Matrix regularization for gauge fields and Seiberg-Witten map

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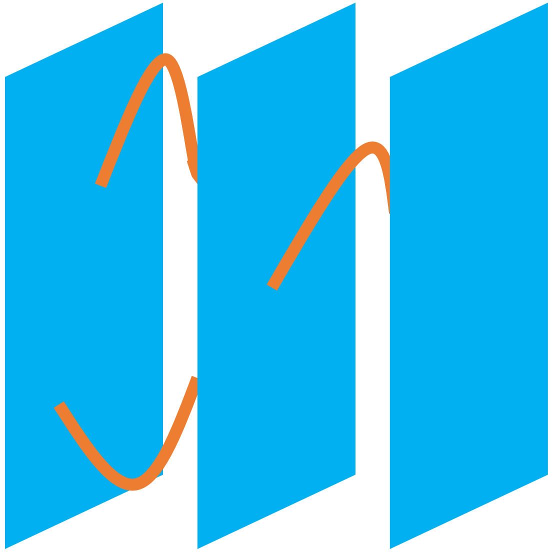
Contents

1. Introduction (6)

string theory ⟷ **matrix model**



geometric object

$X_I =$ 

matrix

Matrix regularization is important!!

The definition of matrix regularization [Hoppe]

Matrix regularization is linear map $T_N : C^\infty(M) \to M_N(\mathbb{C})$ satisfying following equations (M : 2 dim symplectic mfd)

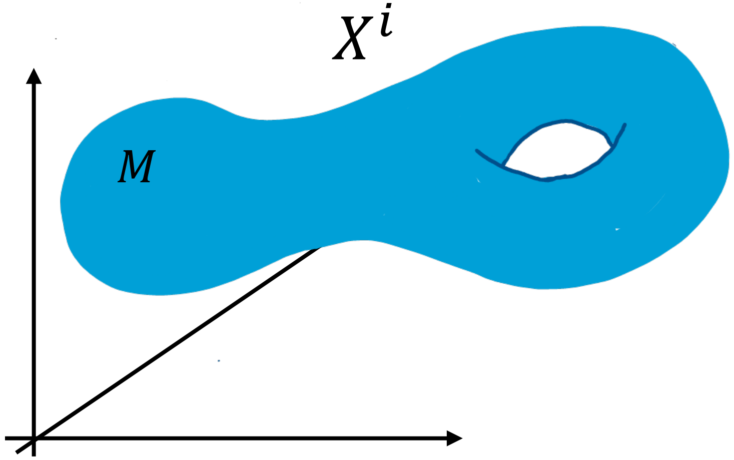
(1). $\lim_{N\to\infty} |T_N(f)T_N(g) - T_N(fg)| = 0$

(2). $\lim_{N\to\infty} |iN[T_N(f), T_N(g)] - T_N(\{f,g\})| = 0 \quad \left(\{f,g\} = W^{\mu\nu}\partial_\mu f \partial_\nu g\right)$

(3). $\lim_{N\to\infty} \frac{1}{N}\mathrm{Tr}T_N(f) = \frac{1}{2\pi}\int_M \omega f \quad (\omega\text{: symplectic form})$

The physical applications of matrix regularization

embedding function



Matrix reg. for *X*

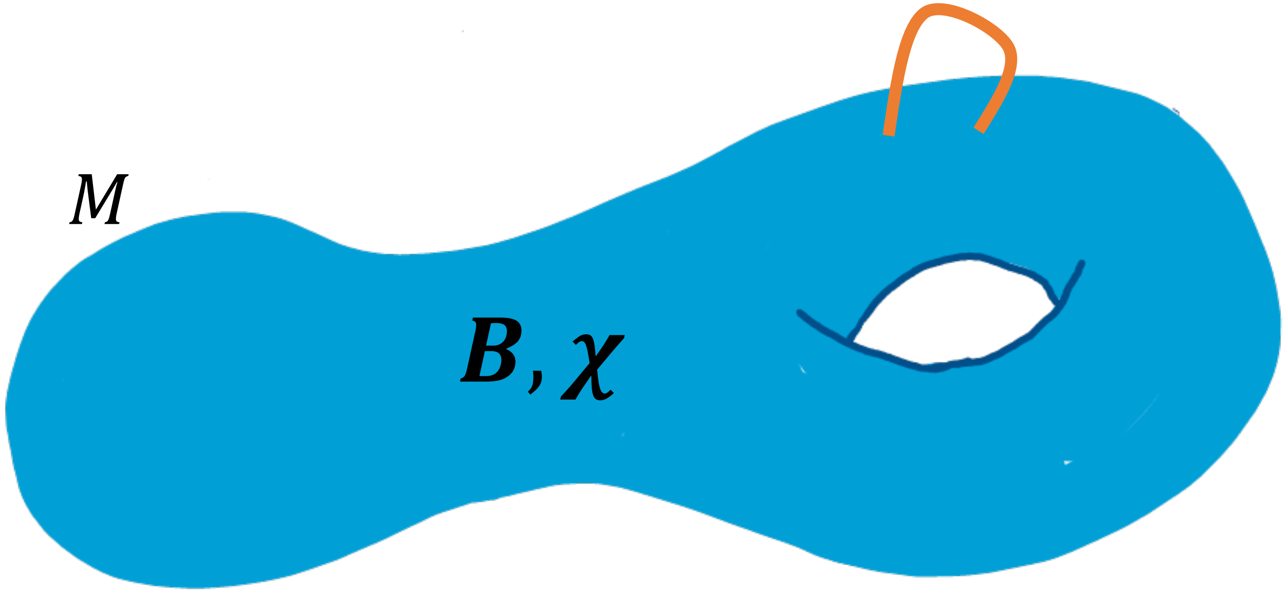
$$T_N(X^i) = \left(\text{Matrix corresponding with } M \right)$$

<u>symmetry</u>

APD

$$\delta X^i = \{\alpha, X^i\} \longrightarrow \delta T_N(X^i) = i[T_N(\alpha), T_N(X^i)]$$

The issue of matrix regularization



Gauge transformation

$$\delta_\alpha B_b = -\nabla_b \beta + i[\beta, B_b\,]$$
$$\delta_\alpha \chi = i[\beta, \chi]$$

D-branes are described as a $U(N)$ bundle with connection B

The issue of matrix regularization

function	→	square matrix	Sec. 2,3
matter field (tensor field, fundamental scalar)	→	rectangular matrix [Adachi-Ishiki-SK][Hawkins]	Sec. 4
gauge field	→	**?**	Sec. 5

application

unified understanding of

gauge theorise on noncommutative space

ref of fuzzy S^2: [Iso-Kimura-Tanaka-Wakatsuki][Watamura-Watamura][Steinacker]

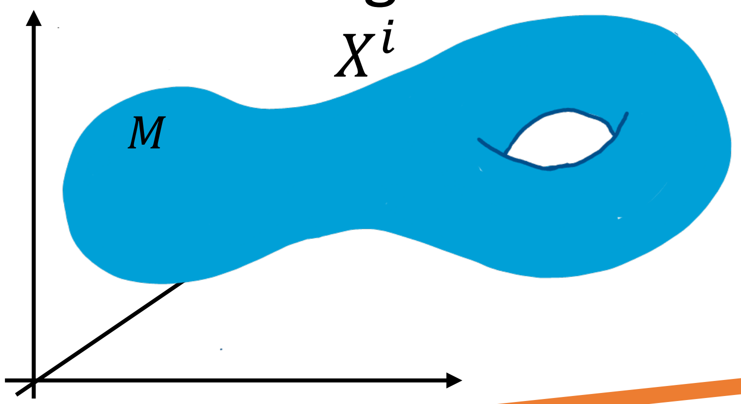
main result

Fields on D-brane
X^i, B, χ

→

Fields on Matrix geometry
$\hat{X}^i(X^i, B, \chi)$

embedding function



→

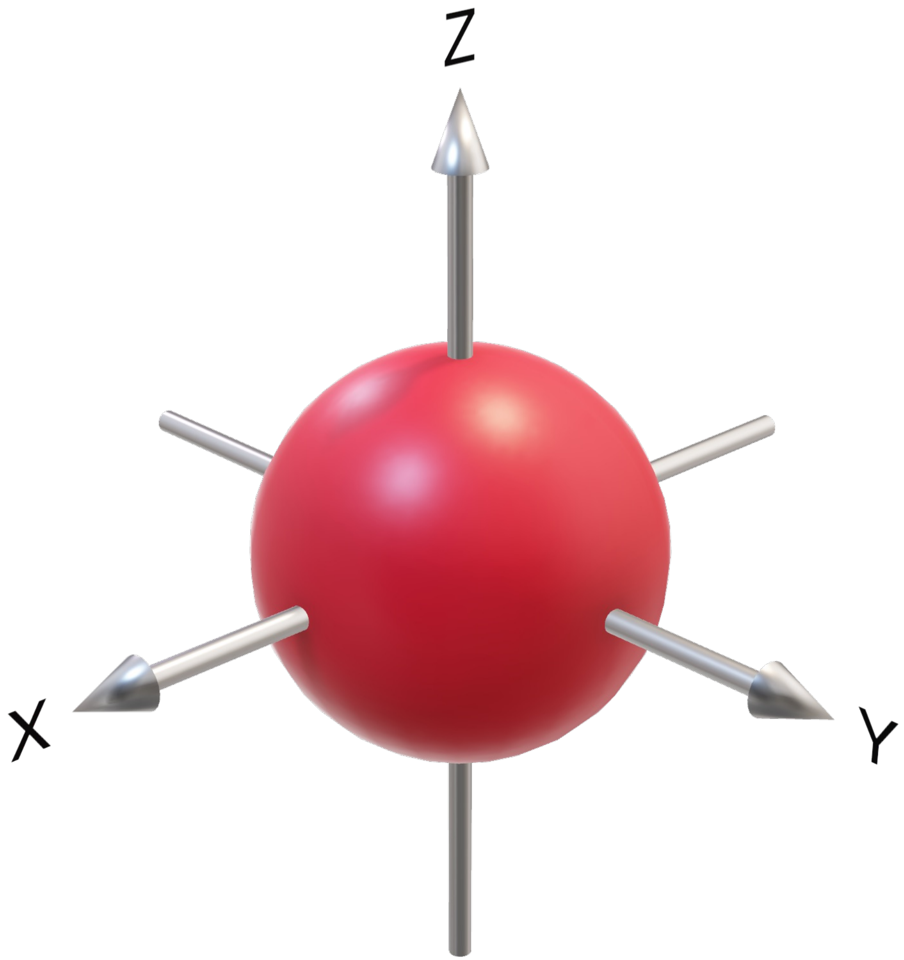
$\hat{X}^i =$

We constructed matrix version of gauge fields by using Berezin-Toeplitz quantization and Seiberg-Witten map

2. matrix regularization (4)

ex) S^2

$$\begin{cases} x_1 = \sin\theta\cos\phi \\ x_2 = \sin\theta\sin\phi \\ x_3 = \cos\theta \end{cases}$$



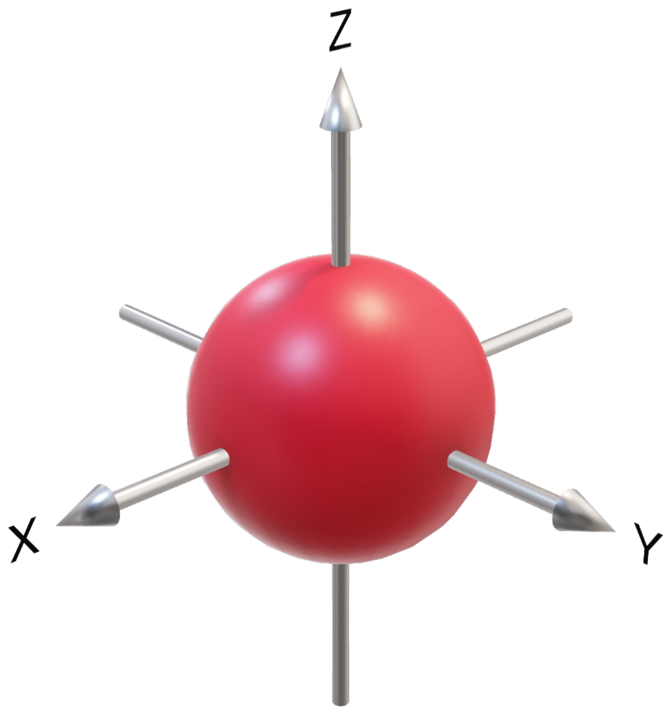
algebraic structure

$$\sum x_i x_i = 1$$
$$\{x_i, x_j\} = \epsilon_{ijk} x_k$$

We can write any function $\varphi(\theta, \phi)$ as

$$\varphi(\theta, \phi) = \sum_{J=0}^{\infty} \sum_{m=-J}^{J} \varphi_{Jm} Y_{Jm}(\theta, \phi)$$

where $Y_{Jm}(\theta,\ \phi)$ is spherical harmonics



$$Y_{Jm}(\theta,\ \phi) = \sum_{l=0}^{J} \sum_{i_k} c^{Jm}_{i_1 i_2 \cdots i_l} x_{i_1} x_{i_2} \cdots x_{i_l}$$

matrix regularization

$$x_i \longrightarrow \hat{x}_i = \frac{1}{\sqrt{\Lambda(\Lambda+1)}} \hat{L}_i$$

$$[\hat{L}_i, \hat{L}_j] = i\epsilon_{ijk}\hat{L}_k$$

spin Λ representation

$$\sum \hat{x}_i \hat{x}_i = \frac{1}{\Lambda(\Lambda+1)} \sum \hat{L}_i \hat{L}_i = 1_{\mathrm{N}}$$

$$[\hat{x}_i, \hat{x}_j] = \frac{i\epsilon_{ijk}}{\sqrt{\Lambda(\Lambda+1)}} \hat{x}_k$$

$$Y_{Jm} = \sum_{l=0}^{J} \sum_{i_k} c^{Jm}_{i_1 i_2 \cdots i_l} x_{i_1} x_{i_2} \cdots x_{i_l} \longrightarrow \hat{Y}_{Jm} = \sum_{l=0}^{J} \sum_{i_k} c^{Jm}_{i_1 i_2 \cdots i_l} \hat{x}_{i_1} \hat{x}_{i_2} \cdots \hat{x}_{i_l}$$

$$\varphi(\theta,\phi)=\sum_{J=0}^{\infty}\sum_{m=-J}^{J}\varphi_{Jm}Y_{Jm}(\theta,\phi)\longrightarrow\hat{\varphi}=\sum_{J=0}^{\Lambda}\sum_{m=-J}^{J}\varphi_{Jm}\hat{Y}_{Jm}$$

$$\{x_i,x_j\}=\epsilon_{ijk}x_k\longrightarrow[\hat{x}_i,\hat{x}_j]=\frac{i\epsilon_{ijk}}{\sqrt{\Lambda(\Lambda+1)}}\hat{x}_k$$

We can obtain the matrix regularization on S^2

→ **general manifold ?**

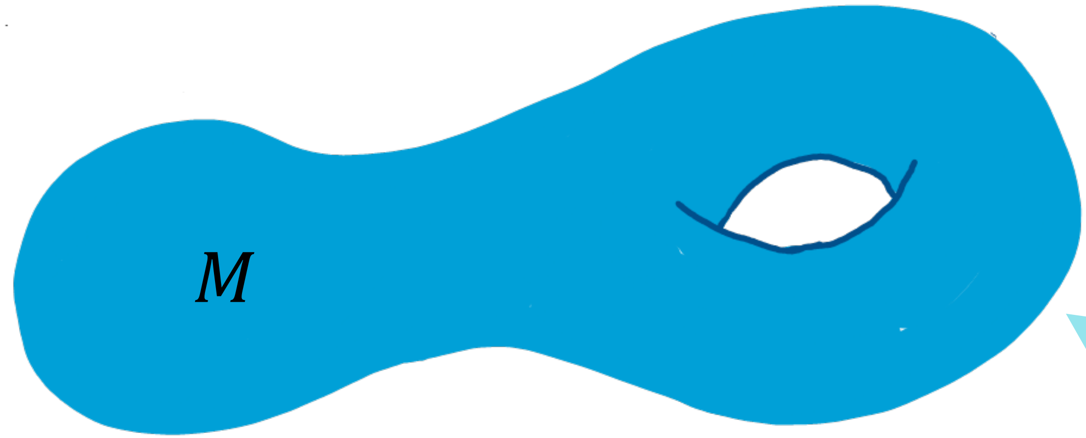
3. Berezin-Toeplitz quantization for function (6)

Berezin-Toeplitz(BT) map is one of the methods of matrix regularization!!

- Applicable to **general (symplectic) manifold**, not just sphere [Ma-Marinescu]
- Applicable to **general fields**, not just functions [Adachi-Ishiki-SK][Hawkins]

setup

M : 2dim connected closed Kähler mfd
(ω : symplectic form)



N charged spinor system with flux

Flux : $F = dA \propto \omega$

consider functions as linear maps

spinor → spinor (function f)

$$f = \sum_{i=1}^{\infty} |\psi_i\rangle f_{ij} \langle\psi_j|$$

$$\left[\begin{array}{c} f_{ij} = \langle\psi_i | f\psi_j\rangle \\ f\psi(x) = f(x)\psi(x) \end{array} \right]$$

the eigenstate of Dirac operator

$$D_0 = i\gamma^a(\partial_a + \Omega_a - iNA_a)$$

$$D_0|\psi_i\rangle = \lambda_i|\psi_i\rangle \qquad \langle\psi_i|\psi_j\rangle = \delta_{ij}$$

projection to finite dimension

$$\dim(\mathrm{Ker}D_0) = N$$

from index theorem etc.

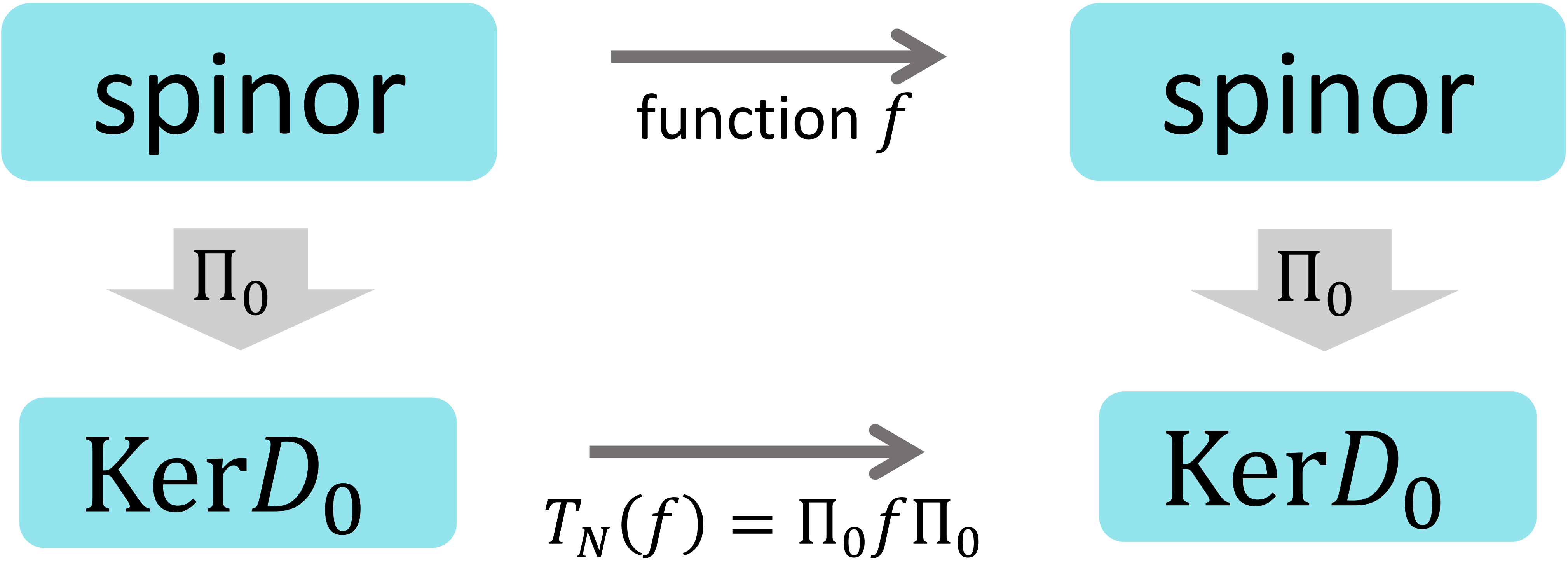
projection from all modes to zero modes

$$\boldsymbol{T(f) = \sum_{I=1}^{N} |\psi_I^0\rangle f_{IJ} \langle \psi_J^0|}$$

zeromodes of Dirac operator $(I, J = 1,2,\cdots,N)$

$$D_0|\psi_I^0\rangle = 0 \qquad \langle \psi_I^0|\psi_J^0\rangle = \delta_{IJ}$$

Let Π_0 be the projection to $\mathrm{Ker}D_0$



We can obtain $N \times N$ matrices from functions

BT map is one of the methods of matrix regularization!!

$$T_N(f)T_N(g) = \sum_{i=0}^{\infty} \frac{1}{N^i} T_N\big(C_i(f,g)\big)$$

We can obtain C_i from $1/N$ expansion for Π_0

$$\begin{cases} C_0(f,g) = fg \\ C_1(f,g) = -\frac{1}{2}(g^{\mu\nu} + iW^{\mu\nu})\partial_\mu f \partial_\nu g \\ \quad\vdots \end{cases}$$

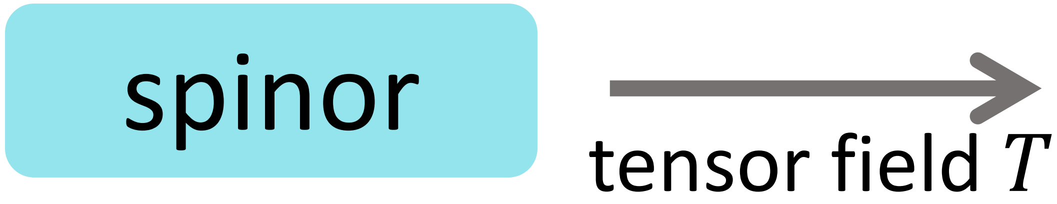
$$\lim_{N\to\infty} |T_N(f)T_N(g) - T_N(fg)| = 0$$

$$\lim_{N\to\infty} |iN[T_N(f), T_N(g)] - T_N(\{f,g\})| = 0$$

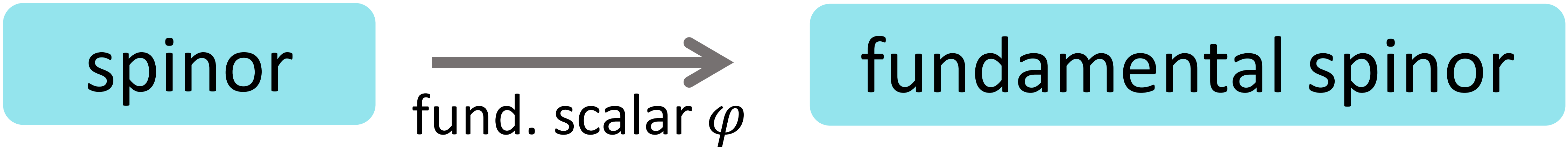
4. Berezin-Toeplitz quantization for matter field (3)

Example of matter fields (not only functions)

ex) tensor field



ex) fundamental scalar



BT quantization for mater fields

$D_1 = i\gamma^a(\partial_a + \Omega_a - iNA_a^{(L)})$

spinor

Π_0

$\mathrm{Ker}D_0$

$\dim(\mathrm{Ker}D_0) = N$

fund. scalar ϕ

$T_N(\phi) = \Pi_1 \phi \Pi_0$

$D_1 = i\gamma^a(\partial_a + \Omega_a - iNA_a^{(L)} - iA_a^{(E)})$

fundamental spinor

Π_1

$\mathrm{Ker}D_1$

$\dim(\mathrm{Ker}D_1) = d^{(E)}N + c_1^{(E)}$

$d^{(E)}$: dim of fiber, $c_1(E)$: first chern ember

$$(1).\ \lim_{N\to\infty} |T_N(f)T_N(\phi) - T_N(f\phi)| = 0$$

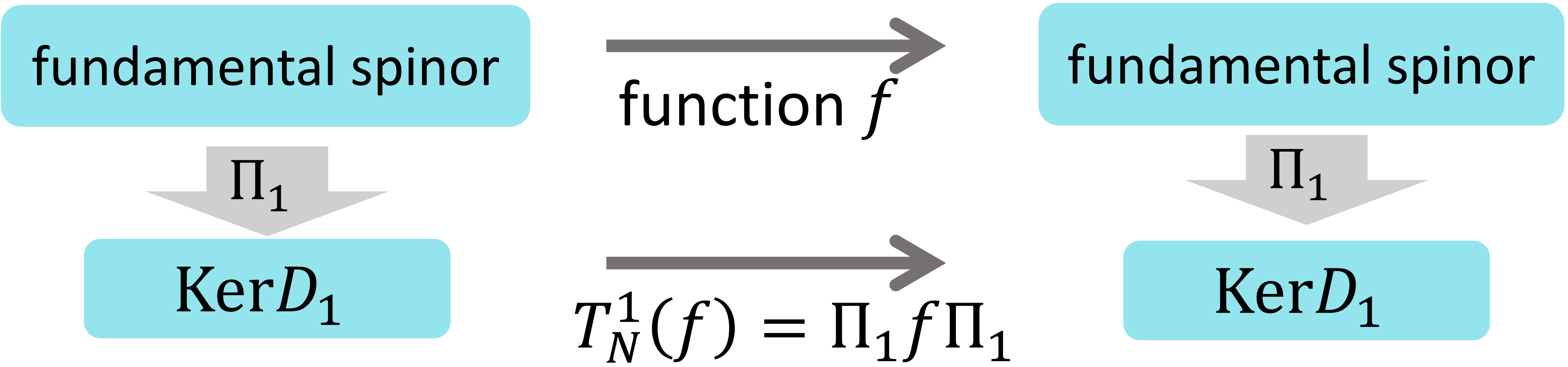
$$(2).\ \lim_{N\to\infty} |iN[T(f),T(\phi)] - T_N(\{f,\phi\})| = 0$$

$$(3).\ \lim_{N\to\infty} \frac{1}{N}\mathrm{Tr}\left(T_N(\phi)^\dagger T_N(\phi)\right) = \frac{1}{2\pi}\int_M \omega\phi_a^\dagger\phi^a$$

Generalized Poisson bracket and Generalized commutator

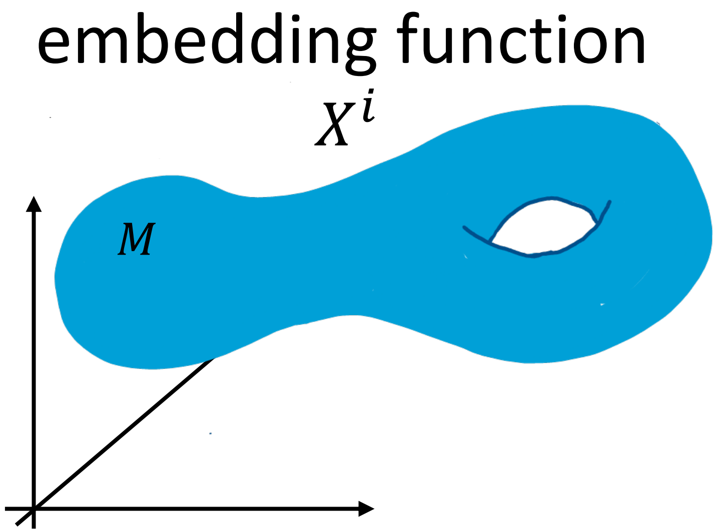
$$\{f,\phi\} := W^{\alpha\beta}\partial_\alpha f\nabla_\beta\phi$$

$$[T(f),T(\phi)] := T_N^1(f)T_N(\phi) - T_N(\phi)T_N(f)$$



5. Berezin-Toeplitz quantization for gauge fields (8)

motivation of matrix regularization for gauge field



$$T_N(X^i) = \begin{pmatrix} \text{Matrix corresponding to } M \end{pmatrix}$$

branes are described as a bundle with connection

→ **matrix version of the bundle structure (B, χ) ?**

How does the matrix include the fields on D-brane ?

Fields on D-brane	? →	Matrix
X^i, B, χ		$\hat{X}^i(X^i, B, \chi)$

<u>Symmetry</u>

$$\delta_\alpha B_b = -\nabla_b \beta + i[\beta, B_b\,] \qquad \xrightarrow{?} \qquad \delta \hat{X}^i = i\left[T_N(\alpha), \hat{X}^i\right]$$
$$\delta_\alpha \chi = i[\beta, \chi]$$

Is there the map compatible with symmetry?

→**Seiberg-Witten map**

Seiberg-Witten map [Seiberg-Witten]

Moyal product

$$f * g(x) := \exp\left(\frac{i}{2}\theta^{\mu\nu}\partial^y_\mu \partial^z_\nu\right) f(y)g(z)\Big|_{y=z=x}$$

$$\theta = \begin{pmatrix} 0 & \hbar & 0 & 0 \\ -\hbar & 0 & 0 & 0 \\ 0 & 0 & 0 & \hbar \\ 0 & 0 & -\hbar & 0 \end{pmatrix}$$

$$= f(x)g(x) + \frac{i}{2}\theta^{\mu\nu}\partial_\mu f(x)\partial_\nu g(x) + O(\theta^2)$$

This multiplication is noncommutative

$$[x^\mu, x^\nu]_* = x^\mu * x^\nu - x^\nu * x^\mu$$
$$= i\theta^{\mu\nu}$$

ex) $U(1)$

commutative $U(1)$ gauge

$$\delta_\lambda A_i = \partial_i \lambda$$

$\xrightarrow{\hat{A}(A)}$

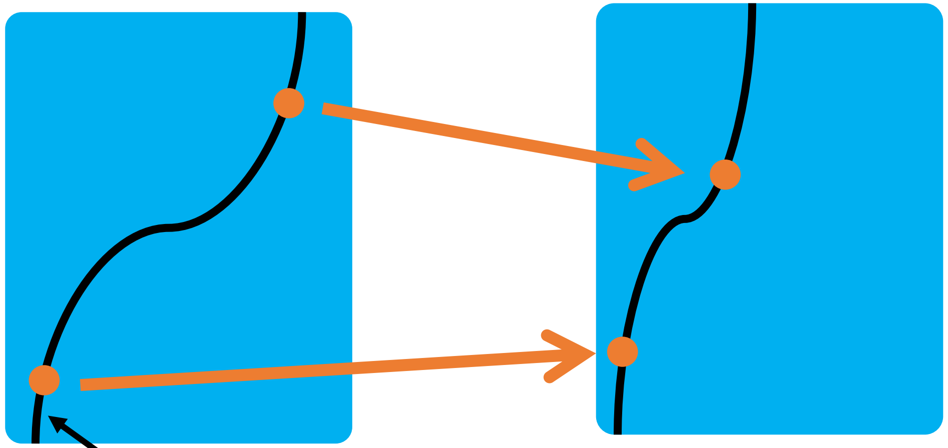
noncommutative $U(1)$ gauge

$$\hat{\delta}_{\hat{\lambda}} \hat{A}_i = \partial_i \hat{\lambda} + i[\hat{\lambda}, \hat{A}_i]_*$$

$$\boldsymbol{\hat{A}(A) + \hat{\delta}_{\hat{\lambda}} \hat{A}(A) = \hat{A}(A + \delta_\lambda A)}$$

Space of gauge fields

Space of gauge fields on NC space



gauge equivalent config.

The solution

$$\hat{A}_i(A) = A_i - \frac{1}{4}\theta^{kl}\{A_k, \partial_l A_i + F_{li}\} + O(\theta^2)$$

$$\hat{\lambda}(\lambda, A) = \lambda + \frac{1}{4}\theta^{ij}\{\partial_i \lambda, A_j\} + O(\theta^2)$$

matrix version of Seiberg-Witten map

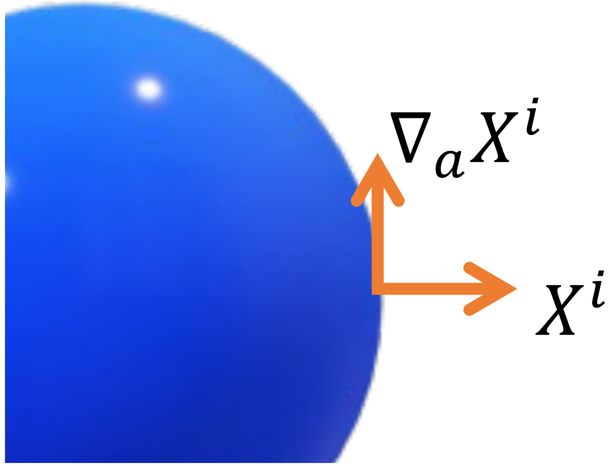
ex) S^2

embedding function

X^i

$(\sum X^i X^i = 1)$

$$T_N(X^i) = \left(\text{fuzzy } S^2 \right)$$



$$\hat{X}^i(B,\chi) = T_N\left(X^i 1_M + \frac{1}{N} W^{ab}(\nabla_a X^i) A_b + \frac{1}{N} X^i \phi \right)$$

Introduce fluctuation of embedding function!!

Is there a map $A(B,\chi), \phi(B,\chi)$ compatible with symmetry ?

Symmetry of A, ϕ

$$\delta \hat{X}^i = i\left[T_N(\alpha), \hat{X}^i\right]$$

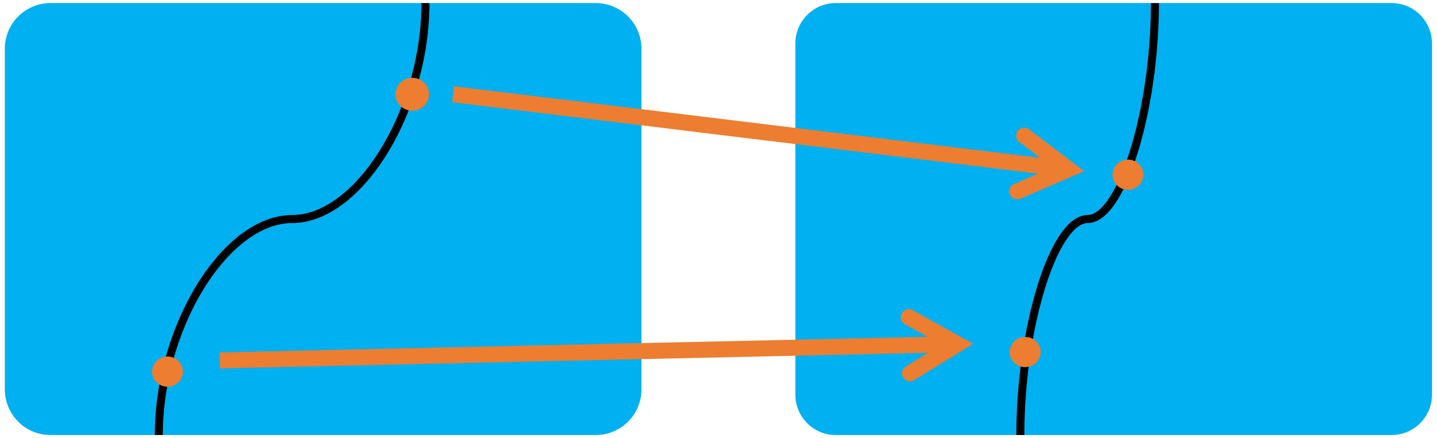
$$\hat{X}^i(B,\chi) = T_N\left(X^i 1_M + \frac{1}{N} W^{ab}\left(\nabla_a X^i\right) A_b + \frac{1}{N} X^i \phi\right)$$

$$\delta_\alpha A_c = -\nabla_c \alpha + i[\alpha, A_c]$$
$$+\frac{1}{N}\left(\frac{1}{4} R(\nabla_c \alpha) - \frac{i}{2} g^{\alpha\beta}[\nabla_\alpha \alpha, \nabla_\beta A_c] - \frac{i}{2} W_{cd} g^{\alpha d}[\nabla_\alpha \alpha, \phi] + \frac{1}{2} W^{\alpha\beta}\{\nabla_\alpha \alpha, \nabla_\beta A_c\} - \frac{1}{2}\{\nabla_\alpha \alpha, \phi\}\right) + O(1/N^2)$$

$$\delta_\alpha \phi = i[\alpha, \phi]$$
$$+i\frac{1}{N}\left(-\frac{i}{2} W^{ab}[\nabla_a \alpha, A_b] - \frac{i}{2} g^{\alpha\beta}[\nabla_\alpha \alpha, \nabla_\beta \phi] - \frac{1}{2} g^{ab}\{\nabla_a \alpha, A_b\} + \frac{1}{2} W^{\alpha\beta}\{\nabla_\alpha \alpha, \nabla_\beta \phi\}\right) + O(1/N^2)$$

solution for SW map

Space of gauge fields



Space of gauge fields on NC space

$$\boldsymbol{A_b(B,\chi) + \delta_\alpha A_b(B,\chi) = A_b\left(B + \delta_\beta B, \chi + \delta_\beta \chi\right)}$$
$$\boldsymbol{\phi(B,\chi) + \delta_\alpha \phi(B,\chi) = \phi\left(B + \delta_\beta B, \chi + \delta_\beta \chi\right)}$$

The solution

$$A_c(B,\chi) = B_c - \frac{1}{N}\left(\frac{R}{4}B_c + \frac{i}{2}\left[B_b, g_{cd}W^{db}\chi - \nabla^b B_c + \frac{1}{2}g^{bd}(\nabla_c B_d - i[B_c,B_d])\right] - \frac{1}{2}\left\{B_b, \delta_c^b\chi - W^{bd}\nabla_d B_c + \frac{1}{2}W^{bd}(\nabla_c B_d - i[B_c,B_d])\right\}\right) + \cdots$$
$$\phi(B,\chi) = \chi - \frac{1}{N}\left(\frac{1}{2}g^{cd}B_cB_d + \frac{i}{2}W^{cd}B_cB_d - \frac{i}{2}g^{cd}[B_c,\nabla_d\chi] + \frac{1}{2}W^{cd}\{B_c,\nabla_d\chi\} + \frac{1}{4}g^{cd}\left[B_c,[B_d,\chi]\right] + \frac{i}{4}W^{cd}\{B_c,[B_d,\chi]\}\right) + \cdots$$
$$\alpha(B,\chi,\beta) = \beta + \frac{1}{N}\left(\frac{i}{4}g^{cd}[\nabla_c\beta, B_d] - \frac{1}{4}W^{cd}\{\nabla_c\beta, B_d\}\right) + \cdots$$

Matrix regularization for gauge theory

ex) cubic matrix model [Ishii-Ishiki-Ohta-Tsuchiya-Shimasaki]

$$S_{\rm mm} = -\frac{1}{g^2}\mathrm{Tr}\left(\hat{X}^i\hat{X}^i + \frac{iN}{3}\epsilon^{ijk}\hat{X}^i[\hat{X}^j,\hat{X}^k]\right)$$

$\downarrow N \to \infty$

$$S_{\rm mm} = \frac{1}{2\pi}\int d^2x\ \mathrm{tr}\Big[(2F_{12}\chi - \chi^2)$$

$$+\frac{1}{N}\left(3F_{12}\chi + \frac{2}{3}(\chi-1/2)^3 + 2(\chi-1/2)^2 - \frac{3}{2}\chi + (F_{12})^2 + 3D_-\chi D_+\chi + (\chi-1/2)^2F_{12} - F_{12}(D_-D_+ + D_+D_-)\chi\right)$$

$$+O(1/N^2)\Big]$$

$D_\pm\phi = \partial_\pm\phi - i[B_\pm,\phi]$

$U(M)$ massive BF theory

6. Summary and discussion (3)

Summary

- By analogy with D-brane, we identified a gauge field in matrix geometry
- We can obtain a correspondence between gauge fields in matrix geometry and gauge fields in commutative space by using SW map
- We applied the matrix regularization to some gauge theories

massive BF theory

$$\int d^2x \ \mathrm{tr}(2F_{12}\chi - \chi^2)$$

Matrix Model

$$-\frac{1}{g^2}\mathrm{Tr}\left(\hat{X}^A\hat{X}^A + \frac{iN}{3}\epsilon^{ABC}\hat{X}^A[\hat{X}^B,\hat{X}^C]\right)$$

outlook

- Regularization of field theory
  → New QCD regularization that can be simulated ?

- There is another formulation by using Wilson line

- From the viewpoint of tachyon condensation, we want to clarify the relationship with D-brane

[Asakawa-Sugimoto-Terashima][Terashima][Asakawa-Ishiki-Matsumoto-Matsuura-Muraki]

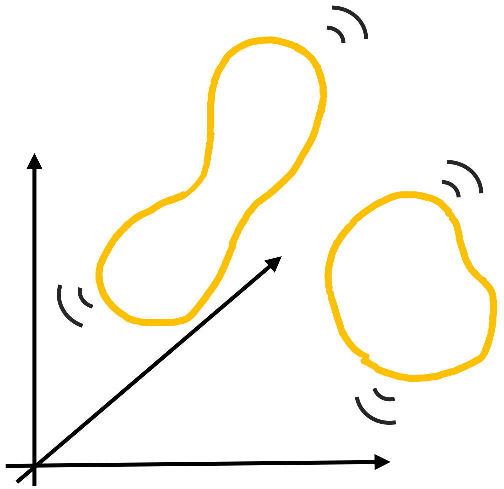
- Given a matrix, we want to extract geometric information inversely
 (inverse problem of matrix regularization)

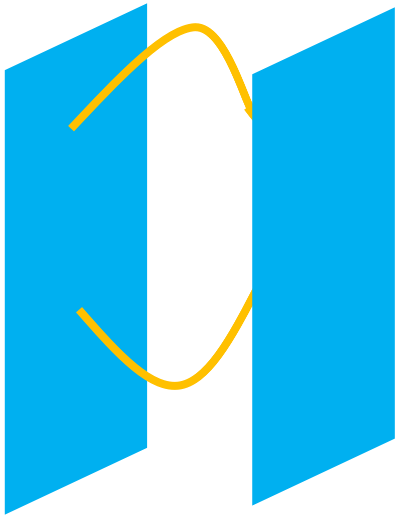
[Ishiki-Matsumoto-Muraki][ishiki][de Wit-Hoppe-Nicolai][Sako]

- We would like to find a relation between contravariant gravity and SW map on a matrix geometry

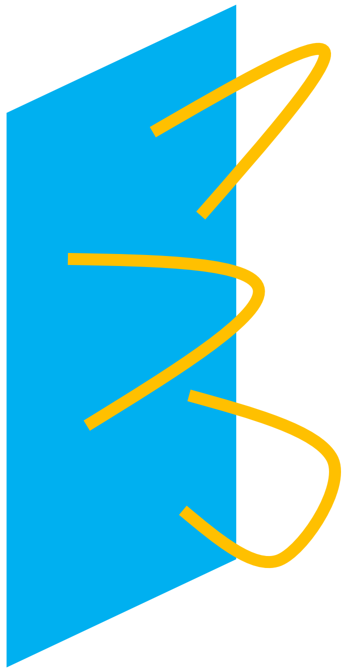
[Kaneko-Muraki-Watamura]

Backup slide





D-brane : extended object which is boundary of open string



D-branes are dynamical
by excitation of open string

Concretely method : Berezin-Toeplitz quantization

M : 2dim connected closed Kähler mfd
(ω : symplectic form)

S : spinor bundle over M

L : line bundle over M with $\omega = F^{(L)}$

- Dirac operator D_0 on $\Gamma\left(S \otimes L^{\otimes N}\right)$ is satisfying

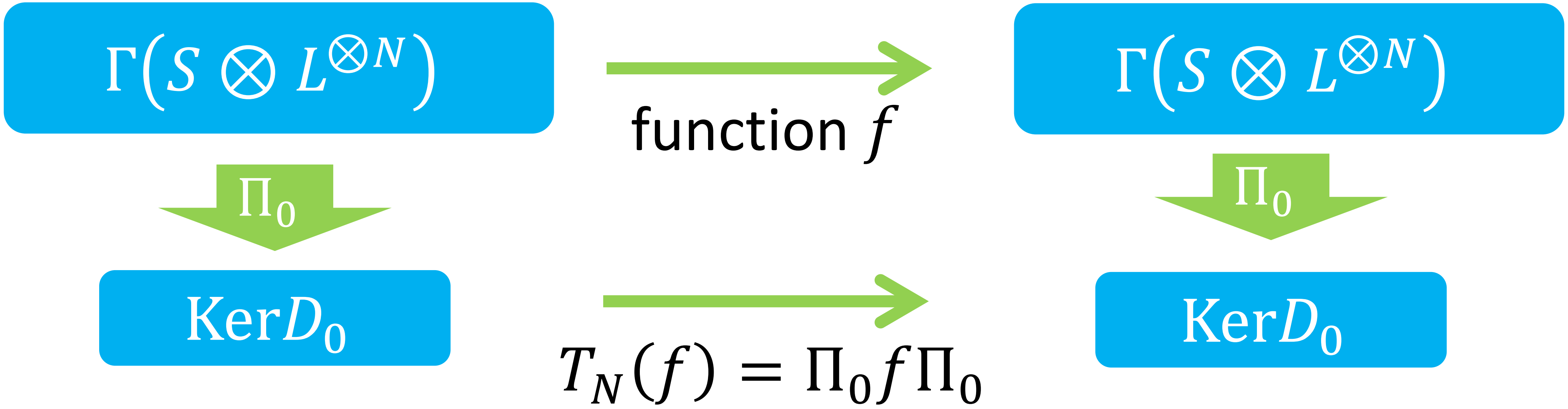
$$\dim(\mathrm{Ker} D_0) = N$$

$D_0 = i\gamma^a(\partial_a + \omega_a - iA_a^{(L)})$

- consider a functions as linear maps

$$f : \Gamma\left(S \otimes L^{\otimes N}\right) \to \Gamma\left(S \otimes L^{\otimes N}\right)$$
$$\psi(x) \mapsto f(x)\psi(x)$$

- projection $\Pi_0 : \Gamma\left(S \otimes L^{\otimes N}\right) \to \mathrm{Ker} D_0$



We can obtain $N \times N$ matrices from functions

$$T_N(f)T_N(g) = \Pi_0 f \Pi_0 g \Pi_0$$

We can obtain C_i by the estimate of $1/N$ expansion for Π

$$\begin{cases} C_0(f,g) = fg \\ C_1(f,g) = -\frac{1}{2}(g^{\mu\nu} + iW^{\mu\nu})\partial_\mu f \partial_\nu g \\ \vdots \end{cases}$$

BT map is one of the methods of matrix regularization!!

Cutoff regularization

$$\varphi(\theta,\phi) = \sum_{J=0}^{\infty}\sum_{m=-J}^{J}\varphi_{Jm}Y_{Jm}(\theta,\phi) \longrightarrow \varphi(\theta,\phi) = \sum_{J=0}^{\Lambda}\sum_{m=-J}^{J}\varphi_{Jm}Y_{Jm}(\theta,\phi)$$

The product $\varphi_1(\theta,\phi)\varphi_2(\theta,\phi)$

$$[\Lambda] \otimes [\Lambda] = [0]\oplus[1]\oplus\cdots\oplus[2\Lambda]$$

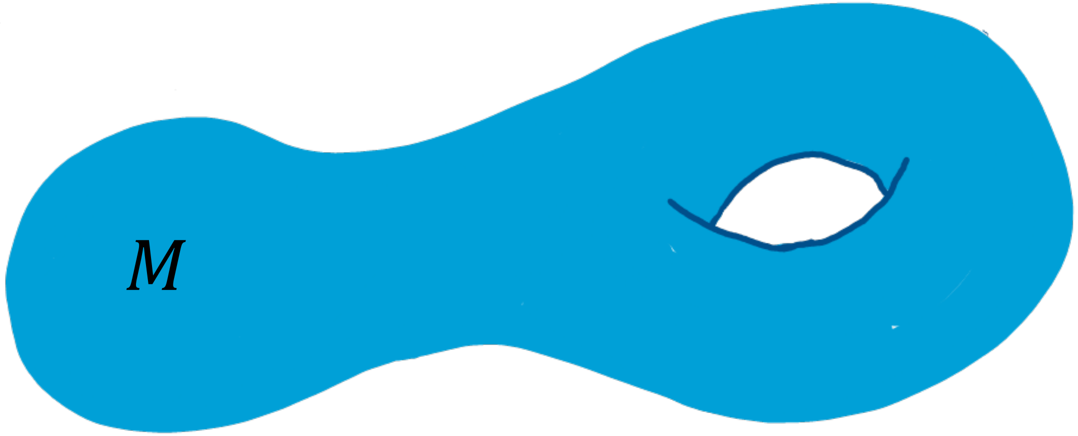
Cutoff regularization is not closed

BT quantization

M : 2dim connected closed Kähler mfd
(ω : symplectic form)

S : spinor bundle over M

L : line bundle over M with $\omega = F^{(L)}$



Charged spinor system with constant flux

Matrix regularization is linear map $T_N : C^\infty(M) \to M_N(\mathbb{C})$ satisfying following equations

(M : 2 dim symplectic mfd)

$$(1).\ T_N(f)T_N(g) = \sum_{i=0}^{\infty} \frac{1}{N^i} T_N\big(C_i(f,g)\big)$$

$$C_0(f,g) = fg$$

$$C_1(f,g) - C_1(g,f) = \frac{1}{iN}\{f,g\}$$

$$(2).\ \lim_{N\to\infty} \frac{1}{N} \mathrm{Tr} T_N(f) = \frac{1}{2\pi}\int_M \omega f \qquad (\omega : \text{volume form})$$

space M $\longrightarrow$ Matrix geometry corresponding with M

$C(M)$
commutative C*algebra

equivalent ↕

space M

▶

$C(\widehat{M})$
noncommutative C*algebra

↓ definition

Noncommutative space $\widehat{M}$

C*algebra : algebras with good norm and involution

ex1) function over topological space $C(M)$

ex2) matrix algebra $M_N(\mathbb{C})$

matrix geometry

$C(\widehat{M}) = M_N(\mathbb{C})$ with a good commutative limit $(N \to \infty)$

$$\boldsymbol{M_N(\mathbb{C}) \to C^\infty(M)}$$

近似

algebraic structure

Matrix regularization is important!! [Hoppe]

The action of matrix model [BFSS,IKKT,DVV]

$$S = \int dt \, \mathrm{Tr} \left[\frac{1}{2} (DX^i)^2 + \frac{1}{4} [X^i, X^j]^2 + (\mathrm{fermion}) \right]$$

[BFSS]

X^i : $N \times N$ Hermitian matrices

The symmetry

$\delta X^i = i[\alpha, X^i]$ (α : $N \times N$ Hermitian matrix)

The matrix model with matrix size $N \to \infty$ is a candidate for the formulation of M-theory

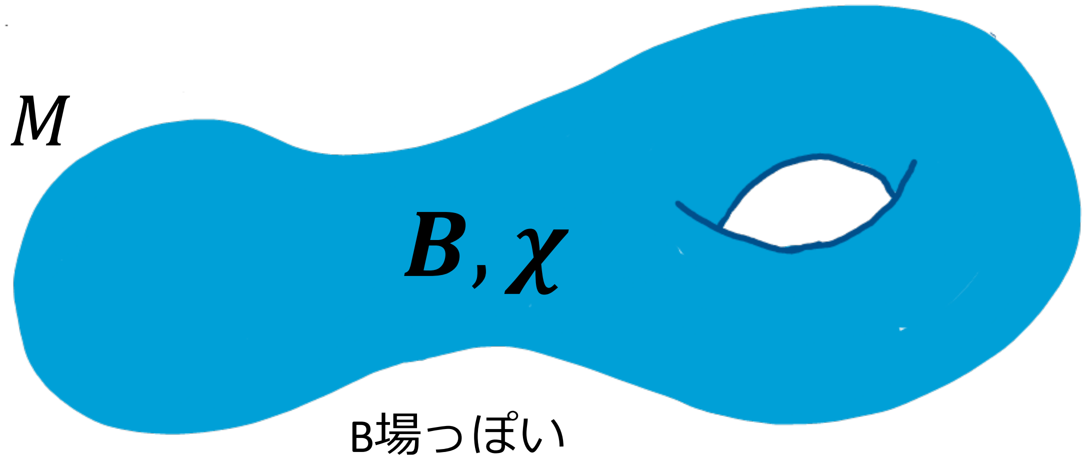
D-brane : extended object which is boundary of open string

D-branes are dynamical because of excitation of open string

Config. of matrix model

D-brane

$X^i =$ 



How are they related to each other ?

We consider a functions as linear maps

Matrix is linear map on finite dim vector space

intuitive understanding

$$f = \int d^2x d^2y \, |x\rangle f_{xy} \langle y|$$

$$f_{xy} = \langle x|f|y\rangle = f(x)\delta(x-y)$$

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