# Strebel differentials and string field theory

International Workshop on String Field Theory and Related Aspects

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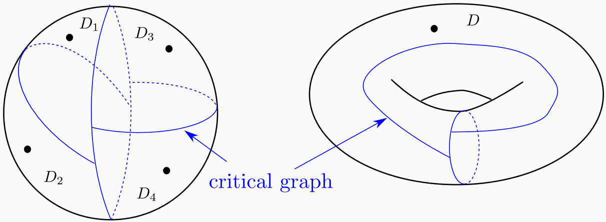
# Summary

- On punctured Riemann surfaces (~worldsheets for string amplitudes), one can define quadratic differentials called Strebel differentials.
- Via Strebel differentials, to any punctured Riemann surface one can associate a critical graph (~local interaction vertex of strings).




- We propose an SFT (for closed bosonic strings) based on such descriptions of Riemann surfaces. (PTEP 2024 (2024) 7, 073B02)

## Summary

- We need to employ the Fokker-Planck formalism to construct such a theory.
- The Fokker-Planck Hamiltonian is given by

$$\begin{aligned}\hat{H}_{\mathrm{FP}} &= -L\hat{\pi}_I\hat{\pi}_{I'}G^{I'I} + L\hat{\phi}^I\hat{\pi}_I \\ &\quad -\frac{1}{2}g_{\mathrm{s}}V^{II'I''}G_{I''K''}G_{I'K'}\hat{\phi}^{K''}\hat{\phi}^{K'}\hat{\pi}_I \\ &\quad -g_{\mathrm{s}}W^{II'I''}G_{I''K''}\hat{\phi}^{K''}\hat{\pi}_{I'}\hat{\pi}_I\end{aligned}$$

$\hat{\phi}^I, \hat{\pi}_I$ : string fields

# Remarks

- The theory given in this talk looks very weird from the physical point of view.
- Unlike the conventional SFT, our theory may not give a formulation from which important physical properties of string theory (unitarity, UV finiteness, background independence etc.) can easily be derived.

- The theory should be considered as a machinery for computing correlation functions of string theory.
- The method used here may be useful in studying the vertices of conventional SFTs.

# Outline

1. Strebel differentials
2. Pants decomposition
3. Schwinger-Dyson equation for strings
4. The Fokker-Planck formalism
5. Conclusions and outlook

# 1. Strebel differentials

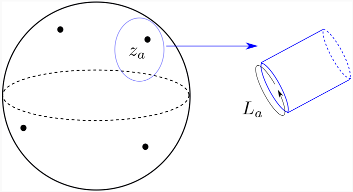
- On a punctured Riemann surface let us consider a quadratic differential $\phi(z)dz^2$ such that
  - near punctures $(z \sim z_a \ (a = 1, \cdots n))$

$$\phi(z)dz^2 \sim -\left(\frac{L_a}{2\pi}\right)^2 \frac{dz^2}{(z-z_a)^2}$$

with $L_a > 0$ and holomorphic for $z \neq z_a$

- A locally flat metric

$$\begin{aligned} ds^2 &= |\phi(z)|\, dz d\bar{z} = dw d\bar{w} \\ w &= \int^z dz' \sqrt{\phi(z')} \end{aligned}$$

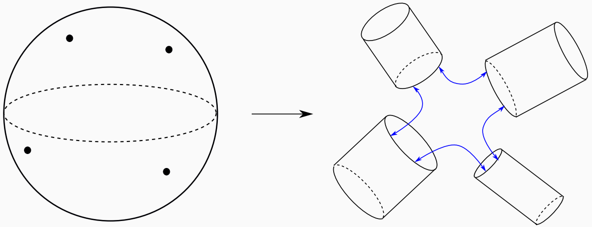


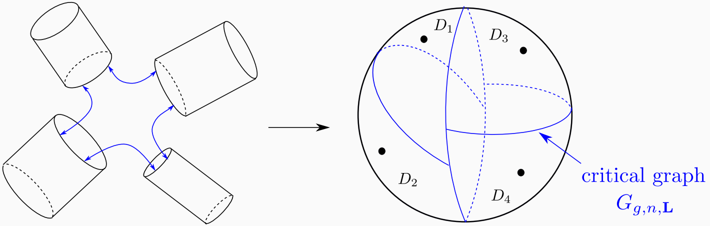
# Strebel's theorem

- Given a punctured Riemann surface $(2g-2+n>0)$ and positive numbers $L_1, \cdots, L_n$, there exists the unique quadratic differential $\phi(z)dz^2$ (Strebel differential) such that
  - for $z \sim z_a$, $\phi(z)dz^2 \sim -\left(\frac{L_a}{2\pi}\right)^2 \frac{dz^2}{(z-z_a)^2}$
  - holomorphic for $z \neq z_a$
  - with the metric

$$\begin{aligned} ds^2 &= |\phi(z)|\, dz d\bar{z} = dw d\bar{w} \\ & w = \int^z dz' \sqrt{\phi(z')} \end{aligned}$$

  the surface looks like

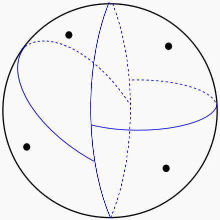


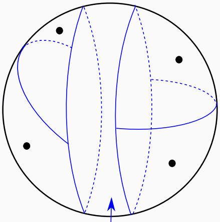



- To any punctured Riemann surface, one can associate a graph called critical graph.
- Moduli spaces of punctured Riemann surfaces can be parametrized by the lengths of the edges of the critical graphs (combinatorial moduli space $\mathcal{M}_{g,n}(\mathbf{L})$).
- Such a description plays important roles in
  - Kontsevich's proof of Witten conjecture
  - studying gauge/string duality (Gopakumar, Razamat, ..., Gopakumar-Mazenc, Gopakumar-Koushik-Komatsu-Mazenc-Sarkar,...)

# Strebel differentials and string field theory

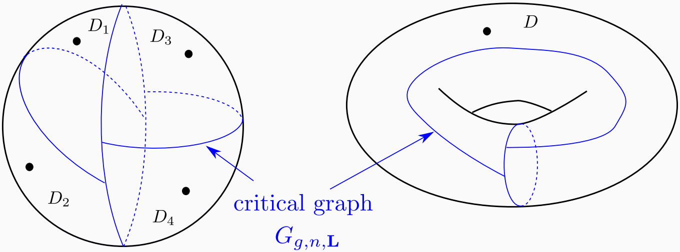
- The critical graphs look like local interaction vertices of closed strings.
- Strebel's theorem implies that any punctured Riemann surface can be described by such an interaction vertex.
- Such a description is not compatible with conventional SFT.






- Strebel differentials were used to construct the interaction vertices of a closed bosonic string field theory in Saadi-Zwiebach, Kugo-Kunitomo-Suehiro.

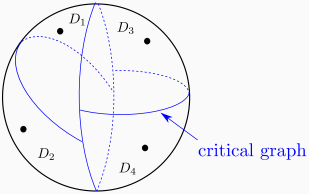
# Strebel differentials and string field theory

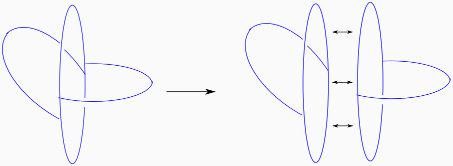



$$A^{i_1 \cdots i_n}_{g,n}(\mathbf{L}) \sim \int_{\mathcal{M}_{g,n}(\mathbf{L})} \langle G_{g,n,\mathbf{L}} | B_{6g-6+2n} | i_1 \rangle \cdots | i_n \rangle$$

- If Strebel differentials are really important in understanding gauge/string duality, it may be worthwhile to construct an SFT from which the amplitudes in this form are reproduced*.
  - string fields will be labeled by $(i, L)$ $(0 < L < \infty)$
- How can one construct such an SFT?

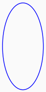
# 2. Pants decomposition



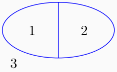



- The critical graphs can be decomposed into three string vertices.
- This coincides with the "pants decomposition" defined by Andersen et al..
- We may be able to construct a theory with

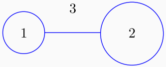
propagator



vertices

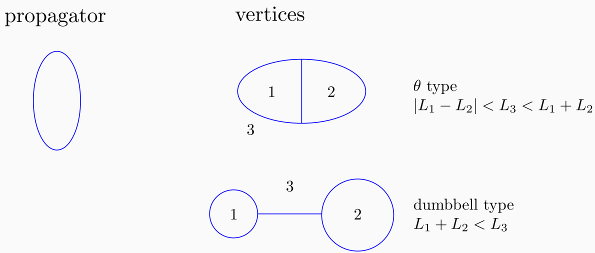



$\theta$ type
$|L_1 - L_2| < L_3 < L_1 + L_2$




dumbbell type
$L_1 + L_2 < L_3$

# SFT action

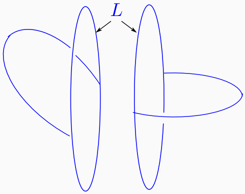



- We may be able to construct an SFT action starting from these:

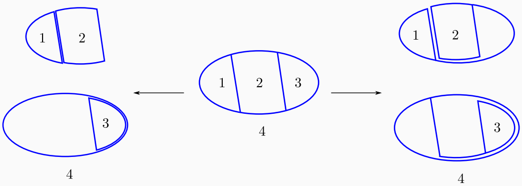
$$\begin{aligned} S[\phi] &= \frac{1}{2}\sum_i \int_0^\infty dL \phi^i(L)\phi^i(L) \\ &\quad + \frac{g_s}{6}\sum_{i_1,i_2,i_3}\int d^3L V_{i_1 i_2 i_3}(L_1,L_2,L_3)\phi^{i_1}(L_1)\phi^{i_2}(L_2)\phi^{i_3}(L_3) \\ &\quad + \cdots \end{aligned}$$

$\cdots$ are fixed so that the amplitudes are reproduced correctly.

# Tree level four point amplitude

$\int_0^\infty dL$ 


- This integral diverges because the moduli space is covered infinitely many times.
  - The pants decomposition of a critical graph is not unique.
  - Different decompositions are transformed to each other by action of the mapping class group.



## The action is not well-defined

$$
\begin{aligned}
S[\phi] \quad = \quad & \frac{1}{2}\sum_i \int_0^\infty dL \phi^i(L)\phi^i(L) \\
& + \frac{g_s}{6} \sum_{i_1, i_2, i_3} \int d^3 L V_{i_1 i_2 i_3}(L_1, L_2, L_3)\phi^{i_1}(L_1)\phi^{i_2}(L_2)\phi^{i_3}(L_3) \\
& + \cdots
\end{aligned}
$$

- $\cdots$ should include divergent counter terms to make the amplitude finite.
- This happens for almost all the amplitudes and $S[\phi]$ is not well-defined.

# 3. Schwinger-Dyson equation for strings

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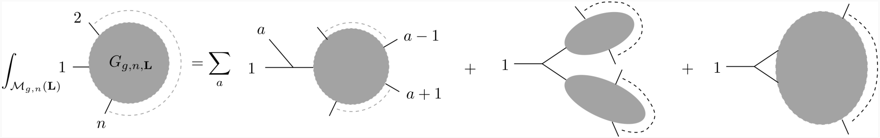
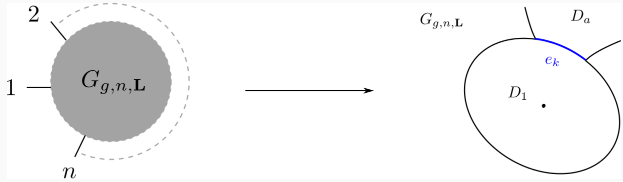
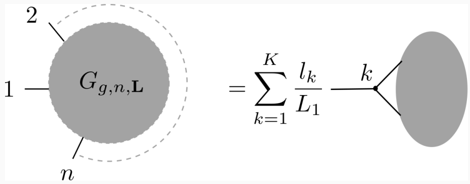



- Although action is ill-defined, one can derive an SD equation. $(3g-3+n>0,\ (g,n)\neq(1,1))$
  - Given a critical graph $G_{g,n,\mathbf{L}}$, we decompose it into a three string vertex one of whose legs is the first external line, and the rest.
  - In our case, it is impossible to uniquely pin down such a vertex, but it is possible to define a finite set of such vertices canonically.
  - Making a weighted sum of the decompositions corresponding to these vertices, we get the right hand side of the above.

# Three string vertices

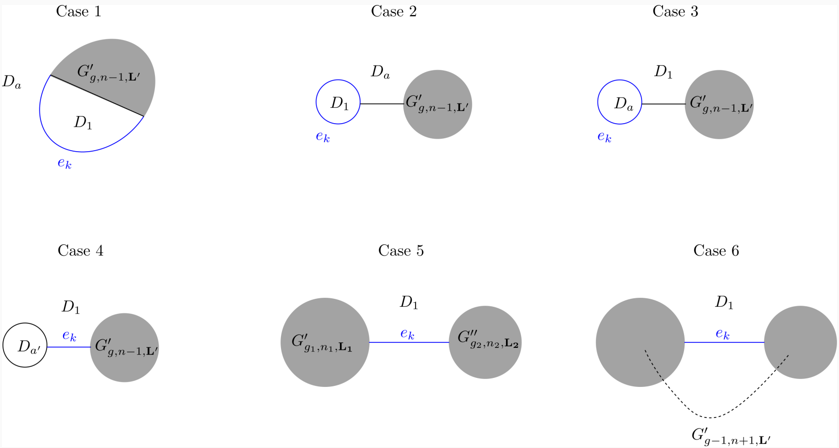



- $\partial D_1$ consists of edges $e_k$ $(k = 1, \cdots, K)$, whose lengths are denoted by $l_k$. They satisfy $\sum_{k=1}^{K} l_k = L_1$.
- $\partial D_1, \partial D_a$ and $e_k$ specify a three string vertex uniquely.



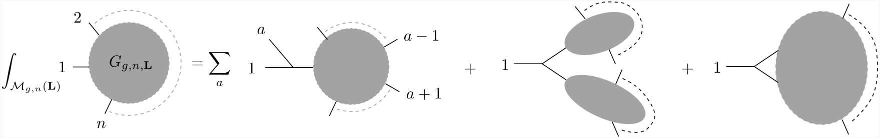
# SD equation

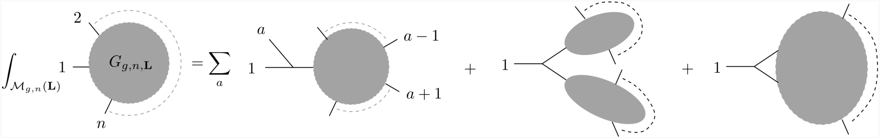
- The configuration falls into one of the following six cases (Bennett et al.)



# SD equation

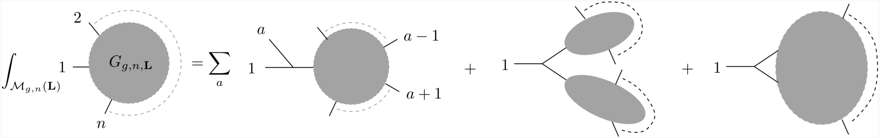
$$
\begin{aligned}
A^{i_1\cdots i_n}_{g,n}(\mathbf{L}) \;=\; & \sum_{a=1}^{n} \varepsilon_a \Bigg[ \int_{|L_1-L_a|}^{L_1+L_a} dx\, \frac{L_1+L_a-x}{2L_1} V^{i_1 i_a}{}_{j}(L_1,L_a,x) A^{j i_2\cdots \hat{i_a}\cdots j_n}_{g,n-1}(x,L_2,\cdots,\hat{L_a},\cdots L_n) && \text{Case 1} \\
& \quad + \theta(L_a-L_1)\int_0^{L_a-L_1} dx\, \frac{L_1}{L_1} V^{i_1 i_a}{}_{j}(L_1,L_a,x) A^{j i_2\cdots \hat{i_a}\cdots j_n}_{g,n-1}(x,L_2,\cdots,\hat{L_a},\cdots L_n) && \text{Case 2} \\
& \quad + \theta(L_1-L_a)\int_0^{L_1-L_a} dx\, \frac{L_1-x}{L_1} V^{i_1 i_a}{}_{j}(L_1,L_a,x) A^{j i_2\cdots \hat{i_a}\cdots j_n}_{g,n-1}(x,L_2,\cdots,\hat{L_a},\cdots L_n) \Bigg] && \text{Case 3 + Case 4} \\
& + \frac{1}{2}\sum_{\text{stable}} \frac{\varepsilon_{I_1 I_2}}{(n_1-1)!(n_2-1)!} \int_0^{L_1} dx \int_0^{L_1-x} dy\, \frac{L_1-x-y}{L_1} V^{i_1}{}_{jj'}(L_1,x,y) A^{j' i_{I_1}}_{g_1,n_1}(y,L_{I_1}) A^{j i_{I_2}}_{g_2,n_2}(x,L_{I_2}) && \text{Case 5} \\
& + \frac{1}{2}\int_0^{L_1} dx \int_0^{L_1-x} dy\, \frac{L_1-x-y}{L_1} V^{i_1}{}_{jj'}(L_1,x,y) A^{j' j i_2\cdots j_n}_{g-1,n+1}(y,x,L_2,\cdots,L_n)\,, && \text{Case 6}
\end{aligned}
$$






- We have employed the combinatorial Fenchel-Nielsen coordinates $(l_s; \tau_s)$ $(s = 1, \cdots, 3g-3+n)$ (Andersen et al.) to describe $\mathcal{M}_{g,n}(\mathbf{L})$.
  - $l_s$: the lengths of the nonperipheral boundaries of the pairs of pants
  - $\tau_s$: twist parameters
- We should take care of the $b$-ghost insertions.
- Although any critical graph can be decomposed into pairs of pants, a graph made by gluing pairs of pants may not be a critical graph.
  - Some twist parameters do not correspond to critical graphs. (nonadmissible twists)
  - Fortunately, nonadmissible twists do not appear on the right hand side. (Andersen et al.).

# SD equation




Introducing a convenient notation, we get

$$A^{I_1\cdots I_n}_{g,n} = \sum_{a=2}^{n} \varepsilon_a B^{I_1 I_a J} G_{JI} A^{I I_2 \cdots \hat{I}_a \cdots I_n}_{g,n-1}$$

$$+ \frac{1}{2} C^{I_1 J' J} G_{JI} G_{J'I'} \left[ A^{I I' I_2 \cdots I_n}_{g-1,n+1} + \sum_{\text{stable}} \frac{\varepsilon_{\mathcal{I}_1 \mathcal{I}_2}}{(n_1-1)!(n_2-1)!} A^{I \mathcal{I}_1}_{g_1,n_1} A^{I' \mathcal{I}_2}_{g_2,n_2} \right]$$

- $I \leftrightarrow (i, \alpha, L)$
  - $i$: labels for states of worldsheet theory
  - $\alpha = \pm$: labels for $b$-ghost insertions*

$$X_I Y^I = X^I Y_I = \sum_i \sum_{\alpha=\pm} \int_0^\infty dL X(i,\alpha,L) Y(i,\alpha,L)$$

$$G_{I_1 I_2} \equiv \delta(L_1 - L_2) \delta_{i_1,i_2} \left[ \delta_{\alpha_1,+} \delta_{\alpha_2,-} + \delta_{\alpha_1,-} \delta_{\alpha_2,+} (-1)^{|\varphi_{i_1}|} \right]$$

# 4. The Fokker-Planck formalism

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- In our theory, the Schwinger-Dyson equation can be derived but the action will be ill-defined.
- We can use the Fokker-Planck formalism to describe the theory.

Euclidean field theory

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = \frac{\int [d\phi] e^{-S[\phi]}\phi(x_1)\cdots\phi(x_n)}{\int [d\phi] e^{-S[\phi]}}$$

- the FP formalism

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = \lim_{\tau\to\infty}\langle 0|e^{-\tau \hat{H}_{\mathrm{FP}}}\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)|0\rangle$$

$$[\hat{\pi}(x),\hat{\phi}(y)] = \delta(x-y)\,, [\hat{\pi},\hat{\pi}] = [\hat{\phi},\hat{\phi}] = 0$$
$$\langle 0|\hat{\phi}(x) = \hat{\pi}(x)|0\rangle = 0$$
$$\hat{H}_{\mathrm{FP}} = -\int dx \left(\hat{\pi}(x) - \frac{\delta S}{\delta\phi(x)}[\hat{\phi}]\right)\hat{\pi}(x)$$

# The FP Hamiltonian and SD equation

- The FP Hamiltonian

$$\hat{H}_{\mathrm{FP}} = -\int dx \underbrace{\left(\hat{\pi}(x) - \frac{\delta S}{\delta\phi(x)}[\hat{\phi}]\right)}_{\equiv\ \hat{T}(x)} \hat{\pi}(x)$$

- SD equation for $e^{W[J]} \equiv \int [d\phi] e^{-S[\phi]+\int dx J(x)\phi(x)}$

$$\begin{aligned} 0 &= \int [d\phi] \frac{\delta}{\delta\phi(x)} \left(e^{-S[\phi]+\int dx J(x)\phi(x)}\right) \\ &= \underbrace{\left(J(x) - \frac{\delta S}{\delta\phi(x)}\left[\frac{\delta}{\delta J(x)}\right]\right)}_{\equiv\ T\left[J(x),\, \frac{\delta}{\delta J(x)}\right]} e^{W[J]} \end{aligned}$$

- $\hat{T}(x)$ satisfies

$$\hat{T}(x) e^{\int dx J(x)\hat{\phi}(x)}|0\rangle = T\left[J(x), \frac{\delta}{\delta J(x)}\right] e^{\int dx J(x)\hat{\phi}(x)}|0\rangle$$

This fact gives a quick way to derive FP Hamiltonian from SD equation.

- The generating functional

$$W[J] \equiv \sum_{g=0}^{\infty}\sum_{n=1}^{\infty} g_s^{2g-2+n}\frac{1}{n!}J_{I_n}\cdots J_{I_1}A_{g,n}^{I_1\cdots I_n}$$

$$A_{0,1}^{I} \equiv 0,\ \ A_{0,2}^{I_1 I_2} \equiv G^{I_1 I_2}$$

- The SD equation for $W[J]$

$$T^I\left[J_K, \frac{\delta}{\delta J_K}\right]e^{W[J]} = 0$$

$$\begin{aligned} T^I\left[J_K, \frac{\delta}{\delta J_K}\right] \equiv\ & L\frac{\delta}{\delta J_I} - LG^{II'}(-1)^{|I'|}J_{I'} \\ & -\frac{1}{2}g_s V^{II'I''}G_{I''K''}G_{I'K'}\frac{\delta^2}{\delta J_{K''}\delta J_{K'}} \\ & -g_s W^{II'I''}G_{I''K''}(-1)^{|I'|}J_{I'}\frac{\delta}{\delta J_{K''}}(-1)^{|I'||I''|} \end{aligned}$$

$$V^{I_1 I_2 I_3} = \begin{cases} (L_1-L_2-L_3)\langle G_{0,3,(L_1,L_2,L_3)}|B^1_{\alpha_1}B^2_{\alpha_2}B^3_{\alpha_3}|\varphi^{\alpha_1}_{i_1}\rangle_1|\varphi^{\alpha_2}_{i_2}\rangle_2|\varphi^{\alpha_3}_{i_3}\rangle_3 & L_2+L_3<L_1 \\ 0 & L_1<L_2+L_3 \end{cases},$$

$$W^{I_1 I_2 I_3} = \begin{cases} 0 & L_1+L_2<L_3 \\ (L_1+L_2-L_3)\langle G_{0,3,(L_1,L_2,L_3)}|B^1_{\alpha_1}B^2_{\alpha_2}B^3_{\alpha_3}|\varphi^{\alpha_1}_{i_1}\rangle_1|\varphi^{\alpha_2}_{i_2}\rangle_2|\varphi^{\alpha_3}_{i_3}\rangle_3 & |L_1-L_2|<L_3<L_1+L_2 \\ \min(L_1,L_2)\langle G_{0,3,(L_1,L_2,L_3)}|B^1_{\alpha_1}B^2_{\alpha_2}B^3_{\alpha_3}|\varphi^{\alpha_1}_{i_1}\rangle_1|\varphi^{\alpha_2}_{i_2}\rangle_2|\varphi^{\alpha_3}_{i_3}\rangle_3 & L_3<|L_1-L_2| \end{cases}.$$

# The FP formalism for strings

- Operators and states

$$
\begin{aligned}
[\hat{\pi}_I, \hat{\phi}^K] &= \delta_I{}^K\,, \\
[\hat{\pi}_I, \hat{\pi}_K] &= [\hat{\phi}^I, \hat{\phi}^K] = 0\,, \\
\langle 0|\hat{\phi}^I &= \hat{\pi}_I|0\rangle = 0\,,
\end{aligned}
$$

- The FP Hamiltonian

$$
\begin{aligned}
\hat{H}_{FP} &= \hat{T}^I \hat{\pi}_I \\
&= -L\hat{\pi}_I\hat{\pi}_{I'}G^{I'I} + L\hat{\phi}^I\hat{\pi}_I \\
&\quad -\frac{1}{2}g_{\rm s}V^{II'I''}G_{I''K''}G_{I'K'}\hat{\phi}^{K''}\hat{\phi}^{K'}\hat{\pi}_I \\
&\quad -g_{\rm s}W^{II'I''}G_{I''K''}\hat{\phi}^{K''}\hat{\pi}_{I'}\hat{\pi}_I\,,
\end{aligned}
$$

- One can prove

$$
e^{W[J]} = \lim_{\tau\to\infty}\langle 0|e^{-\tau\hat{H}_{\rm FP}}e^{J_I\hat{\phi}^I}|0\rangle
$$

perturbatively using

$$
\hat{T}^I e^{J_K\hat{\phi}^K}|0\rangle = T^I\left[J_K, \frac{\delta}{\delta J_K}\right]e^{J_K\hat{\phi}^K}|0\rangle
$$

# BRST symmetry

- The correlation functions are BRST invariant

$$e^{W[J]} = \lim_{\tau\to\infty} \langle 0| e^{-\tau \hat{H}_{\rm FP}} e^{J_I \hat{\phi}^I} |0\rangle$$

$$\lim_{\tau\to\infty} \langle 0| e^{-\tau \hat{H}_{\rm FP}} \hat{Q} = \hat{Q}|0\rangle = 0$$

- $\hat{H}_{\rm FP}$ itself is not BRST invariant

$$\begin{aligned} \left[\hat{Q}, \hat{H}_{\rm FP}\right] &= \left[\hat{Q}, \hat{T}^I \hat{\pi}_I\right] \\ &= \left[\hat{Q}, \hat{T}^I\right] \hat{\pi}_I + \hat{T}^I \left[\hat{Q}, \hat{\pi}_I\right] \end{aligned}$$

$$\lim_{\tau\to\infty} \langle 0| e^{-\tau \hat{H}_{\rm FP}} \left[\hat{Q}, \hat{T}^I\right] = \lim_{\tau\to\infty} \langle 0| e^{-\tau \hat{H}_{\rm FP}} \hat{T}^I = 0$$

- BRST invariant Hamiltonian can be obtained by introducing auxiliary fields

$$\hat{H}_{\rm FP} \to \hat{H}_{\rm FP} + \left[\hat{Q}, \hat{T}^I\right] \lambda_I^Q + \hat{T}^I \lambda_I^T$$

  - This modification does not change the correlation functions.

# 5. Conclusions and outlook

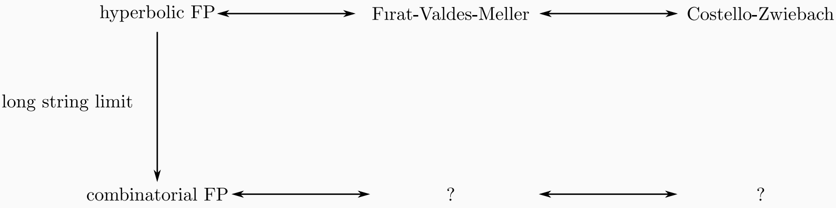
# 5. Conclusions and outlook

- We have constructed an SFT for closed bosonic strings based on the Strebel differentials via the Fokker-Planck formalism.

$$\begin{aligned}\hat{H}_{\rm FP} &= -L\hat{\pi}_I\hat{\pi}_{I'}G^{I'I} + L\hat{\phi}^I\hat{\pi}_I \\ &\quad -\frac{1}{2}g_{\rm s}V^{II'I''}G_{I''K''}G_{I'K'}\hat{\phi}^{K''}\hat{\phi}^{K'}\hat{\pi}_I \\ &\quad -g_{\rm s}W^{II'I''}G_{I''K''}\hat{\phi}^{K''}\hat{\pi}_{I'}\hat{\pi}_I\,,\end{aligned}$$

- gauge fixed version
- Superstrings?
- Implication for conventional SFT?

# Outlook

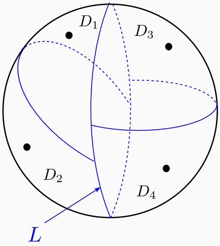

hyperbolic FP
Fırat-Valdes-Meller
Costello-Zwiebach
long string limit
combinatorial FP
?
?

# Backup

# Amplitudes*

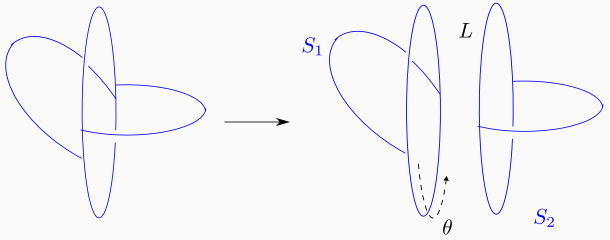
$$A^{i_1 \cdots i_n}_{g,n}(\mathbf{L}) \sim \int_{\mathcal{M}_{g,n}(\mathbf{L})} \langle G_{g,n,\mathbf{L}} | B_{6g-6+2n} | i_1 \rangle \cdots | i_n \rangle$$

- Since $\mathcal{M}_{g,n}(\mathbf{L})$ is homeomorphic to $\mathcal{M}_{g,n}$, $A^{i_1 \cdots i_n}_{g,n}(\mathbf{L})$ coincide with the on-shell amplitudes when $|i_a\rangle$ are on-shell.
  - The integration over $L \sim 0$ region yields the $s$-channel poles.




- There are observables involving integration over the length. They give the nonamputated correlation functions.

# $b$-ghost insertions*




$$b(\partial_L) = b_{S_1}(\partial_L) + b_{S_2}(\partial_L)$$
$$b(\partial_\theta) = -2\pi i b_0^-$$

- We put $b(\partial_\theta)b_{S_1}(\partial_L)$ on $S_1$ and $b(\partial_\theta)b_{S_2}(\partial_L)$ on $S_2$.
- String states with these insertions are labeled by $\alpha = -$, and those with no insertions are labeled by $\alpha = +$.