition function in $(2,p)$ minimal string $M(p)$, $p \in 2\mathbb{N}-1$

- $M(1)$ $\Leftrightarrow$ Topological gravity (Witten-Kontsevich)
- $M(3)$ $\Leftrightarrow$ Pure gravity
- $M(\infty)$ $\Leftrightarrow$ Jackiw-Teitelboim (JT) gravity (2D dilaton gravity)

3 approaches (recursion relations) in this talk

1. Chekhov-Eynard-Orantin topological recursion (CEO TR) ("B-model/matrix model, complex geometry")
2. Andersen-Borot-Orantin topological recursion (ABO TR) ("A-model, symplectic geometry")
3. Virasoro constraints/quantum Airy structure ("Algebra")

- 1, 2 are recursions on worldsheet genus, and 1 $\Leftrightarrow$ 2 (Laplace dual)
- 1 or 2 $\Rightarrow$ 3 (all genus generating function)

$(2, p)$ minimal string =

Liouville CFT + $(2, p)$ minimal CFT + bc-ghosts

- Liouville CFT

$$S_{bulk} = \int_\Sigma \left[\frac{1}{4\pi} \partial_a\phi\partial^a\phi + \frac{Q}{4\pi} R\,\phi + \mu\, \mathrm{e}^{2b\phi} \right]$$

The central charge is $c_L = 1 + 6Q^2$, $Q = b + b^{-1}$.

- $(2, p)$ minimal CFT has the central charge

$$c_M = 1 - \frac{3(p-2)^2}{p},$$

(e.g. $c_M(p = 1) = -2$, $c_M(p = 3) = 0$, $c_M(p = \infty) = -\infty$)
and the conformal anomaly cancellation $c_L + c_M = 26$ leads to
$c_M = 1 - 6(b - b^{-1})^2$, $b = \sqrt{2/p}$.

FZZT brane

The Liouville CFT admits FZZT branes (boundary condition) parametrized by the boundary cosmological constant μ_B in
[Fateev-Zamolodchikov-Zamolodchikov, Teschner '00]

$$S_{boundary} = \int_{\partial\Sigma} \left[\frac{Q}{2\pi} K\,\phi + \mu_B\, \mathrm{e}^{b\phi} \right]$$

The marked disk partition function defines the spectral curve $y = y(x)$:

$$\partial_{\mu_B} Z(\mu_B) = \left\langle c\, \mathrm{e}^{b\phi} \right\rangle_{disk} \sim y(x = \mu_B)$$

$(2, p)$ minimal string spectral curve

[Seiberg-Shih '03, Saad-Shenker-Stanford '19]

$$x(z) = \frac{1}{2} z^2, \quad y(z) = \frac{1}{2\pi} \sin\left(\frac{p}{2} \arccos\left(1 - \frac{8\pi^2 z^2}{p^2} \right) \right)$$

$y_{p=1}(z) = z$, $y_{p=3}(z) = z - 16\pi^2 z^3/27$, $y_{p=\infty}(z) = \sin(2\pi z)/2\pi$.

Dual 1-matrix model interpretation of $(2,p)$ minimal string

Hermitian $N \times N$ 1-matrix models are toy models of 2D quantum gravity, and actually the correspondence

$$\partial_{\mu_B} Z(\mu_B) = \left\langle c\, \mathrm{e}^{b\phi} \right\rangle_{disk} \quad \leftrightarrow \quad \lim_{N\to\infty} \frac{1}{N} \left\langle \mathrm{Tr} \frac{1}{x - M} \right\rangle_{Matrix} \sim y(x)$$

provides a matrix model method to compute FZZT brane partition functions with higher genus/multi-marked point topology by CEO TR only from the spectral curve input.

- The spectral curve $y = y(x)$ in the matrix model interpretation encodes the eigenvalue density $\rho(x)$:

$$y(x) \sim \rho(x) = \lim_{N\to\infty} \frac{1}{N} \langle \mathrm{Tr}\, \delta(x - M) \rangle_{Matrix}, \quad x \in \{\text{set of cuts}\}$$

- The parameter $x = \mu_B$ is the deformation parameter of the FZZT brane, and then the spectral curve $y = y(x)$ describes the moduli space of the brane.

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2. CEO TR

Hermitian $N \times N$ 1-matrix model

$$Z = \int dM \, \mathrm{e}^{-N \mathrm{Tr} V(M)} \sim \int_{\mathbb{R}^N} \prod_{\ell=1}^{N} d\lambda_\ell \left(\prod_{i<j} (\lambda_i - \lambda_j)^2 \right) \mathrm{e}^{-N \sum_{i=1}^{N} V(\lambda_i)},$$

where $V(x) = \sum_{n \geq 0} g_n x^n$.

- We are interested in the (connected) correlators

$$W(x_1, \ldots, x_h) = \langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_h) \rangle_{Matrix}^{(c)}, \quad \mathcal{O}(x) := \mathrm{Tr} \frac{1}{x - M}$$

- The large N expansions (worldsheet genus expansions):

$$\log Z \sim \sum_{g \geq 0} N^{2-2g} F_g, \quad W(x_1, \ldots, x_h) \sim \sum_{g \geq 0} N^{2-2g-h} W_{g,h}(x_1, \ldots, x_h)$$

Proposition [Eynard '04]

The matrix model loop equation

$$\int_{\mathbb{R}^N} \prod_{\ell=1}^{N} d\lambda_\ell \sum_{i=1}^{N} \partial_{\lambda_i} \left[\frac{1}{x-\lambda_i} \mathcal{O}(x_1)\cdots\mathcal{O}(x_h) \left(\prod_{i<j} (\lambda_i - \lambda_j)^2 \right) \mathrm{e}^{-N\sum_{i=1}^{N} V(\lambda_i)} \right] = 0$$

is rewritten as the CEO TR defined below.

Spectral curve (input) $\mathcal{C} = (C; x, y, B)$

consists of a Riemann surface C, meromorphic functions $x = x(z), y = y(z), z \in C$ such that the zeros of dx are simple and different from zeros of dy, and a bidifferential B on $C^{\otimes 2}$ which is defined to be holomorphic except $z_1 = z_2$ and

- $B(z_1, z_2) \sim \frac{dx(z_1)\otimes dx(z_2)}{(x(z_1)-x(z_2))^2} +$ (holomorphic) for $z_1 \to z_2$
- $\oint_{A_i} B(z_1, z_2) = 0, \quad i = 1, \ldots, \#$ genus of C

CEO TR ['07]

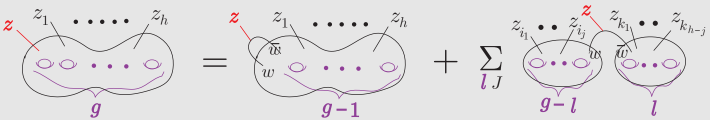
For $\mathcal{C} = (C; x, y, B)$, the CEO TR defines meromorphic multidifferentials $\omega_{g,h}(z_1, \ldots, z_h) = W_{g,h}(x_1, \ldots, x_h) dz_1 \otimes \cdots \otimes dz_h$, $(g, h) \neq (0,1), (0,2)$ by

$$\omega_{g,h+1}(z, z_H) = \sum_{\alpha \in Ram(\text{zeros of } dx)} \mathop{\mathrm{Res}}_{w=\alpha} \frac{\frac{1}{2}\int_w^{\overline{w}} B(\cdot, z)}{y(w)dx(w) - y(\overline{w})dx(\overline{w})} \Bigg[\omega_{g-1,h+2}(w, \overline{w}, z_H) + \sum_{\substack{g_1+g_2=g \\ H_1 \cup H_2 = H}}^{\text{no } (0,1)} \omega_{g_1, 1+|H_1|}(w, z_{H_1}) \omega_{g_2, 1+|H_2|}(\overline{w}, z_{H_2}) \Bigg]$$

where $\omega_{0,2} = B$, and $\overline{w}$ is the conjugate point of w near $\alpha \in Ram$.

- The annulus amplitude is given by $W_{0,2}(z_1, z_2) dz_1 \otimes dz_2 = B(z_1, z_2) - \frac{dx(z_1)dx(z_2)}{(x(z_1) - x(z_2))^2}$

Graphical representation



The genus g free energies $F_{g\geq 2}$ are also determined by

$$F_{g\geq 2} = \frac{1}{2-2g} \sum_{\alpha\in Ram} \operatorname*{Res}_{w=\alpha} \left(\int^{w} y(z)dx(z) \right) \omega_{g,1}(w),$$

and F_0, F_1 are separately obtained.

The CEO TR determines $\omega_{g,h}$ from $\omega_{0,1} = ydx$ and $\omega_{0,2} = B$, recursively, on $-\chi = 2g - 2 + h$:

$$\begin{aligned}(g,h) = (0,1),(0,2) &\longrightarrow (0,3),(1,1) \longrightarrow (0,4),(1,2)\\ &\longrightarrow (0,5),(1,3),(2,1) \longrightarrow (0,6),(1,4),(2,2)\\ &\longrightarrow (0,7),(1,5),(2,3),(3,1) \longrightarrow \cdots\end{aligned}$$

E.g.

$$\omega_{1,1}(z) = \sum_{\alpha \in Ram} \operatorname*{Res}_{w=\alpha} K(z,w)\, \omega_{0,2}(w, \overline{w})$$

$$\omega_{0,3}(z, z_1, z_2) = \sum_{\alpha \in Ram} \operatorname*{Res}_{w=\alpha} K(z,w)\, 2\omega_{0,2}(w, z_1)\omega_{0,2}(\overline{w}, z_2)$$

$$\omega_{1,2}(z, z_1) = \sum_{\alpha \in Ram} \operatorname*{Res}_{w=\alpha} K(z,w) \left[\omega_{0,3}(w, \overline{w}, z_1) + \omega_{1,1}(w)\omega_{0,2}(\overline{w}, z_1) + \omega_{1,1}(\overline{w})\omega_{0,2}(w, z_1)\right]$$

$$\omega_{0,4}(z, z_1, z_2, z_3) = \sum_{\alpha \in Ram} \operatorname*{Res}_{w=\alpha} K(z,w) \left[\omega_{0,3}(w, z_1, z_2)\omega_{0,2}(\overline{w}, z_3) + \omega_{0,3}(w, z_1, z_3)\omega_{0,2}(\overline{w}, z_2) + \omega_{0,3}(w, z_2, z_3)\omega_{0,2}(\overline{w}, z_1) + (w \leftrightarrow \overline{w})\right]$$

$$\omega_{2,1}(z) = \sum_{\alpha \in Ram} \operatorname*{Res}_{w=\alpha} K(z,w) \left[\omega_{1,2}(w, \overline{w}) + \omega_{1,1}(w)\omega_{1,1}(\overline{w})\right]$$

Here

$$K(z,w) = \frac{\frac{1}{2}\int_w^{\overline{w}} B(\cdot, z)}{y(w)dx(w) - y(\overline{w})dx(\overline{w})}$$

is the recursion kernel.

The CEO TR provides an framework to quantize 2D gravity and is applicable to enumerative problems as:

- certain intersection numbers on the moduli space of Riemann surfaces (topological gravity);
- the Weil-Peterson volumes of moduli spaces of bordered Riemann surfaces (JT gravity);
- topological string amplitudes on local toric Calabi-Yau 3-folds with certain branes (remodeling the B-model);
- and so on...

Remark that the CEO TR can be applicable even when we have no matrix model description, and what we need is only an initial input $\mathcal{C} = (C; x, y, B)$.
In the following, $C = \mathbb{P}^1$ is assumed.

Example: Topological gravity (Witten-Kontsevich)

The spectral curve $\mathcal{C}^{\mathrm{A}}$, referred to as Airy spectral curve, is

$$x(z) = \frac{1}{2}z^2, \quad y(z) = z \quad \left(\frac{1}{2}y^2 = x\right), \quad B(z_1, z_2) = \frac{dz_1 \otimes dz_2}{(z_1 - z_2)^2}$$

- $\mathcal{C}^{\mathrm{A}}$ is obtained from the Gaussian spectral curve $\frac{1}{2}y^2 = x^2 - c$, for the Gaussian matrix model with potential $V(x) = x^2$, by a scaling limit (“a constant shift of x” + “zooming in to a branch point”).
- The CEO TR for $\mathcal{C}^{\mathrm{A}}$ computes intersection numbers

$$\omega^{\mathrm{A}}_{g,h}(z_1, \ldots, z_h) = \sum_{d_1,\ldots,d_h \geq 0} F^{\mathrm{A}(g)}_{d_1,\ldots,d_h} \otimes_{i=1}^{h} \frac{dz_i}{z_i^{2d_i+2}},$$

$$F^{\mathrm{A}(g)}_{d_1,\ldots,d_h} = \prod_{i=1}^{h}(2d_i + 1)!! \int_{\overline{\mathcal{M}}_{g,h}} \psi_1^{d_1} \cdots \psi_h^{d_h}, \quad \left(\sum_{i=1}^{h} d_i = 3g - 3 + h\right),$$

where $\overline{\mathcal{M}}_{g,h}$ is the compactified moduli space of stable curves of genus g with h marked points, and ψ_i is the first Chern class of the line bundle over $\overline{\mathcal{M}}_{g,h}$ with the fiber over the ith marked point.

E.g. (zeros of $dx(z) = d\left(z^2/2\right) = zdz$: $z = 0$)

$$\omega^{\mathrm{A}}_{1,1}(z_1) = \frac{dz_1}{8z_1^4}$$

$$\omega^{\mathrm{A}}_{0,3}(z_1, z_2, z_3) = \otimes_{i=1}^{3} \frac{dz_i}{z_i^2}$$

$$\omega^{\mathrm{A}}_{1,2}(z_1, z_2) = \left(\sum_{i=1}^{2} \frac{5}{8z_i^4} + \frac{3}{8z_1^2 z_2^2}\right) \otimes_{i=1}^{2} \frac{dz_i}{z_i^2}$$

$$\omega^{\mathrm{A}}_{0,4}(z_1, \cdots, z_4) = \left(\sum_{i=1}^{4} \frac{3}{z_i^2}\right) \otimes_{i=1}^{4} \frac{dz_i}{z_i^2}$$

$$\omega^{\mathrm{A}}_{2,1}(z_1) = \frac{105 dz_1}{128 z_1^{10}}$$

$$\omega^{\mathrm{A}}_{0,5}(z_1, \cdots, z_5) = \left(\sum_{i=1}^{5} \frac{15}{z_i^4} + \sum_{1 \le i < j \le 5} \frac{18}{z_i^2 z_j^2}\right) \otimes_{i=1}^{5} \frac{dz_i}{z_i^2}$$

$$\omega^{\mathrm{A}}_{1,3}(z_1, z_2, z_3) = \left(\sum_{i=1}^{3} \frac{35}{8z_i^6} + \sum_{1 \le i, j \le 3} \frac{15}{4z_i^2 z_j^4} + \frac{9}{4z_1^2 z_2^2 z_3^2}\right) \otimes_{i=1}^{3} \frac{dz_i}{z_i^2}$$

For applying the CEO TR to the $(2,p)$ minimal string spectral curve $\mathcal{C}^{\mathrm{M}(p)}$:

$$x(z)=\frac{1}{2}z^2,\quad y(z)=\frac{1}{2\pi}\sin\left(\frac{p}{2}\arccos\left(1-\frac{8\pi^2z^2}{p^2}\right)\right),\quad B(z_1,z_2)=\frac{dz_1\otimes dz_2}{(z_1-z_2)^2},$$

the following proposition can be used.

Proposition ★ [Dunin-Barkowski-Orantin-Shadrin-Spitz '12]

For the KdV spectral curve $\mathcal{C}^{\mathrm{KdV}}$:

$$x(z)=\frac{1}{2}z^2,\quad y(z)=z+\sum_{d\geq 2}u_d\,z^d,\quad B(z_1,z_2)=\frac{dz_1\otimes dz_2}{(z_1-z_2)^2},$$

the CEO TR computes

$$\omega_{g,h}^{\mathrm{KdV}}(z_1,\ldots,z_h)=\sum_{d_1,\ldots,d_h\geq 0}F_{d_1,\ldots,d_h}^{\mathrm{KdV}(g)}\otimes_{i=1}^{h}\frac{dz_i}{z_i^{2d_i+2}},$$

and then

$$F_{d_1,\ldots,d_h}^{\mathrm{KdV}(g)}=\sum_{m\geq 0}\frac{(-1)^m}{m!}\sum_{b_1,\ldots,b_m\geq 2}\left(\prod_{j=1}^{m}\frac{u_{2b_j-1}}{2b_j+1}\right)F_{d_1,\ldots,d_h,b_1,\ldots,b_m}^{\mathrm{A}(g)},$$

where the sum over m and b_j satisfies $\sum_{i=1}^{h}d_i=3g-3+h+m-\sum_{j=1}^{m}b_j$.

In particular, for $\mathcal{C}^{\mathrm{M}(p)}$:

$$y(z) = z + \sum_{d=2}^{\frac{p+1}{2}} \frac{(-2\pi^2)^{d-1}}{(2d-1)!!\,(d-1)!} \prod_{i=1}^{d-1} \left(1 - \frac{(2i-1)^2}{p^2}\right) z^{2d-1},$$

we have

$$\omega_{g,h}^{\mathrm{M}(p)}(z_1,\ldots,z_h) = \sum_{d_1,\ldots,d_h \geq 0} F_{d_1,\ldots,d_h}^{\mathrm{M}(p)\,(g)} \otimes_{i=1}^{h} \frac{dz_i}{z_i^{2d_i+2}},$$

and

$$\begin{aligned} &F_{d_1,\ldots,d_h}^{\mathrm{M}(p)\,(g)} \\ &= \sum_{m\geq 0} \frac{(-1)^m}{m!} \sum_{b_1,\ldots,b_m \geq 2} \left(\prod_{j=1}^{m} \frac{(-2\pi^2)^{b_j-1}}{(2b_j+1)!!\,(b_j-1)!} \prod_{i=1}^{b_j-1} \left(1 - \frac{(2i-1)^2}{p^2}\right)\right) F_{d_1,\ldots,d_h,b_1,\ldots,b_m}^{\mathrm{A}(g)} \\ &= \left(\prod_{i=1}^{h}(2d_i+1)!!\right) \int_{\overline{\mathcal{M}}_{g,h}} \exp\left(-\sum_{b\geq 1} \frac{(-2\pi^2)^b}{b!} \left(\prod_{k=1}^{b}\left(1 - \frac{(2k-1)^2}{p^2}\right)\right)\psi^{b+1}\right) \psi_1^{d_1} \cdots \psi_h^{d_h} \end{aligned}$$

By this formula, the computation of the FZZT brane partition functions in the $(2,p)$ minimal string boils down to the computation for the $(2,1)$ minimal string (topological gravity).

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3. ABO TR

ABO TR ['17]

For initial data

$$\mathrm{A}(L_1,L_2,L_3),\quad \mathrm{B}(L_1,L_2,\ell),\quad \mathrm{C}(L_1,\ell,\ell'),\quad \mathrm{D}(\sigma),$$

the ABO TR defines a volume polynomial $V_{g,h}(L_1,\ldots,L_h)$ with $2g-2+h>0$, on the moduli space $\mathcal{M}_{g,h}(L_1,\ldots,L_h)$ of connected bordered Riemann surfaces of genus g with h boundaries of lengths $L_1,\ldots,L_h$, by

$$V_{g,h+1}(L,L_H)=\sum_{m=1}^{h}\int_{\mathbb{R}_+}\mathrm{B}(L,L_m,\ell)\,V_{g,h}(\ell,L_{H\setminus\{m\}})\,\ell d\ell$$
$$+\frac{1}{2}\int_{\mathbb{R}_+^2}\mathrm{C}(L,\ell,\ell')\,P_{g,h+1}(\ell,\ell',L_H)\,\ell\ell' d\ell d\ell',$$

where

$$P_{g,h+1}(\ell,\ell',L_H)=V_{g-1,n+2}(\ell,\ell',L_H)+\sum_{\substack{g_1+g_2=g\\ H_1\cup H_2=H}}^{\mathrm{no}\,(0,1),(0,2)}V_{g_1,1+|H_1|}(\ell,L_{H_1})\,V_{g_2,1+|H_2|}(\ell',L_{H_2}),$$

$$V_{0,3}(L_1,L_2,L_3)=\mathrm{A}(L_1,L_2,L_3),\quad V_{1,1}(L_1)=V\mathrm{D}(L_1)=\int_{\mathcal{M}_{1,1}(L_1)}\mathrm{D}(\sigma)\,d\mu_{\mathrm{WP}}(\sigma)$$

- The ABO TR is a generalization of the Mirzakhani's recursion ['06, '07] with

$$A^{\mathrm{WP}}(L_1,L_2,L_3)=1,\quad VD^{\mathrm{WP}}(L_1)=\frac{\pi^2}{12}+\frac{1}{48}L_1^2,$$
$$B^{\mathrm{WP}}(L_1,L_2,\ell)=\frac{1}{2L_1}\int_0^{L_1}\left(H^{\mathrm{WP}}(\ell,t+L_2)+H^{\mathrm{WP}}(\ell,t-L_2)\right)dt,$$
$$C^{\mathrm{WP}}(L_1,\ell,\ell')=\frac{1}{L_1}\int_0^{L_1}H^{\mathrm{WP}}(\ell+\ell',t)dt,$$
$$H^{\mathrm{WP}}(x,y):=\frac{1}{1+\mathrm{e}^{\frac{x-y}{2}}}+\frac{1}{1+\mathrm{e}^{\frac{x+y}{2}}},$$

for the Weil-Petersson volume

$$V_{g,h}^{\mathrm{WP}}(L_1,\ldots,L_h)=\int_{\mathcal{M}_{g,h}(L_1,\ldots,L_h)}\exp\omega^{\mathrm{WP}}=\int_{\overline{\mathcal{M}}_{g,h}}\exp\left(2\pi^2\kappa_1+\sum_{i=1}^{h}\frac{L_i^2}{2}\psi_i\right),$$

where ω^{WP} is the Weil-Petersson symplectic form, and κ_1 is a tautological class on $\overline{\mathcal{M}}_{g,h}$ defined by considering a forgetful map $\pi:\overline{\mathcal{M}}_{g,h+1}\to\overline{\mathcal{M}}_{g,h}$ for ψ_{h+1}^2.

- A Laplace transform of $V_{g,h}^{\mathrm{WP}}$: [Eynard-Orantin '07]

$$W_{g,h}^{\mathrm{WP}}(z_1,\ldots,z_h)=\int_{\mathbb{R}_+^h}V_{g,h}^{\mathrm{WP}}(L_1,\ldots,L_h)\,\mathrm{e}^{-\sum_{i=1}^h z_iL_i}\prod_{i=1}^{h}L_idL_i$$

is shown to give the CEO TR correlator for $x(z)=z^2/2,\ y(z)=\sin(2\pi z)/2\pi$.

By inverting the Laplace transforms

$$W_{g,h}^{\mathrm{M}(p)}(z_1,\dots,z_h) = \int_{\mathbb{R}_+^h} V_{g,h}^{\mathrm{M}(p)}(L_1,\dots,L_h)\,\mathrm{e}^{-\sum_{i=1}^h z_i L_i} \prod_{i=1}^h L_i dL_i$$

for the $(2,p)$ minimal string correlators $W_{g,h}^{\mathrm{M}(p)}$, we find that $V_{g,h}^{\mathrm{M}(p)}$ are obtained for the initial data:

$$\mathrm{A}^{\mathrm{M}(p)}(L_1,L_2,L_3) = 1, \quad V\mathrm{D}^{\mathrm{M}(p)}(L_1) = \frac{\pi^2}{12}\left(1-\frac{1}{p^2}\right) + \frac{1}{48}L_1^2,$$

$$\mathrm{B}^{\mathrm{M}(p)}(L_1,L_2,\ell) = \frac{1}{2L_1}\int_0^{L_1}\left(H^{\mathrm{M}(p)}(\ell,t+L_2) + H^{\mathrm{M}(p)}(\ell,t-L_2)\right)dt,$$

$$\mathrm{C}^{\mathrm{M}(p)}(L_1,\ell,\ell') = \frac{1}{L_1}\int_0^{L_1} H^{\mathrm{M}(p)}(\ell+\ell',t)\,dt,$$

$$H^{\mathrm{M}(p)}(x,y) := -\sum_{j=1}^{(p-1)/2}(-1)^j\cos\left(\frac{\pi}{p}j\right)\left(\mathrm{e}^{-u_j(x+y)}\,\theta(x+y) + \mathrm{e}^{-u_j(x-y)}\,\theta(x-y)\right)$$

$$+\sum_{j=0}^{(p-1)/2}(-1)^j\cos\left(\frac{\pi}{p}j\right)\left(\mathrm{e}^{u_j(x+y)}\,\theta(-x-y) + \mathrm{e}^{u_j(x-y)}\,\theta(y-x)\right),$$

where $u_j = \frac{p}{2\pi}\sin\left(\frac{\pi}{p}j\right)$, and θ is the Heaviside step function.

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4. Virasoro constraints

Assume that the volume polynomials are expanded as

$$V_{g,h}(L_1,\ldots,L_h) = \sum_{d_1,\ldots,d_h\geq 0} F^{(g)}_{d_1,\ldots,d_h} \prod_{i=1}^{h} \frac{L_i^{2d_i}}{(2d_i+1)!},$$

or equivalently, by the Laplace transforms

$$W_{g,h}(z_1,\ldots,z_h) = \int_{\mathbb{R}_+^h} V_{g,h}(L_1,\ldots,L_h)\, \mathrm{e}^{-\sum_{i=1}^{h} z_i L_i} \prod_{i=1}^{h} L_i dL_i,$$

assume

$$W_{g,h}(z_1,\ldots,z_h) = \sum_{d_1,\ldots,d_h\geq 0} F^{(g)}_{d_1,\ldots,d_h} \prod_{i=1}^{h} \frac{1}{z_i^{2d_i+2}},$$

the ABO TR (CEO TR) gives

$$F^{(g)}_{d,d_1,\ldots,d_h} = \sum_{m=1}^{h}\sum_{b\geq 0} \mathrm{B}^{d}_{d_m,b}\, F^{(g)}_{b,d_1,\ldots,\widehat{d}_m,\ldots,d_h}$$
$$+\frac{1}{2}\sum_{a,b\geq 0} \mathrm{C}^{d}_{a,b}\left(F^{(g-1)}_{a,b,d_1,\ldots,d_h} + \sum_{\substack{g_1+g_2=g\\ H_1\cup H_2=H}}^{\mathrm{no}\,(0,1),(0,2)} F^{(g_1)}_{d,d_{H_1}}\, F^{(g_2)}_{b,d_{H_2}}\right)$$

Here $\mathrm{B}^{a_1}_{a_2,a}$, $\mathrm{C}^{a_1}_{a,b}$ are defined by

$$\int_{\mathbb{R}_+} \mathrm{B}(L_1, L_2, \ell)\, \frac{\ell^{2a}}{(2a+1)!}\, \ell d\ell = \sum_{a_1, a_2 \geq 0} \mathrm{B}^{a_1}_{a_2,a}\, \frac{L_1^{2a_1}}{(2a_1+1)!}\, \frac{L_2^{2a_2}}{(2a_2+1)!},$$

$$\int_{\mathbb{R}_+^2} \mathrm{C}(L_1, \ell, \ell')\, \frac{\ell^{2a}}{(2a+1)!}\, \frac{\ell'^{2b}}{(2b+1)!}\, \ell\ell' d\ell d\ell' = \sum_{a_1 \geq 0} \mathrm{C}^{a_1}_{a,b}\, \frac{L_1^{2a_1}}{(2a_1+1)!}$$

Considering the all genus generating function

$$Z(\hbar; \mathbf{t}) = \exp\left(\sum_{g\geq 0, n\geq 1} \hbar^{2g-2} \sum_{d_1,\dots,d_n \geq 0} F^{(g)}_{d_1,\dots,d_n} \frac{t_{d_1}\cdots t_{d_n}}{n!} \right),$$

one finds (from the ABO TR):

Proposition [Kontsevich-Soibelman, Andersen-Borot-Chekhov-Orantin '17]

$$\widehat{L}_k Z(\hbar; \mathbf{t}) = 0, \quad k \geq -1,$$

$$\widehat{L}_k := -\frac{1}{2}\partial_{k+1} + \frac{1}{4\hbar^2} \sum_{a,b\geq 0} \mathrm{A}^{k+1}_{a,b}\, t_a t_b + \frac{1}{2} \sum_{a,b\geq 0} \mathrm{B}^{k+1}_{a,b}\, t_a \partial_b + \frac{\hbar^2}{4} \sum_{a,b\geq 0} \mathrm{C}^{k+1}_{a,b}\, \partial_a \partial_b + \frac{1}{2}\mathrm{D}^{k+1},$$

where $\mathrm{A}^{a_1}_{a_2,a_3} = F^{(0)}_{a_1,a_2,a_3}$, $\mathrm{D}^{a_1} = F^{(1)}_{a_1}$.

Example: Topological gravity (Witten-Kontsevich)

The $\widehat{L}_k$ operators for the topological gravity are [Fukuma-Kawai-Nakayama, Dijkgraaf-Verlinde-Verlinde '1991]

$$\widehat{L}_k^{\mathrm{A}} = -\frac{1}{2}\partial_{k+1} + \sum_{a\geq 0}\left(a+\frac{1}{2}\right)t_a\partial_{a+k} + \frac{\hbar^2}{4}\sum_{\substack{a,b\geq 0\\ a+b=k-1}}\partial_a\partial_b + \frac{1}{4\hbar^2}t_0^2\,\delta_{k,-1} + \frac{1}{16}\,\delta_{k,0},$$

and satisfy the Virasoro relations

$$\left[\widehat{L}_k^{\mathrm{A}}, \widehat{L}_\ell^{\mathrm{A}}\right] = (k-\ell)\,\widehat{L}_{k+\ell}^{\mathrm{A}}, \qquad k,\ell\geq -1.$$

Then the Virasoro constraints lead to [Alexandrov '10]

$$\widehat{L}_k^{\mathrm{A}} Z^{\mathrm{A}} = 0 \;\;\rightsquigarrow\;\; Z^{\mathrm{A}}(x\hbar; x\mathbf{t}) = \sum_{k\geq 0} x^k Z_k^{\mathrm{A}}(\hbar;\mathbf{t}) = \mathrm{e}^{x\widehat{W}^{\mathrm{A}}}\cdot 1,$$

where the cut-and-join operator $\widehat{W}^{\mathrm{A}}$ is

$$\widehat{W}^{\mathrm{A}} = \frac{1}{3}\sum_{a,b\geq 0}(2a+1)(2b+1)\,t_a t_b\partial_{a+b-1} + \frac{\hbar^2}{6}\sum_{a,b\geq 0}(2a+2b+5)\,t_{a+b+2}\partial_a\partial_b + \frac{t_0^3}{6\hbar^2} + \frac{t_1}{8}$$

Proposition ★ for the KdV spectral curve, with $y(z) = z + \sum_{d\geq 2} u_d z^d$, leads to:

Proposition

The generating function of $F^{\mathrm{KdV}(g)}_{d_1,\ldots,d_n}$,

$$\log Z^{\mathrm{KdV}}(\hbar;\mathbf{t}) = \sum_{g\geq 0, n\geq 1} \hbar^{2g-2} \sum_{\substack{d_1,\ldots,d_n\geq 0\\ |\mathbf{d}|\leq 3g-3+n}} F^{\mathrm{KdV}(g)}_{d_1,\ldots,d_n} \frac{t_{d_1}\cdots t_{d_n}}{n!},$$

is obtained just by shifting the variables in Z^{A} as

$$t_d \quad \to \quad t_d - \frac{u_{2d-1}}{2d+1} \qquad \text{for } d\geq 2$$

This proposition can be used to compute the FZZT brane partition functions in the $(2,p)$ minimal string by the Virasoro constraints for the topological gravity.

Contents

5. Conclusion

Conclusion

- Applied and formulated the CEO TR, ABO TR, and Virasoro constraints for the $(2, p)$ minimal string.
- We can similarly consider the supersymmetric analogue of the $(2, p)$ minimal string.
- We can also include a massless scalar field on the Riemann surface (mathematically related to “Masur-Veech type twist”).

Some outlooks

- Other models including general (p, q) minimal string?
- Non-perturbative aspects of the CEO TR and ABO TR (resurgence and large genus expansion)?