Numerical Study of the Massive Schwinger Model with Lüscher's "Admissible" Gauge Fields

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(collaborated with Tetsuya Onogi)

Contents

1. Introduction

1.1 Chiral symmetries on the lattice

Ginsparg-Wilson relation

[Phys.Rev.D25,2649(1982)]

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D \tag{1}$$

This relation gives redefinition of exact chiral symmetry at classical level.

Admissibility Condition

[Nucl.Phys.B549,295(1999)]

$$\|1 - U^P_{\mu\nu}(x)\| \le \epsilon \quad \text{for all} \;\; x, \mu, \nu \tag{2}$$

"admissible" gauge fields keep topological structure on lattice space.
$\longrightarrow$ proper treatment of quantum anomaly.

1.2 Effects of admissibility condition

- **Topological charge (in QED on T^2)**

  continuum theory

$$\frac{1}{2\pi}\int_{T^2} d^2x \left[\frac{1}{2}\epsilon_{\mu\nu}F_{\mu\nu}(x)\right] = M \quad \text{integer} \tag{3}$$

  lattice theory (under admissibility condition)

$$\frac{1}{2\pi}\sum_{x,\mu,\nu}\left[-\frac{i}{2}\epsilon_{\mu\nu}\ln U^P_{\mu\nu}(x)\right] = M \quad \text{integer} \tag{4}$$

- **Locality**

  Overlap Dirac operator D is always local under admissibility.

- **Range of gauge action**

  Lüscher's gauge action which satisfies admissibility:

$$S_G = \begin{cases} \sum_{x,\mu,\nu} \dfrac{\beta(1-\mathrm{Re}U^P_{\mu\nu}(x))}{1-(1-\mathrm{Re}U^P_{\mu\nu}(x))/\epsilon} & \text{if admissible} \\ \infty & \text{otherwise} \end{cases} \tag{5}$$

  which gives value in

$$0 \le S_G < \infty. \tag{6}$$

  This range is the same as in continuum theories.

- **Range of field strength**

  The space of field strength $F_{\mu\nu} \equiv -i \ln U^P_{\mu\nu}(x)$ is contractible.

  Wilson's action

$$-\frac{\pi}{a^2} \leq F_{\mu\nu} < \frac{\pi}{a^2} \tag{7}$$

$$\Downarrow$$

  Lüscher's action

$$-\frac{\pi\epsilon'}{a^2} < F_{\mu\nu} < \frac{\pi\epsilon'}{a^2} \tag{8}$$

  where $|\epsilon'| < 1$.

Is "admissibility condition" applicable to numerical simulations ?

The purpose of my study is to answer the question.

1.3 The massive Schwinger model

It's interesting to examine the "admissibility" in 2-flavor QED on T^2 (the massive Schwinger model) because it has following properties:

- **Confinement**
- **Nontrivial topological structure**
  This theory has a topological charge and vacuum dependent on phase θ. (θ-vacuum)
- **Bosonization picture**
  This theory can be understood analytically in continuum space even in strong coupling limit by bosonization.
- **Mesons**
  At $\theta \sim 0$, isotriplet meson has light mass dependent on θ;
  $$m_\pi = 2.07 e^{1/3} (m \cos \frac{\theta}{2})^{2/3} \tag{9}$$
  (where m is fermion mass and e is electric charge.) while isosinglet meson has heavier mass;
  $$m_\eta = e\sqrt{\frac{2}{\pi}} + (O(m)\text{corrections}) \tag{10}$$
  This shows U(1)problem in 2 dimension.

1.4 Our work

- "Admissibility condition" is applicable to simulation.
- A new method of summing over different topological sectors.
- All the results are consistent with the continuum theory.

2. Lattice simulations

2.1 Methods

- **Action**
  We take Lüscher's action combined with domain-wall fermion action;

$$S = \beta S_G + S_F, \tag{11}$$

$$S_G = \begin{cases} \displaystyle\sum_{x,\mu,\nu} \frac{(1-\mathrm{Re}U^P_{\mu\nu}(x))}{1-(1-\mathrm{Re}U^P_{\mu\nu}(x))/\epsilon} & \text{if admissible} \\ \infty & \text{otherwise} \end{cases} \tag{12}$$

$$S_F = \sum_{x,x'}\sum_{s,s'}\sum_{i=1}^{2}\left[\bar{\psi}^i_s(x)D_{DW}(x,s;x',s')\psi^i_{s'}(x') + \phi^{i*}_s(x)D_{AP}(x,s;x',s')\phi^i_{s'}(x')\right], \tag{13}$$

$$\beta = 1/e^2 \quad ,$$
D_{DW}, D_{AP} : domain-wall fermion operators
ϕ^i : Pauli-Villars fields

- **Algorithm**
  We use the Hybrid Monte Carlo (HMC) method with pseudofermions and conjugate gradient (CG) algorithm for the inversion of the fermion matrix. We also check the admissibility at the Metropolis test.

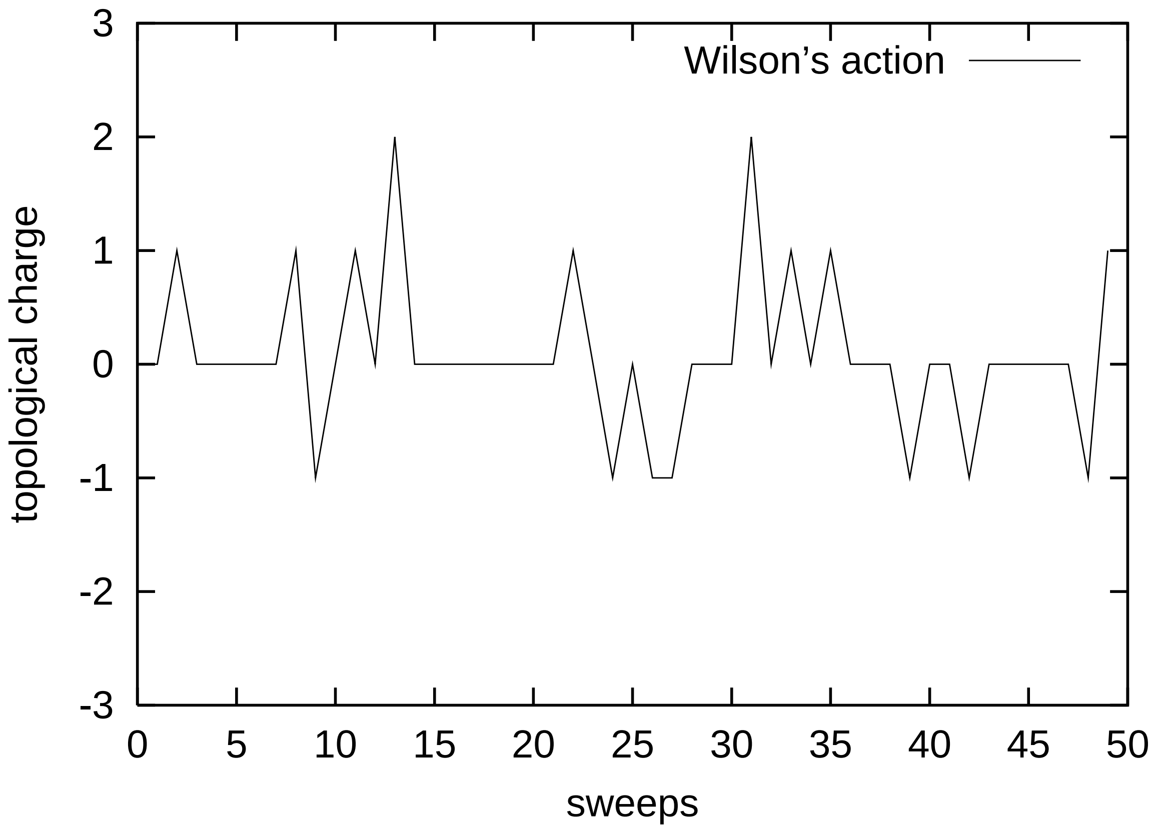
- **Parameters**
  - size : $16 \times 16 \times 6$
  - $\epsilon = 1.0$
  - β : 0.5
  - fermion mass m : $0.1, 0.2, 0.3, 0.4$
  - molecular dynamics stepsize $\Delta\tau$: 0.02
  - 50 molecular dynamics steps per one trajectory.
  - Configurations are updated per 10 trajectories.

2.2 Topological Sectors

Initial configuration is given as follows;

$$\begin{aligned} U_1^{cl[N]}(x,y) &= \exp\left\{-\frac{2\pi N i}{L}\delta_{x,L-1}x\right\}, \\ U_2^{cl[N]}(x,y) &= \exp\left\{\frac{2\pi N i}{L^2}y\right\}, \end{aligned} \tag{14}$$

where L is lattice size. This configuration gives constant background electric fields over the torus and minimizes the gauge action in the sector with topological charge N.



Wilson's action
topological charge
sweeps
3
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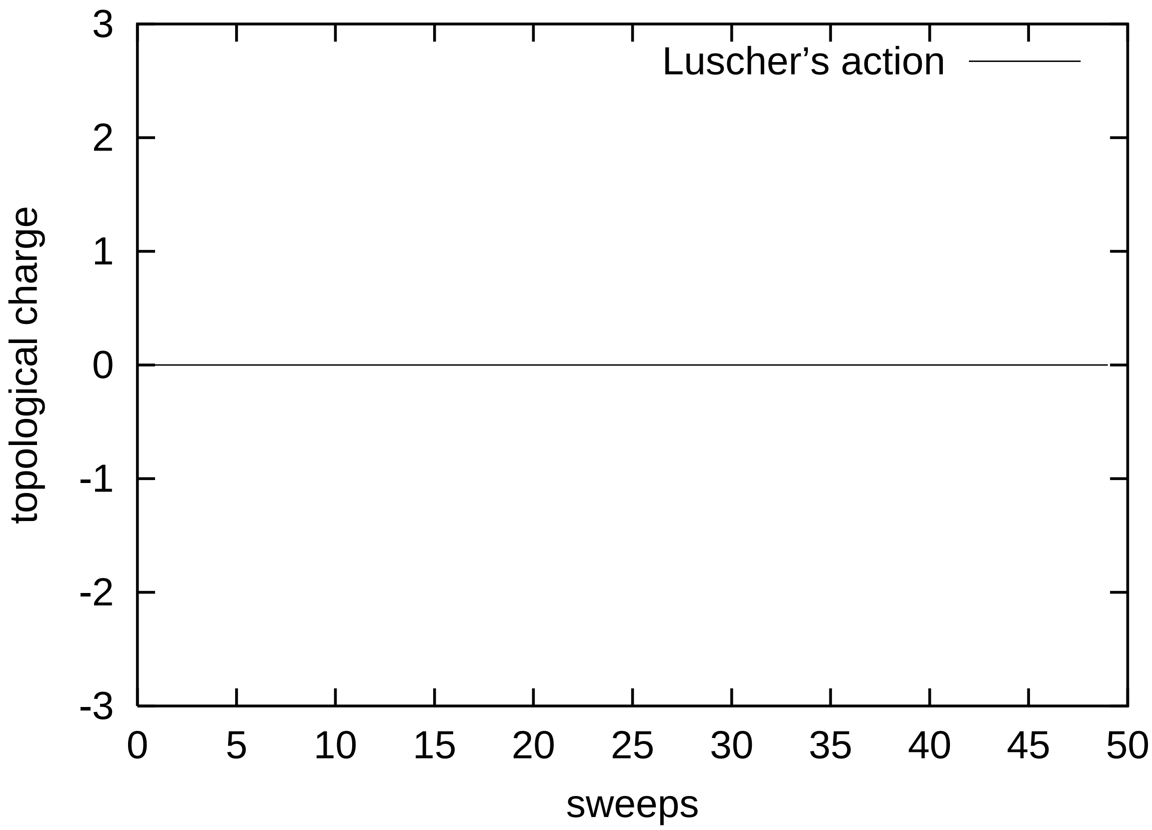

Luscher's action
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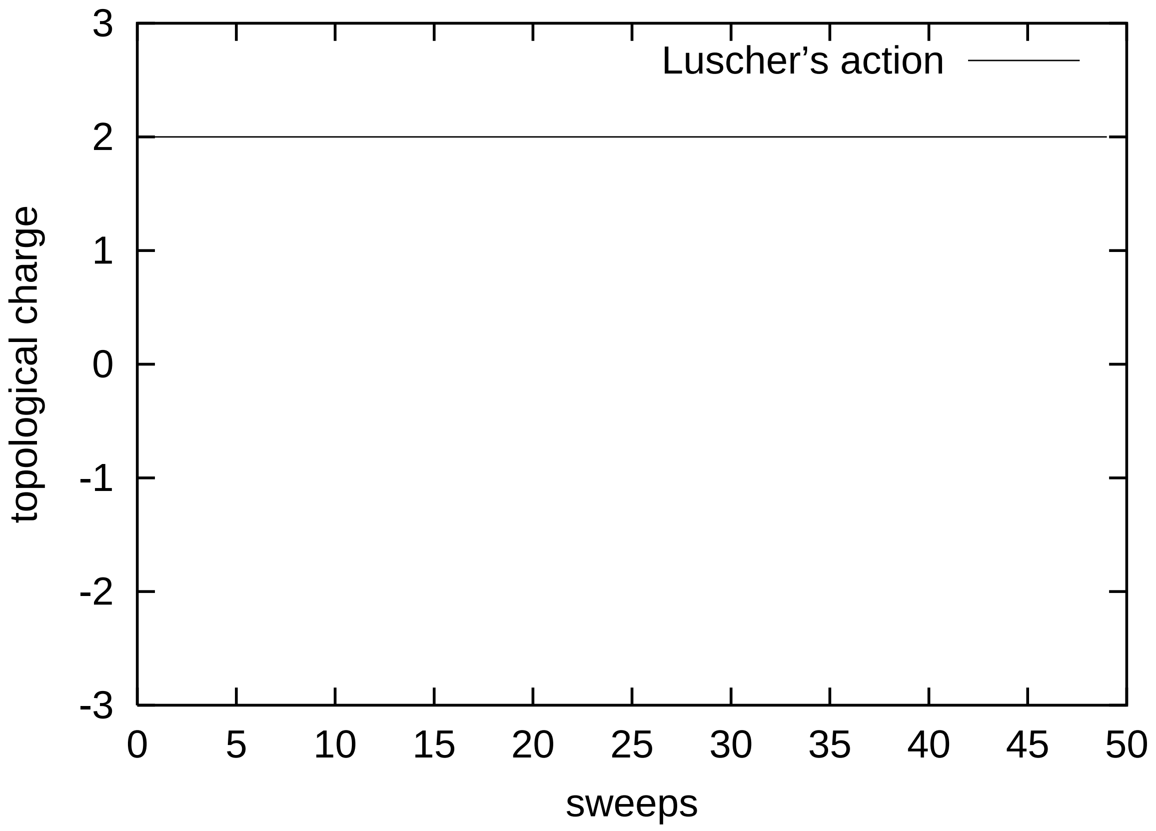

As the figures show, topological structure is not broken on the lattice. We generate 500 configurations for each topological sector ($|N| < 5$) and measure the expectation values of meson propagators in each sector.

2.3 A new method of summing over different topological sectors

Full expectation value of an operator O in θ-vacuum is written as follows,

$$\langle O\rangle_{\beta,m}^{full} = \frac{\Sigma_{N=-\infty}^{+\infty} e^{iN\theta}\langle O\rangle_{\beta,m}^{N} R^{N}(\beta,m)}{\Sigma_{N=-\infty}^{+\infty} e^{iN\theta} R^{N}(\beta,m)}, \tag{15}$$

where

$$\langle O\rangle_{\beta,m}^{N} = \frac{\int DU_{\mu}^{N} D\bar{\psi} D\psi\, O e^{-\beta S_G - S_F}}{Z_N(\beta,m)}, \tag{16}$$

$$Z_N(\beta,m) = \int DU_{\mu}^{N} D\bar{\psi} D\psi e^{-\beta S_G - S_F} \tag{17}$$

and

$$R^{N}(\beta,m) = \frac{Z_N(\beta,m)}{Z_0(\beta,m)}. \tag{18}$$

We call $R^{N}(\beta,m)$ the reweighting factor.

$Z_N(\beta,m)$ satisfies following differential equation.

$$\frac{\partial Z_N(\beta,m)}{\partial\beta} / Z_N(\beta,m) = -\langle S_G\rangle_{\beta,m}^{N}. \tag{19}$$

By integrating over β again, $Z_N(\beta,m)$ is expressed as,

$$Z_N(\beta,m) = Z_N(\infty,m)\exp\left(\int_{\beta}^{\infty} d\beta' \langle S_G\rangle_{\beta',m}^{N}\right). \tag{20}$$

Then, the reweighting factor $R^N(\beta, m)$ is expressed as,

$$R^N(\beta, m) \tag{21}$$
$$= \frac{Z_N(\infty,m)}{Z_0(\infty,m)} e^{\left[\int_\beta^\infty d\beta' \left(\langle S_G\rangle^N_{\beta',m} - \langle S_G\rangle^0_{\beta',m}\right)\right]}$$

$$= \exp(-\beta S^N_{G\,min}) \frac{\int d\nu_1 d\nu_2 \det(D^N_{DW})^2/\det(D^N_{AP})^2}{\int d\nu_1 d\nu_2 \det(D^0_{DW})^2/\det(D^0_{AP})^2}$$
$$\times \exp\left[\int_\beta^\infty d\beta' \left(\langle S_G - S^N_{G\,min}\rangle^N_{\beta',m} - \langle S_G\rangle^0_{\beta',m}\right)\right] \tag{22}$$

$S^N_{G\,min}$: minimum of the gauge action in sector with N given by constant background fields.

D^N_{DW}, D^N_{AP} : Dirac operators with constant background.

Note that the integrand vanishes rapidly as $\beta' \to \infty$, so the integral over β' converges.

We can evaluate the full path integrals on θ-vacuum by obtaining $\langle O\rangle^N_{\beta,m}$ and $R^N(\beta, m)$ in each sector.

- **Calculation of $S^N_{G\,min}$**

  The classical minima of the gauge action $S^N_{G\,min}$ are evaluated easily.

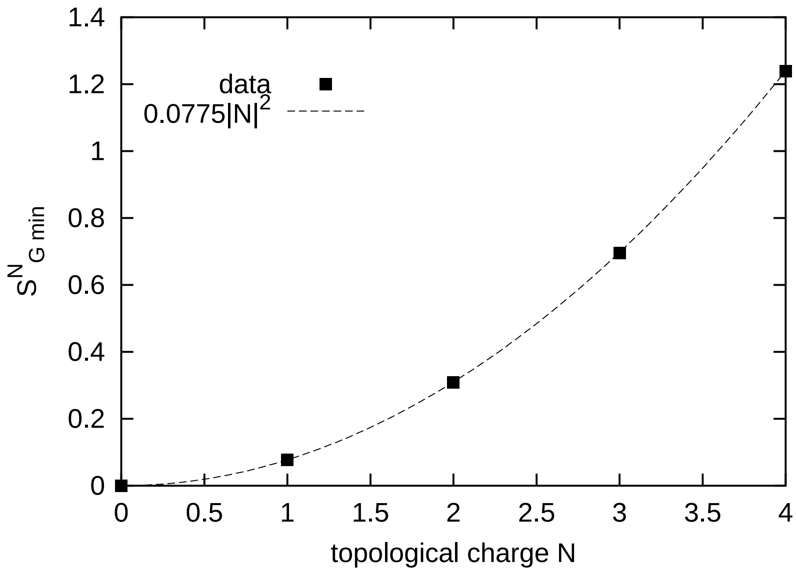

Figure 1: Minimum action in each topological sector : The behavior is consistent with continuum theory.

- **Calculation of fermion determinants**

  The fermion determinant $\det D^2$ on classical background in the sector with N is numerically calculated using the Householder method and the QL method. The value of

$$Det^N \equiv \frac{\int d\nu_1 d\nu_2 \det(D^N_{DW})^2 / \det(D^N_{AP})^2}{\int d\nu_1 d\nu_2 \det(D^0_{DW})^2 / \det(D^0_{AP})^2} \qquad (23)$$

  is approximated by the weighted sum over the discrete set of points uniformly distributed.

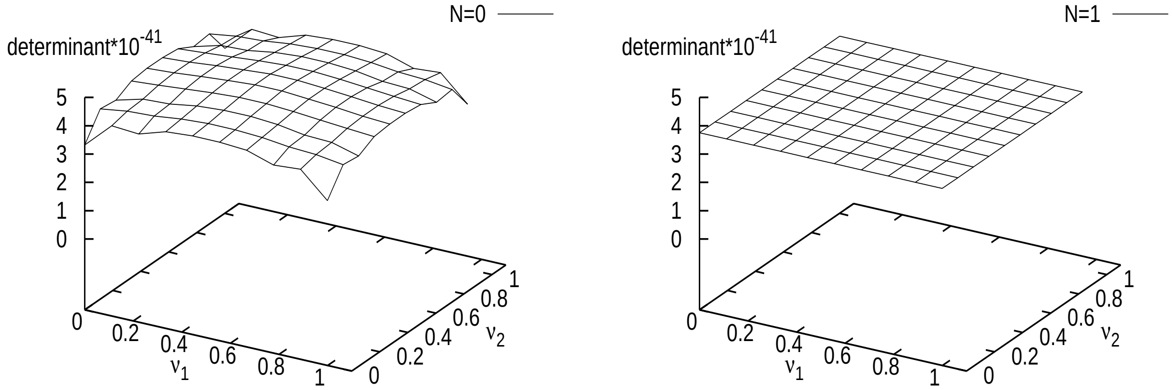

Figure 2: ν dependence of $\det(D_{DW}^{N})^2/\det(D_{AP}^{N})^2$.

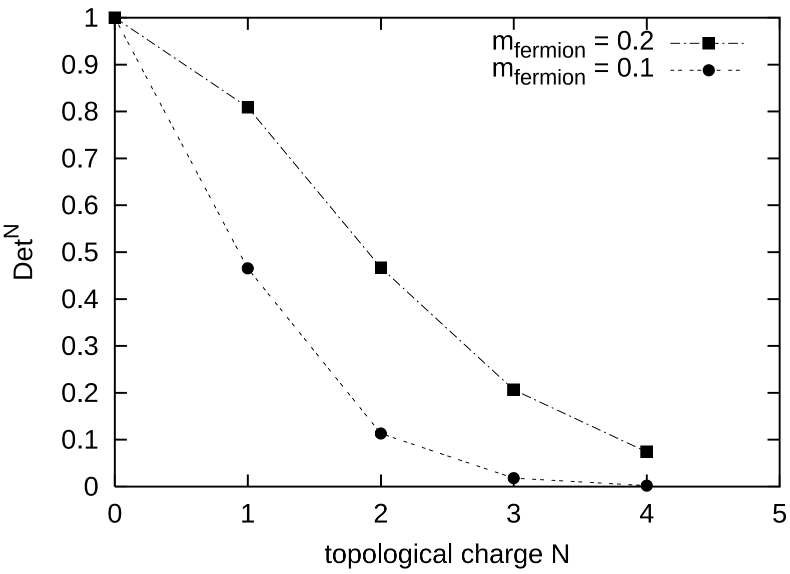

Figure 3: $|N|$ dependence of Det^N

- The integral of

$$S^N_{sub}(\beta', m) \equiv \langle S_G - S^N_{G\ min} \rangle^N_{\beta',m} - \langle S_G \rangle^0_{\beta',m}. \quad (24)$$

is approximated by a weighted sum over the discrete set of points at $\beta' = 0.5, 1.0, 1.5, 2.0$. For each β', we evaluate $S^N_{subtr}(\beta', m)$ by sampling more than 5000 configurations.

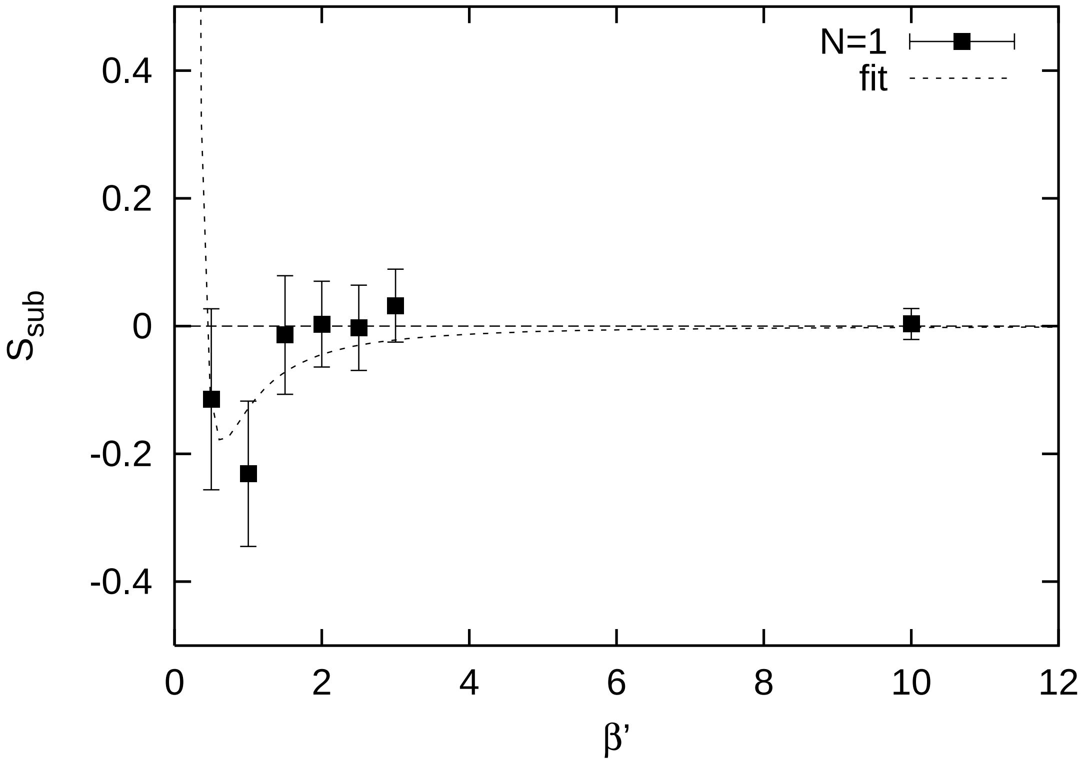

Figure 4: S^1_{sub}

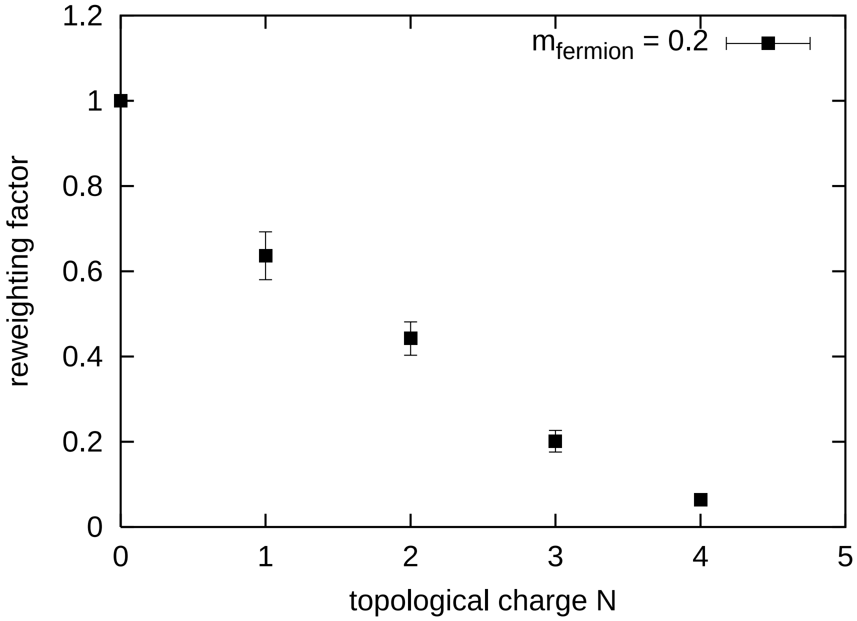

Figure 5: The total reweighting factor $R^N(0.5, 0.2)$

- **Advantages of our method**
  - Unphysical configurations are suppressed.
  - We can get higher topological sectors even in weak coupling.
  - Once each topological sector can be computed separately, we can obtain a θ dependence at once.

3. Meson Masses

- **π meson (isotriplet meson)**
  We evaluate the total expectation value on θ-vacuum as

$$\sum_y \langle \pi(x,y)\pi(0,0)\rangle_{full} = \sum_{N=-4}^{4} e^{iN\theta} \sum_y \langle \pi(x,y)\pi(0,0)\rangle^N_{\beta,m} R^N(\beta,m). \quad (25)$$

Here we have ignored $|N| > 4$ sectors since they only give contributions less than 1.2 % of zero sector.

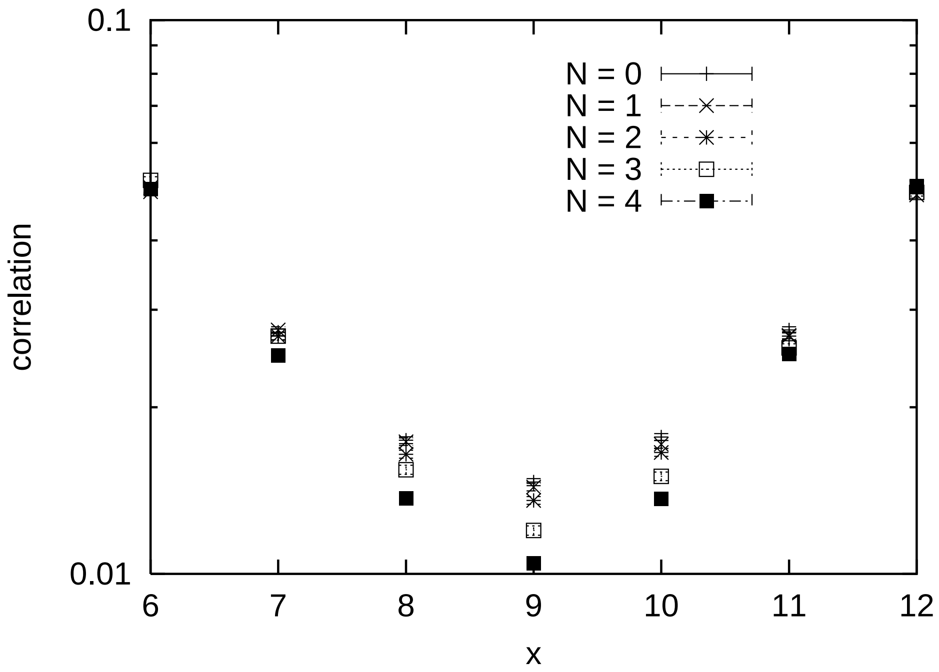

Figure 6: The pion propagator in each sector for $m = 0.2$.

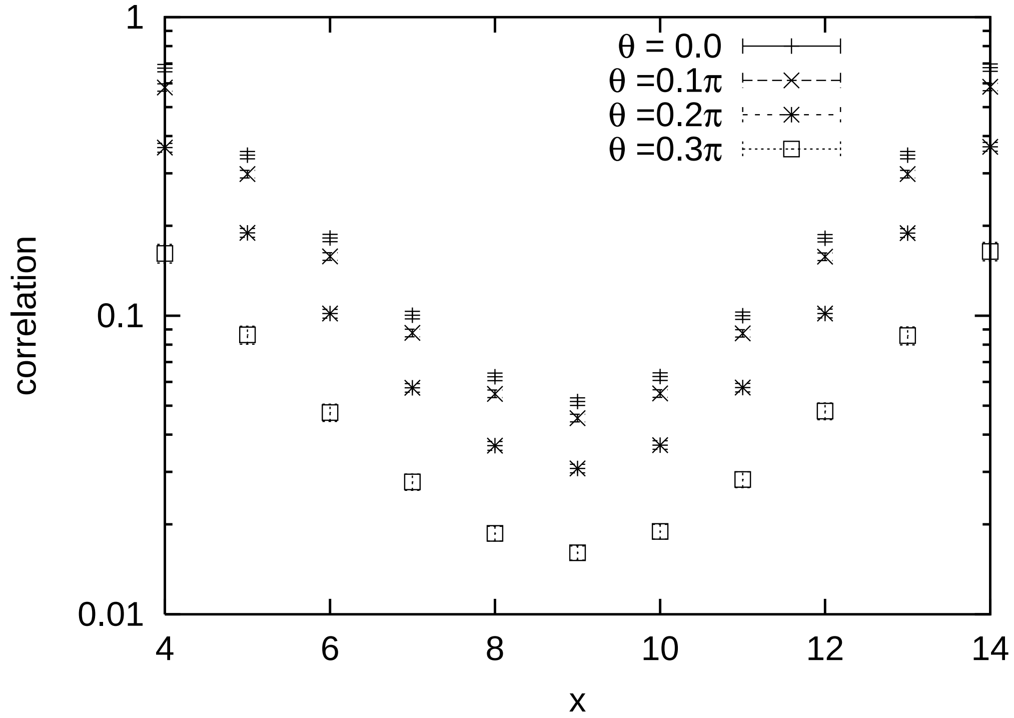

Figure 7: full propagetors of pion at $m = 0.2$.

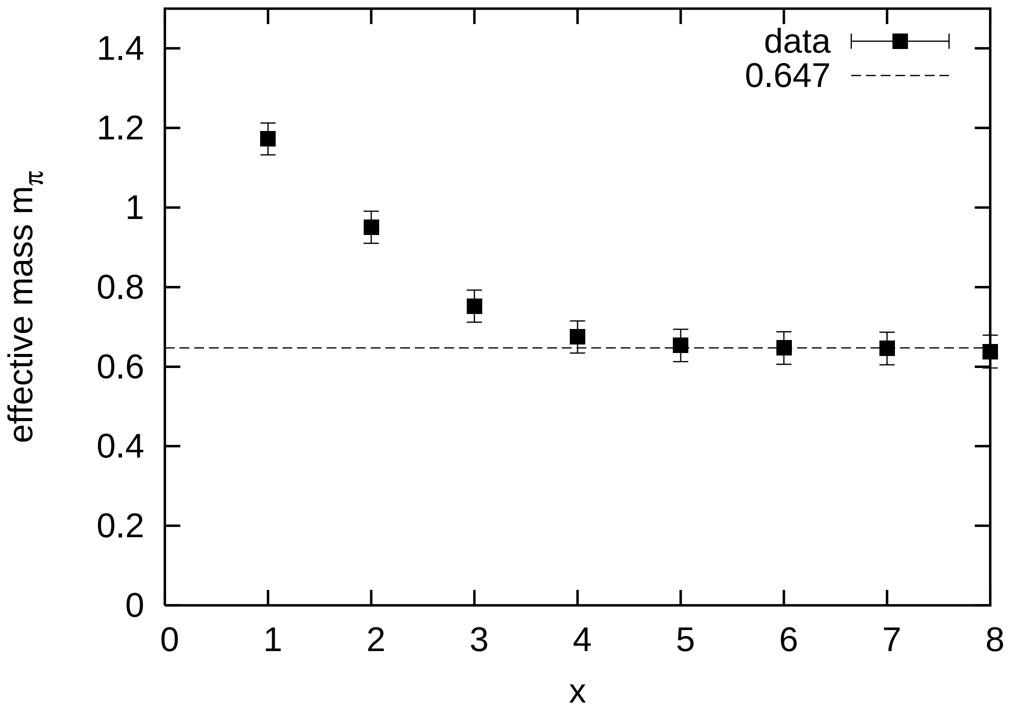

Figure 8: effective mass of pion at $m = 0.2, \theta = 0$.

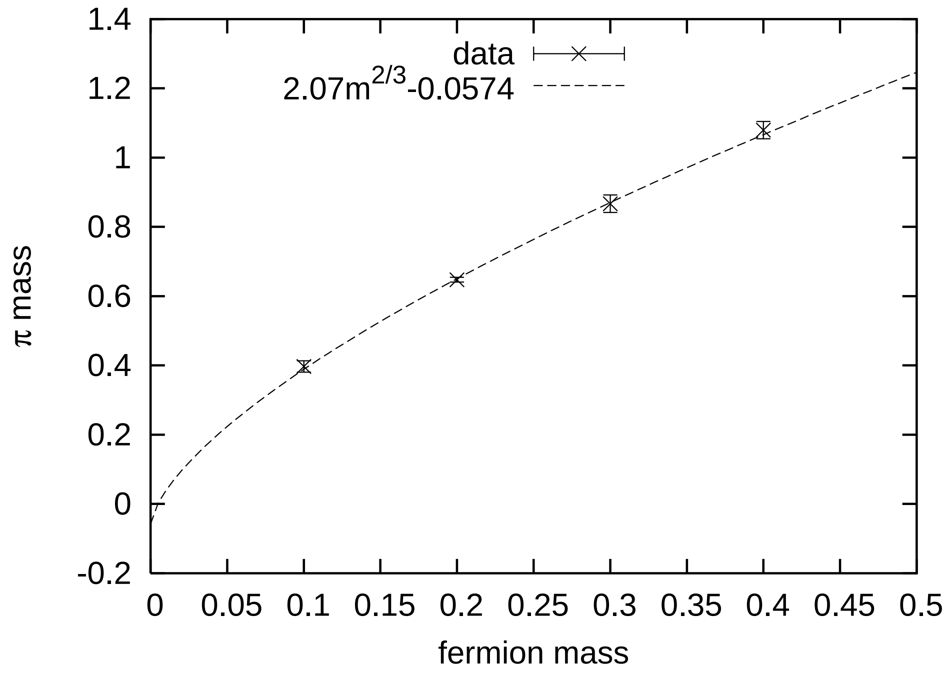

Figure 9: The fermion mass dependence of the pion mass for $\theta = 0$. The chiral behavior is consistent with that of continuum theory. ($m_\pi = -0.057 \pm 0.060$)

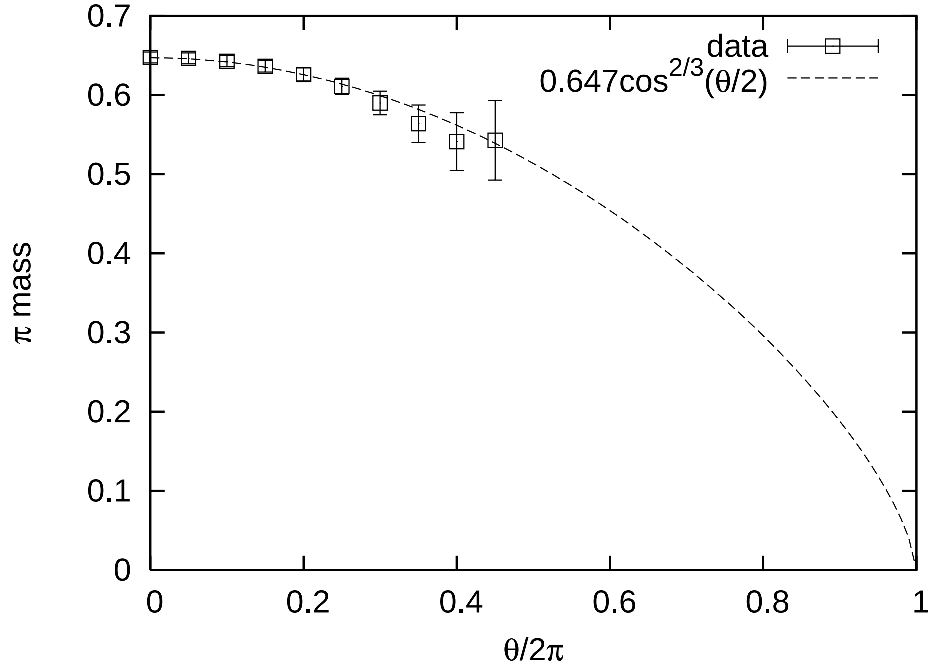

Figure 10: θ dependence of the pion mass at $m = 0.2$. the pion mass is proportional to $\cos(\theta/2)^{2/3}$, which is in complete agreement with the continuum results.

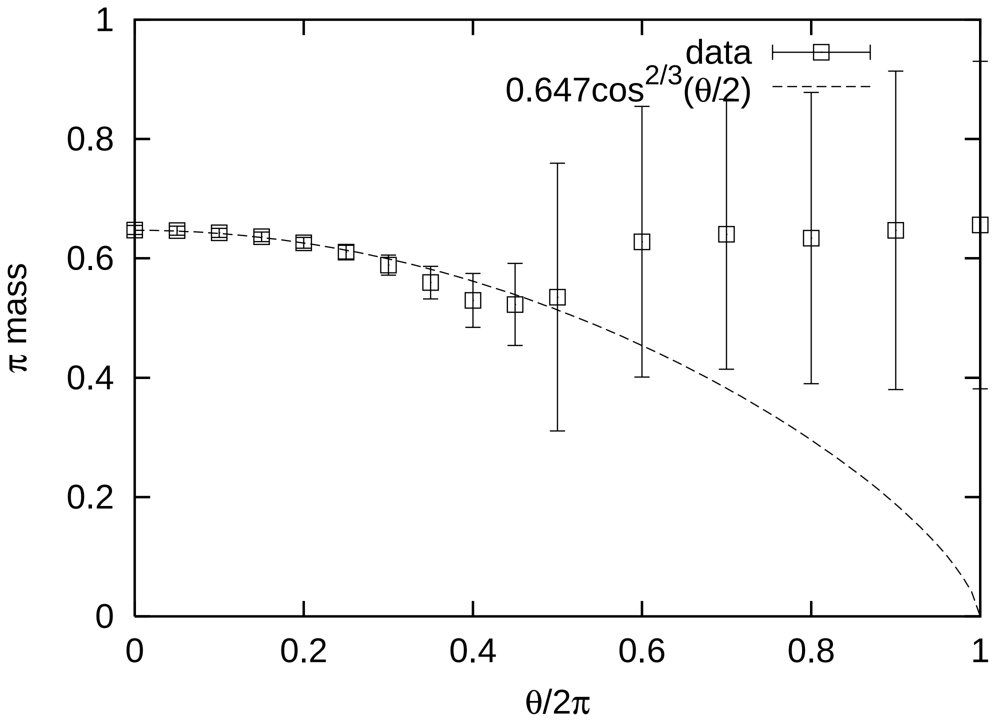

Figure 11: θ dependence of the pion mass at $m = 0.2$.

- η **meson (isosinglet meson)** The η propagator consists of two parts;

$$\langle \eta\eta \rangle = -2\langle tr\left(\gamma_3 \frac{1}{D}\gamma_3 \frac{1}{D}\right)\rangle + 4\langle tr\left(\gamma_3 \frac{1}{D}\right) tr\left(\gamma_3 \frac{1}{D}\right)\rangle \tag{26}$$

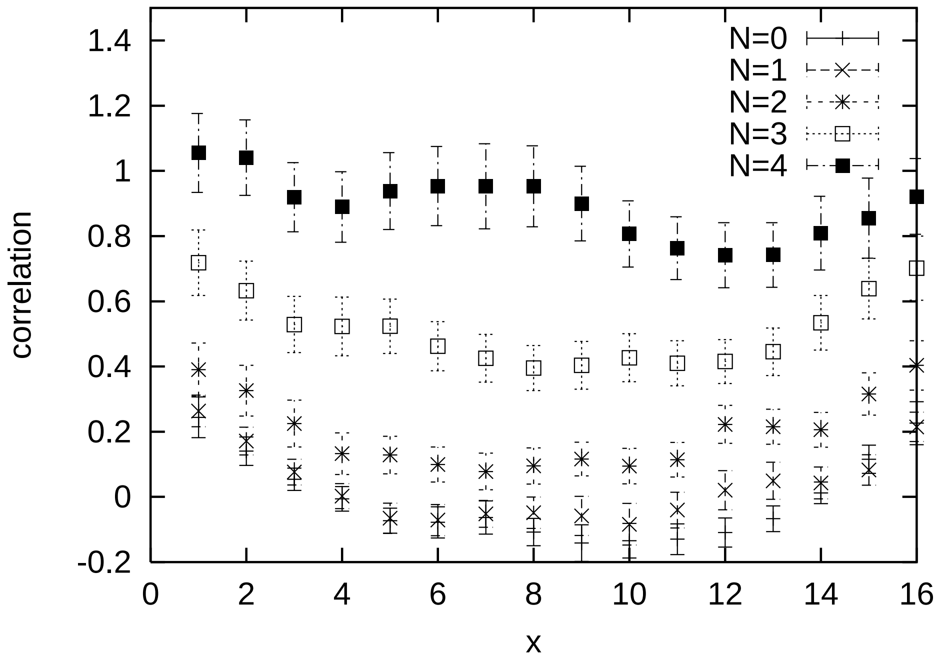

Figure 12: Disconnected propagators of η in each sector at $m = 0.2$.

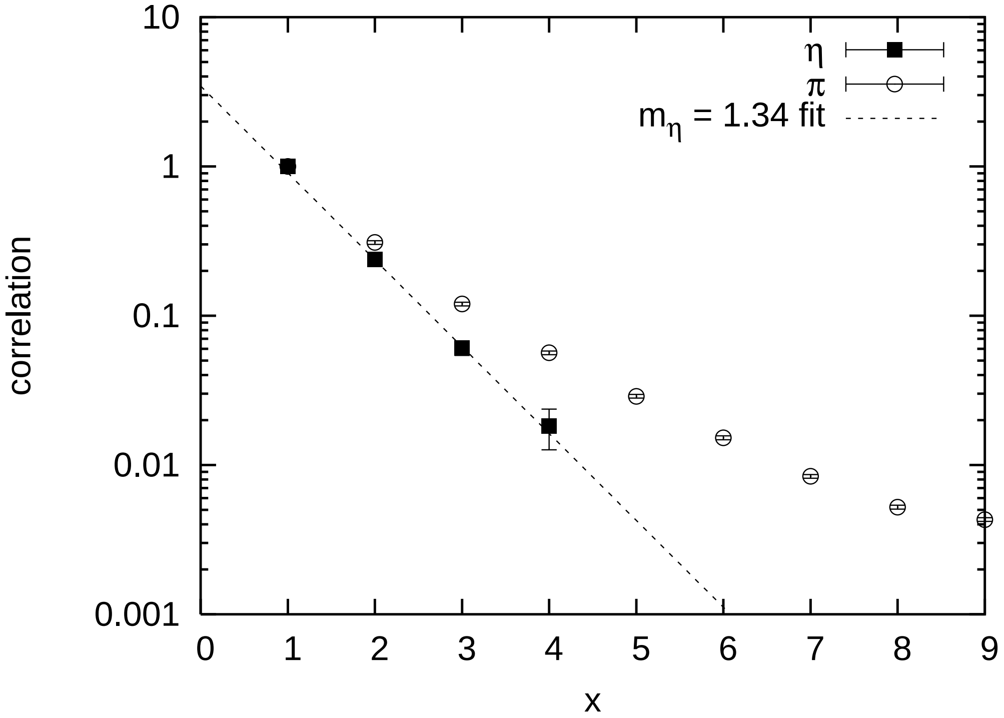

Figure 13: The full propagator of η meson at $m = 0.2$ and $\theta = 0$. The propagators are normalized by the value at $x = 1$.

We may say that η meson is heavier than π meson but studies are needed for quantitative analysis.

4. Summary and discussion

4.1 Summary

We performed a numerical simulation of 2-flavor QED on T^2 with Lüscher's action combined with domain-wall fermion action.

We also proposed and examined a new method of summing over different topological sectors.

- "Admissibility condition" is applicable to numerical simulation. We can obtain configurations in each topological sector.
- Our new method of summing over topological sectors is feasible and we can evaluate θ dependence of meson masses.
- All the results are qualitatively consistent with continuum theory and chiral limit is also good.

4.2 Application to 4dim QCD

Theoretically, SU(2) lattice gauge fields under the admissibility condition on T^4 have also topological charge;

$$\begin{aligned} Q &= \tfrac{1}{24\pi} \textstyle\sum_{n,\mu,\nu,\rho,\sigma} \epsilon_{\mu\nu\rho\sigma} \\ &\Big\{3 \textstyle\int_{p(n+\hat{\mu}+\hat{\nu},\mu,\nu)} d^2x Tr[P^n_{n+\hat{\mu}+\hat{\nu},\mu\nu}\partial_\rho(P^n_{n+\hat{\mu}+\hat{\nu},\mu\nu})^{-1}(R^n_{n+\hat{\mu},\mu;\nu})^{-1}\partial_\sigma R^n_{n+\hat{\mu},\mu;\nu}] \\ &\quad -3 \textstyle\int_{p(n+\hat{\nu},\mu,\nu)} d^2x Tr[P^n_{n+\hat{\nu},\mu\nu}\partial_\rho(P^n_{n+\hat{\nu},\mu\nu})^{-1}(R^n_{n,\mu;\nu})^{-1}\partial_\sigma R^n_{n,\mu;\nu}] \\ &\quad - \textstyle\int_{f(n+\hat{\mu},\mu)} d^3x Tr[S^n_{n+\hat{\mu},\mu}\partial_\nu(S^n_{n+\hat{\mu},\mu})^{-1}S^n_{n+\hat{\mu},\mu}\partial_\rho(S^n_{n+\hat{\mu},\mu})^{-1}S^n_{n+\hat{\mu},\mu}\partial_\sigma(S^n_{n+\hat{\mu},\mu})^{-1}] \\ &\quad + \textstyle\int_{f(n,\mu)} d^3x Tr[S^n_{n,\mu}\partial_\nu(S^n_{n,\mu})^{-1}S^n_{n,\mu}\partial_\rho(S^n_{n,\mu})^{-1}S^n_{n,\mu}\partial_\sigma(S^n_{n,\mu})^{-1}]\Big\} \qquad (27) \end{aligned}$$

$$\Rightarrow_{a\to 0} \ -\frac{1}{16\pi^2}\int_{T^4} d^4x Tr[F_{\mu\nu} * F_{\mu\nu}] \quad \text{(integer)} \ (28)$$

Reweighting factors will also be more complicated. But it's worth to try application to QCD !!!

Action with admissibility and Ginsparg-Wilson relation may be standard in the future...

4.3 Outlook

- More precise and quantitative study of 2dim QED
- Application to SU(2) pure gauge theory on T^4
- Application to QCD

Appendix : Quenched results

In quenched approximation with Wilson action, fermion zero modes are all neglected so much unphysical configurations are generated. On the other hand, Lüscher's action never allow topology changes.

As figures show, Lüscher's action gives consistent pion mass and good chiral limit even in quenched approximation at very strong coupling.

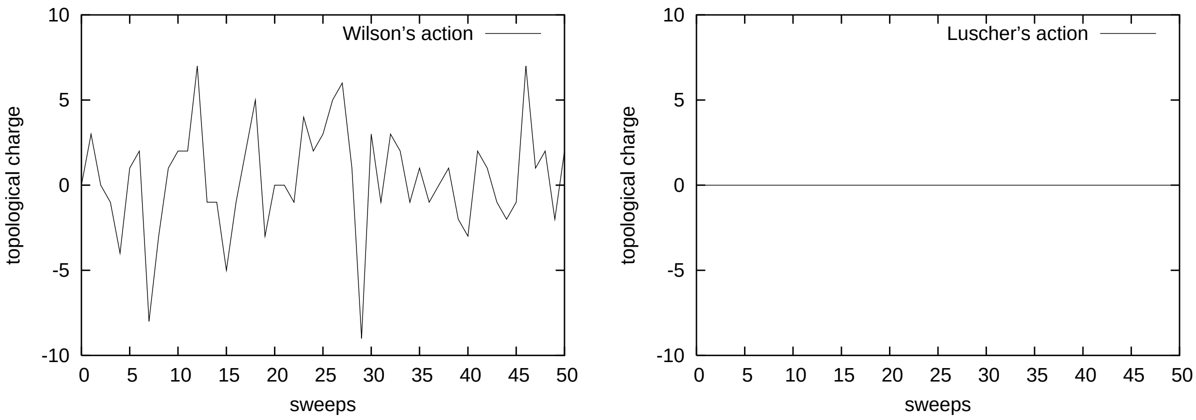

Figure 14: topological charge: Lüscher's gauge action shows no topology change.

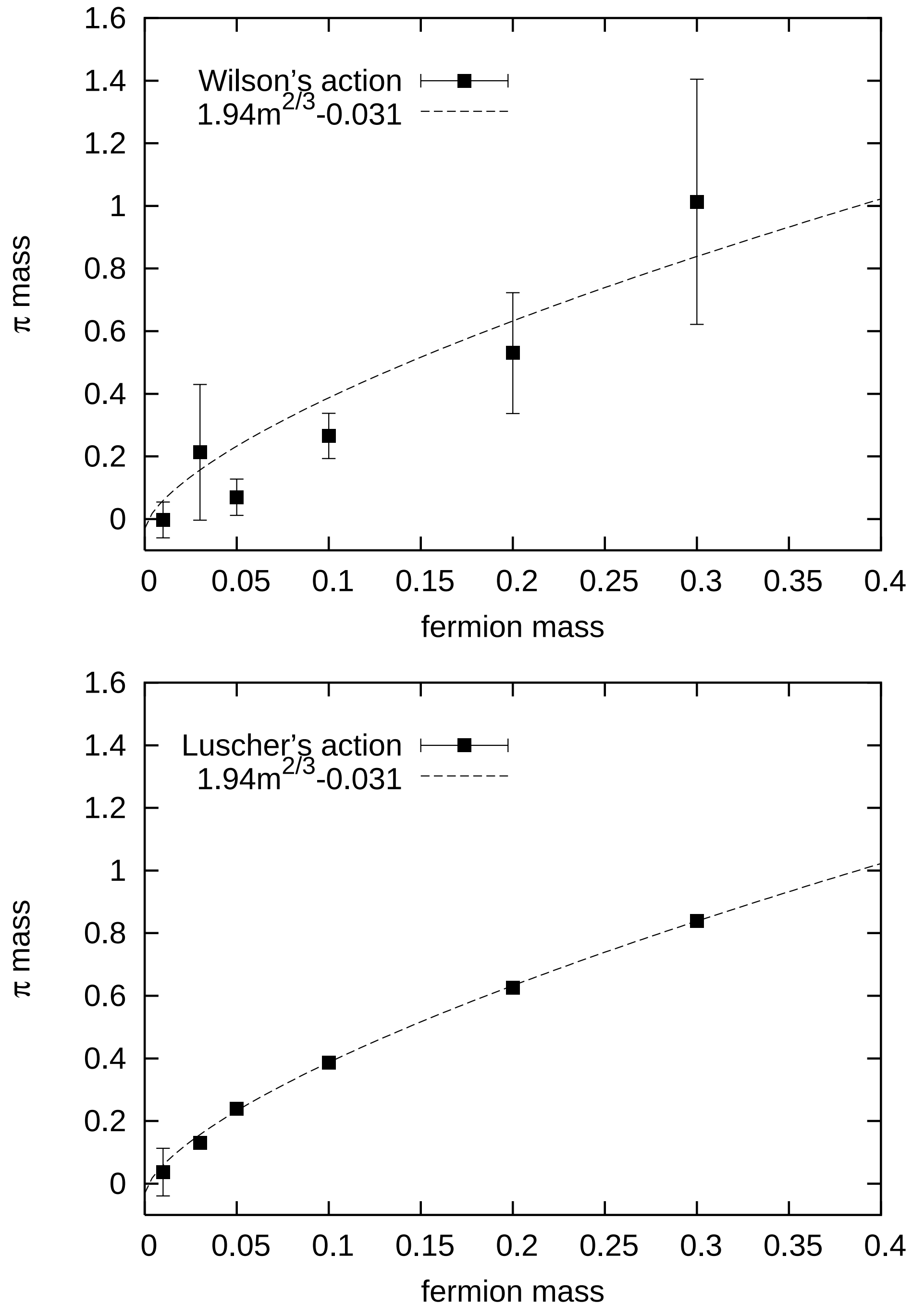

Figure 15: pion mass in quenched calculation