Comments on the Takahashi-Tanimoto tachyon vacuum solution

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Classical solutions of the cubic SFT

$$Q\Psi_{\rm cl} + \Psi_{\rm cl}^2 = 0$$

- Tachyon vacuum solution (Schnabl, Okawa, Erler, Erler-Schnabl, ...)
- Marginal deformation (Kiermaier-Okawa-Rastelli-Zwiebach, Schnabl, Fuchs-Kroyter-Potting, ...)
- Relevant deformation (Bonora-Maccaferri-Tolla, ...)
- $\cdots$
- "Any background" solution (Erler-Maccaferri)

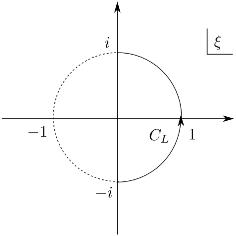
Takahashi-Tanimoto (TT) solutions

tachyon vacuum solution

$$\Psi_{\mathrm{TT}} = \int_{C_L} \frac{d\xi}{2\pi i} \left(\left(e^{h} - 1 \right) j_{\mathrm{B}}(\xi) - (\partial h)^2 e^{h} c(\xi) \right) I$$

$$j_{\mathrm{B}} = cT^{\mathrm{m}} + bc\partial c + \frac{3}{2}\partial^2 c$$

$$e^{h(\xi)} = -\frac{1}{4}\left(\xi - \frac{1}{\xi} \right)^2$$

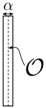


Identity-based solutions

$$\Psi_{\mathrm{cl}} = \mathcal{O} I$$

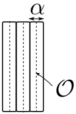
- I: identity string field

$$\Psi_{\mathrm{cl}} = \lim_{\alpha \to 0}$$



- Impossible to calculate observables

$$E \sim \int \Psi_{\mathrm{cl}}^3 = \lim_{\alpha \to 0}$$



SFT around the identity-based solutions

$$\Psi \to \Psi_{\mathrm{cl}} + \Psi$$

$$\begin{aligned} S' &= -\frac{1}{g^2}\int\left[\frac{1}{2}\Psi Q\Psi + \Psi\Psi_{\mathrm{cl}}\Psi + \frac{1}{3}\Psi\Psi\Psi\right] \\ &= -\frac{1}{g^2}\int\left[\frac{1}{2}\Psi Q'\Psi + \frac{1}{3}\Psi\Psi\Psi\right] \end{aligned}$$

$$Q'\Psi = Q\Psi + \{\Psi_{\mathrm{cl}}, \Psi\}_*$$

- In the case of identity-based solutions, Q' can be expressed by using local fields on the worldsheet. For TT solution

$$Q' = \oint \frac{d\xi}{2\pi i}\left(e^h j_{\mathrm{B}}(\xi) - (\partial h)^2 e^h c(\xi)\right)$$

Evidences

There are many evidences for the claim that Ψ_{TT} describes tachyon vacuum:

- No physical open string excitation around the background Ψ_{TT} (Kishimoto-Takahashi, Inatomi-Kishimoto-Takahashi)
- Open string amplitudes vanish (Takahashi-Zeze)
- Existence of an unstable solution around the background Ψ_{TT} (Takahashi, Kishimoto-Takahashi)

In this talk

- I would like to add one more to the list of these evidences.
  - I will consider the Erler-Schnabl solution in the SFT around the TT solution.
  - I will calculate the observables of the solution and the results indicate that the TT solution corresponds to the tachyon vacuum.
- I will study the SFT around the TT solution and discuss how we should calculate various quantities.

c.f. Takahashi's talk

Outline

Erler-Schnabl solution

Tachyon vacuum solution

$$\begin{aligned} \Psi_{\mathrm{ES}} &= \frac{1}{1+K}\left(c+Q\left(Bc\right)\right) \\ B &= \int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}b\left(z\right)I \\ c &= c\left(0\right)I \\ K &= QB \end{aligned}$$

All the conditions in Sen's conjectures are checked

- homotopy operator $A = B\frac{1}{1+K}$, s.t. $QA = 1$
- $E\left[\Psi_{\mathrm{ES}}\right] = -\frac{V}{2\pi^2}$

Erler-Schnabl solution around the TT solution

One can construct ES solution in the SFT around the TT solution

$$S' = -\frac{1}{g^2}\int\left[\frac{1}{2}\Psi Q'\Psi + \frac{1}{3}\Psi\Psi\Psi\right]$$

$$\begin{aligned}\Psi'_{\mathrm{ES}} &= \frac{1}{1+K'}\left(c + Q'\left(Bc\right)\right) \\ K' &= Q'B \\ &= K + \{\Psi_{\mathrm{TT}}, B\}\end{aligned}$$

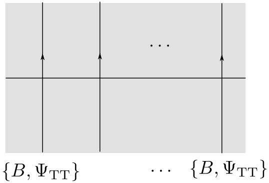
- With the homotopy operator $A' = B\frac{1}{1+K'}$, this solution will correspond to the tachyon vacuum.

Remark

Here we assume that $\frac{1}{1+K'}$ is well-defined with the definition

$$
\begin{aligned}
\frac{1}{1+K'} &= \frac{1}{1+K+\{B,\Psi_{\mathrm{TT}}\}} \\
&= \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{1+K}\{B,\Psi_{\mathrm{TT}}\}\right)^n\frac{1}{1+K}
\end{aligned}
$$

$\frac{1}{1+K'} = \int \cdots$



Observables

We will calculate the observables of the Ψ'_{ES}

$$E\left[\Psi'_{\mathrm{ES}}\right] = -S\left[\Psi'_{\mathrm{ES}}\right] = E_{\Psi'_{\mathrm{ES}}} - E_{\mathrm{TT}}$$

$$\mathrm{Tr}_V \Psi'_{\mathrm{ES}} = \langle Vc\rangle_{\Psi'_{\mathrm{ES}}} - \langle Vc\rangle_{\mathrm{TT}}$$

and show that they vanish.

- Assuming Ψ'_{ES} corrsponds to the tachyon vacuum, this implies that the TT solution also coresponds to the tachyon vacuum.

Remark

- Recently Maccaferri gives a way to construct a regular solution out of an identity-based solution by a gauge transformation

$$\Psi_{\mathrm{TT}} \quad \to \quad \Psi_{\mathrm{M}} = UQU^{-1} + U\Psi_{\mathrm{TT}}U^{-1}$$

$$U = 1 + B\frac{1}{1+K}\Psi_{\mathrm{TT}}$$

The observables become

$$\begin{aligned} E\left[\Psi_{\mathrm{M}}\right] &= E\left[\Psi_{\mathrm{ES}}\right] - E\left[\Psi'_{\mathrm{ES}}\right] \\ \mathrm{Tr}_V \Psi_{\mathrm{M}} &= \mathrm{Tr}_V \Psi_{\mathrm{ES}} - \mathrm{Tr}_V \Psi'_{\mathrm{ES}} \end{aligned}$$

What we will show $\left(E\left[\Psi'_{\mathrm{ES}}\right] = \mathrm{Tr}_V \Psi'_{\mathrm{ES}} = 0\right)$ implies that Ψ_{M} is a tachyon vacuum solution.

c.f. Takahashi's talk

§2 Observables

From

$$\Psi'_{\mathrm{ES}} = \frac{1}{1+K'}\left(c + Q'\left(Bc\right)\right)$$

one can derive

$$\begin{aligned} E\left[\Psi'_{\mathrm{ES}}\right] &= -\frac{1}{6}\mathrm{Tr}\left[\frac{1}{1+K'}c\frac{1}{1+K'}Q'c\right] \\ \mathrm{Tr}_V\Psi'_{\mathrm{ES}} &= \mathrm{Tr}_V\left[\frac{1}{1+K'}c\right] \end{aligned}$$

We would like to show that the RHS vanish.

Proof

One can show

$$\begin{aligned} \mathrm{Tr}_V\left[\frac{1}{1+K'}c\right] &= 0 \\ \mathrm{Tr}\left[\frac{1}{1+K'}c\frac{1}{1+K'}Q'c\right] &= 0 \end{aligned}$$

by using $Q'\left(\frac{1}{\pi^2}b\right) = 1$, $Q'c = 0$.

$$\begin{aligned} \mathrm{Tr}_V\left[\frac{1}{1+K'}c\right] &= \mathrm{Tr}_V\left[\frac{1}{\sqrt{1+K'}}Q'\left(\frac{1}{\pi^2}b\right)\frac{1}{\sqrt{1+K'}}c\right] \\ &= 0 \end{aligned}$$

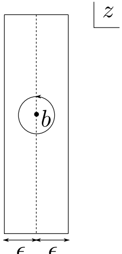
$Q'\left(\frac{1}{\pi^2}b\right)=1,\ Q'c=0$

Treating them more rigorously, these should be expressed as

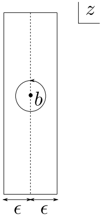
$$\begin{aligned} e^{-\epsilon K}Q'\left(\frac{1}{\pi^2}b\right)e^{-\epsilon K} &= e^{-2\epsilon K} \\ e^{-\epsilon K}Q'ce^{-\epsilon K} &= 0 \end{aligned}$$

With the $e^{-\epsilon K}$'s, we have worldsheet with no operator insertions

and



$e^{-\epsilon K} Q' \left(\frac{1}{\pi^2} b\right) e^{-\epsilon K} = e^{-2\epsilon K}$



$$
\begin{aligned}
Q'\left(\frac{1}{\pi^2} b\right) &= \oint_0 \frac{dz}{2\pi i}\left(-\frac{\sin^2 \pi z}{\cos^2 \pi z} j_{\mathrm{B}}(z) + \frac{4\pi^2}{\cos^4 \pi z} c(z)\right) \frac{1}{\pi^2} b(0) \\
&= 1
\end{aligned}
$$

- $e^{-\epsilon K} (Q' c) e^{-\epsilon K} = 0$ can be proven in the same way.

$\mathrm{Tr}_V\left[\frac{1}{1+K'}c\right] = 0$

$$\begin{aligned}\mathrm{Tr}_V\left[\frac{1}{1+K'}c\right] &= \mathrm{Tr}_V\left[\frac{1}{\sqrt{1+K'}}Q'\left(\frac{1}{\pi^2}b\right)\frac{1}{\sqrt{1+K'}}c\right] \\ &= 0\end{aligned}$$

$$\begin{aligned}\frac{1}{\sqrt{1+K'}} &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^\infty dt t^{-\frac{1}{2}}e^{-t}e^{-tK'} \\ e^{-tK'} &= \sum_{n=0}^{\infty}(-1)^n\int_0^\infty dt_1\cdots\int_0^\infty dt_{n+1} \\ &\quad\times\delta\left(\sum_{i=1}^{n+1}t_i - t\right)e^{-t_1K}\{B,\Psi_{\mathrm{TT}}\}e^{-t_2K}\cdots\{B,\Psi_{\mathrm{TT}}\}e^{-t_{n+1}K}\end{aligned}$$

$Q'\left(\frac{1}{\pi^2}b\right) = 1$, $Q'(c) = 0$ can be used safely.

Remark

- Actually, since $Q'c = 0$, $\Psi'_{\rm ES}$ becomes identity-based

$$\Psi'_{\rm ES} = \frac{1}{1+K'}\left(c + Q'\left(Bc\right)\right) = c$$

- One can avoid this by replacing

$$c \to c_y = c\left(iy\right)I, \;\; \left(y \neq 0\right)$$

$\mathrm{Tr}_V\left[\frac{1}{1+K'}c_y\right]$, $\mathrm{Tr}\left[\frac{1}{1+K'}c_y\frac{1}{1+K'}Q'c_y\right]$ are independent of y, and we get the same answers for the observables

§3 SFT around the TT solution

We have shown

$$\begin{aligned} E\left[\Psi'_{\mathrm{ES}}\right] &= -\frac{1}{6}\mathrm{Tr}\left[\frac{1}{1+K'}c\frac{1}{1+K'}Q'c\right]=0 \\ \mathrm{Tr}_V\,\Psi'_{\mathrm{ES}} &= \mathrm{Tr}_V\left[\frac{1}{1+K'}c\right]=0 \end{aligned}$$

using the definition

$$\begin{aligned} e^{-tK'} \equiv& \sum_{n=0}^{\infty}(-1)^n\int_0^{\infty}dt_1\cdots\int_0^{\infty}dt_{n+1} \\ & \times\delta\left(\sum_{i=1}^{n+1}t_i-t\right)e^{-t_1K}\left\{B,\Psi_{\mathrm{TT}}\right\}e^{-t_2K}\cdots\left\{B,\Psi_{\mathrm{TT}}\right\}e^{-t_{n+1}K} \end{aligned}$$

SFT around the TT solution

$$S' = -\frac{1}{g^2} \int \left[\frac{1}{2} \Psi Q' \Psi + \frac{1}{3} \Psi \Psi \Psi \right]$$

Since Q' is given by a local operator, we should be able to show

$$\begin{aligned} E\left[\Psi'_{\mathrm{ES}}\right] &= -\frac{1}{6} \mathrm{Tr}\left[\frac{1}{1+K'} c \frac{1}{1+K'} Q' c\right] = 0 \\ \mathrm{Tr}_V \Psi'_{\mathrm{ES}} &= \mathrm{Tr}_V \left[\frac{1}{1+K'} c\right] = 0 \end{aligned}$$

by dealing with the operator $K' = Q'B$ more directly.

We find that doing so is a little bit nontrivial.

Similarity transformation

Since K' itself is still difficult to deal with, we use the relation discovered by Kishimoto-Takahshi

$$Q' = -\frac{1}{4} U Q U^{-1}$$

$$U = e^{-q(\lambda)} U_2$$

$$q(\lambda) = 2 \sum_{n=1}^{\infty} \frac{1}{n} q_{-2n}$$

$$j_{\mathrm{gh}}(\xi) = \sum_{m} \xi^{-m-1} q_m$$

U_2:bc-shift operator

$$
\begin{aligned}
U_2 c_n U_2^{-1} &= c_{n+2} \\
U_2 b_n U_2^{-1} &= b_{n-2} \\
U_2 \phi^{\mathsf{m}} U_2^{-1} &= \phi^{\mathsf{m}} \\
U_2 \left|0\right\rangle &= b_{-3} b_{-2} \left|0\right\rangle \\
\left\langle 0\right| U_2^{-1} &= \left\langle 0\right| c_{-1} c_0
\end{aligned}
$$

- U_2 is of ghost number -2

Useful relations

$$Q' = -\frac{1}{4} U Q U^{-1}$$

$$\begin{aligned}
U c(\xi) U^{-1} &= \frac{\left(\xi^2 - 1\right)^2}{\xi^2} c(\xi) \\
U b(\xi) U^{-1} &= \frac{\xi^2}{\left(\xi^2 - 1\right)^2} b(\xi) \\
U |0\rangle &= \frac{1}{16} \partial b b(1) \partial b b(-1) c_0 c_1 |0\rangle \\
U^{-1} |0\rangle &= \frac{1}{16} \partial c c(1) \partial c c(-1) b_{-3} b_{-2} |0\rangle \\
\langle 0| U &= \langle 0| b_2 b_3 \\
\langle 0| U^{-1} &= \langle 0| c_{-1} c_0
\end{aligned}$$

Useful relations

$$Q' = -\frac{1}{4} U Q U^{-1}$$

$$\begin{aligned} U\,|I\rangle &= \frac{1}{32}\partial bb\,(1)\,|I\rangle \\ U^{-1}\,|I\rangle &= 2\partial cc\,(1)\,|I\rangle \\ \langle I|\,U &= 0 \\ \langle I|\,U^{-1} &= 0 \end{aligned}$$

Using the similarity transformation and these relations, it should be possible to calculate various quantities.

Q' cohomology

- Kishimoto-Takahashi

$$Q' = -\frac{1}{4} U Q U^{-1}$$

the representative state of the cohomology of Q'

$$U c V(0)|0\rangle \quad : \quad \mathrm{gh}\# = -1$$

$$U \partial c c V(0)|0\rangle \quad : \quad \mathrm{gh}\# = 0$$

- In conflict with the homotopy operator $A = \frac{1}{\pi^2} b$?

$$\{Q', b(1)\} = 1$$

$UcV(0)|0\rangle\,,\ U\partial ccV(0)|0\rangle$

- These are outside of the Fock space (Inatomi-Kishimoto-Takahashi)

$$UcV(0)|0\rangle = \frac{1}{32}\partial bb(1)\,\partial bb(-1)\,\partial^2 c\partial ccV(0)|0\rangle$$

$$AUcV(0)|0\rangle = b(1)\,UcV(0)|0\rangle = 0$$

- $\{Q', b(1)\} = 1$ is not correct with $\partial bb(1)$.
- Having some worldsheet with no operator insertions is crucial for $Q'A = 1$.

Observables

$$E\left[\Psi'_{\mathrm{ES}}\right] = -\frac{1}{6}\mathrm{Tr}\left[\frac{1}{1+K'}c\frac{1}{1+K'}Q'c\right] = 0$$

$$\mathrm{Tr}_V\,\Psi'_{\mathrm{ES}} = \mathrm{Tr}_V\left[\frac{1}{1+K'}c\right] = 0$$

are derived from $e^{-\epsilon K}Q'\left(\frac{1}{\pi^2}b\right)e^{-\epsilon K} = e^{-2\epsilon K},\; e^{-\epsilon K}Q'ce^{-\epsilon K} = 0.$

Let us see if we can show

$$e^{-\epsilon K'}Q'\left(\frac{1}{\pi^2}b\right)e^{-\epsilon K'} = e^{-2\epsilon K'}$$

$$e^{-\epsilon K'}Q'ce^{-\epsilon K'} = 0$$

instead.

$e^{-\epsilon K'} Q' \left(\frac{1}{\pi^2} b\right) e^{-\epsilon K'} = e^{-2\epsilon K'}$

$$\left(e^{-\epsilon K'}\left|I\right\rangle\right) * Q'\left(b\left(1\right)\left|I\right\rangle\right) * \left(e^{-\epsilon K'}\left|I\right\rangle\right) = e^{-2\epsilon K'}\left|I\right\rangle$$

Inserting $Q' = -\frac{1}{4} U Q U^{-1}$, the left hand side becomes

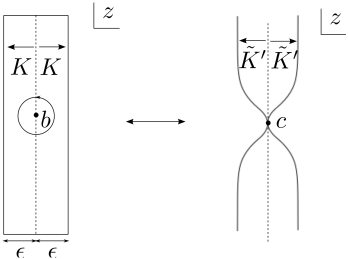
$$\begin{aligned} & -\frac{1}{4}\left(e^{-\epsilon K'}\left|I\right\rangle\right) * UQU^{-1}\left(b\left(1\right)\left|I\right\rangle\right) * \left(e^{-\epsilon K'}\left|I\right\rangle\right) \\ & = -\frac{1}{4} U\left(e^{-\epsilon \tilde{K}'}\left|I\right\rangle * Q\left(2c\left(1\right)\left|I\right\rangle\right) * e^{-\epsilon \tilde{K}'}\left|I\right\rangle\right) \\ & \to -\frac{1}{4} U\left(e^{-\epsilon \tilde{K}'} Q\left(2\pi c\right) e^{-\epsilon \tilde{K}'}\right) \end{aligned}$$

where

$$\tilde{K}' = -\int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} T\left(z\right) \tan^2 \pi z$$

$\tilde{K}' = -\int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} T(z) \tan^2 \pi z$

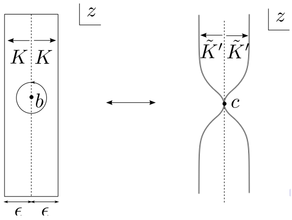
We can use the method of Kiermaier-Sen-Zwiebach to show that $\tilde{K}'$ does not move the points on the boundary.



We cannot prove $e^{-\epsilon K'} Q' \left(\frac{1}{\pi^2} b\right) e^{-\epsilon K'} = e^{-2\epsilon K'}$.

Regularization

- The homotopy operator seems to be crucial for $\Psi_{\rm TT}$ to be a tachyon vacuum solution.
- The surface should be defined as a limit of regular surfaces anyway.
- We propose a regularization such that the homotopy operator becomes well-defined.

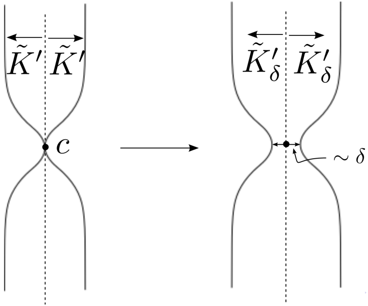


Regularization

Replace $\tilde{K}'$ by

$$\tilde{K}'_\delta = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} T(z) \frac{-\sin^2 \pi z + \delta}{\cos^2 \pi z}$$

and take the limit $\delta \to 0$



Homotopy operator

With this prescription,

$$\begin{aligned} e^{-\epsilon K'} Q' \left(\frac{1}{\pi^2} b \right) e^{-\epsilon K'} &= -\frac{1}{4} U \left(e^{-\epsilon \tilde{K}'} Q \left(2\pi c \right) e^{-\epsilon \tilde{K}'} \right) \\ &\to -\frac{1}{4} U \left(\lim_{\delta \to 0} e^{-\epsilon \tilde{K}'_\delta} Q \left(2\pi c \right) e^{-\epsilon \tilde{K}'_\delta} \right) \\ &= \frac{\pi}{4} U \left(\lim_{\delta \to 0} e^{-\epsilon \tilde{K}'_\delta} \partial c c e^{-\epsilon \tilde{K}'_\delta} \right) \\ &= e^{-2\epsilon K'} \end{aligned}$$

One also has $e^{-\epsilon K'} Q' c e^{-\epsilon K'} = 0$ and we can derive

$$\mathrm{Tr}_V \left[\frac{1}{1+K'} c \right] = \mathrm{Tr} \left[\frac{1}{1+K'} c \frac{1}{1+K'} Q' c \right] = 0$$

§4 Conclusions and discussions

- We have calculated the observables of the Erler-Schnabl solution around the TT solution. The results imply that TT solution corresponds to the tachyon vacuum.
- We explain how to deal with kinetic operator of the SFT around the TT solution.
- We will be able to calculate various quantities from the SFT around the TT solution. We may be able to see its relation to the VSFT. (Drukker, Drukker-Okawa)