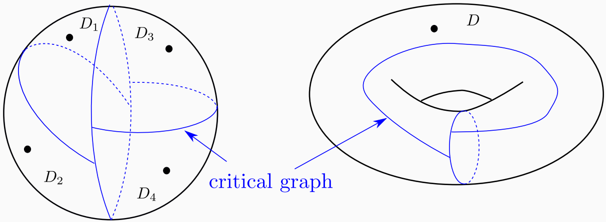
Strebel differentials and string field theory

International Workshop on String Field Theory and Related Aspects

Nobuyuki Ishibashi (University of Tsukuba)

27 March 2025

- On punctured Riemann surfaces (~Feynman graphs of strings), one can define quadratic differentials called Strebel differentials.
- Through Strebel differentials, one can represent any punctured Riemann surface by a critical graph (~local interaction vertex of strings).



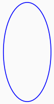
- We propose an SFT (for closed bosonic strings) based on such descriptions of Riemann surfaces. (PTEP 2024 (2024) 7, 073B02)

Fokker-Planck formalism

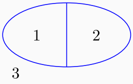
- We need to employ the Fokker-Planck formalism to construct such a theory.
- The Fokker-Planck Hamiltonian we propose is

$$
\begin{aligned}
\hat{H}_{\rm FP} \quad = \quad & -L\hat{\pi}_I\hat{\pi}_{I'}G^{I'I} + L\hat{\phi}^I\hat{\pi}_I \\
& -\frac{1}{2}g_{\rm s}V^{II'I''}G_{I''K''}G_{I'K'}\hat{\phi}^{K''}\hat{\phi}^{K'}\hat{\pi}_I \\
& -g_{\rm s}W^{II'I''}G_{I''K''}\hat{\phi}^{K''}\hat{\pi}_{I'}\hat{\pi}_I \, ,
\end{aligned}
$$

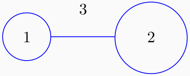
propagator



vertices



θ type
$|L_1 - L_2| < L_3 < L_1 + L_2$



dumbbell type
$L_1 + L_2 < L_3$

Remarks

- The SFT given in this talk looks very weird from the physical point of view.
- Unlike the conventional SFT, the SFT may not give a formulation from which important physical properties of string theory (unitarity, UV finiteness, background independence etc.) can easily be derived.
- The SFT should be considered as a machinery for computing correlation functions of string theory.

This talk

1. Strebel differentials
2. Combinatorial pants decomposition
3. Schwinger-Dyson equation for strings
4. The Fokker-Planck formalism
5. Conclusions and outlook

1. Strebel differentials

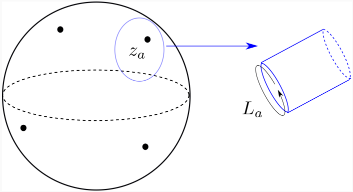
- On a punctured Riemann surface let us consider a quadratic differential $\phi(z)dz^2$ such that
  - near punctures $(z \sim z_a \ (a = 1, \cdots n))$

$$\phi(z)dz^2 \sim -\left(\frac{L_a}{2\pi}\right)^2 \frac{dz^2}{(z - z_a)^2}$$

with $L_a > 0$ and holomorphic for $z \neq z_a$

- A locally flat metric

$$\begin{aligned} ds^2 &= |\phi(z)|\, dz d\bar{z} = dw d\bar{w} \\ w &= \int^z dz' \sqrt{\phi(z')} \end{aligned}$$

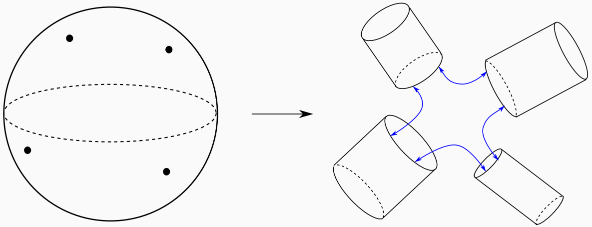


Strebel's theorem

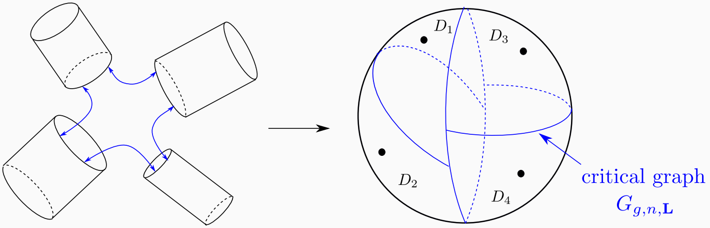
- Given a punctured Riemann surface $(2g-2+n>0)$ and positive numbers $L_1, \cdots, L_n$, there exists the unique quadratic differential $\phi(z)dz^2$ (Strebel differential) such that
  - for $z \sim z_a$, $\phi(z)dz^2 \sim -\left(\frac{L_a}{2\pi}\right)^2 \frac{dz^2}{(z-z_a)^2}$
  - holomorphic for $z \neq z_a$
  - with the metric

$$ds^2 = |\phi(z)|\, dz d\bar{z} = dw d\bar{w}$$
$$w = \int^z dz' \sqrt{\phi(z')}$$

  the surface looks like



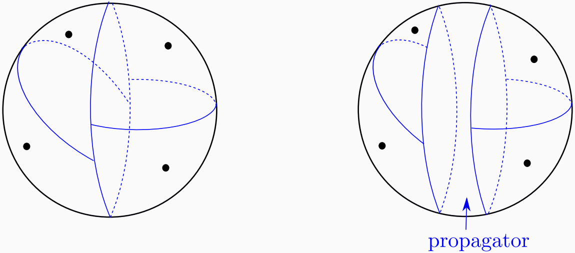
Strebel differentials



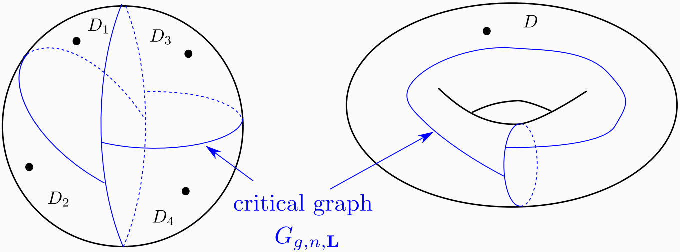
- To any punctured Riemann surface, there corresponds a (metric ribbon) graph called critical graph.
- Moduli spaces of punctured Riemann surfaces can be described by the lengths of the edges of the critical graphs (combinatorial moduli space $\mathcal{M}_{g,n}(\mathbf{L})$).
- Such a description plays important roles in
  - Kontsevich's proof of Witten conjecture
  - studying the free field limit of AdS/CFT (Gopakumar, ...)

Strebel differentials and string field theory

- The critical graphs look like local interaction vertices of closed strings.
- Strebel's theorem implies that any punctured Riemann surface can be described by such an interaction vertex.
- Such a description is not compatible with conventional SFT.



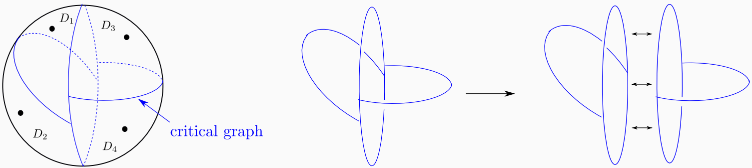
- Strebel differentials (with some restrictions) were used to construct the interaction vertices of a closed bosonic string field theory in Saadi-Zwiebach, Kugo-Kunitomo-Suehiro, Kugo-Suehiro.



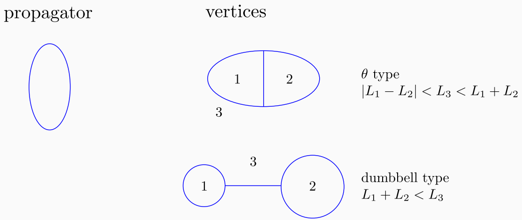
$$A_{g,n}^{i_1 \cdots i_n}(\mathbf{L}) \sim \int_{\mathcal{M}_{g,n}(\mathbf{L})} \langle G_{g,n,\mathbf{L}} | B_{6g-6+2n} | i_1 \rangle \cdots | i_n \rangle$$

- If Strebel differentials are really important in describing the free field limit in AdS/CFT, it may be worthwhile to construct an SFT in which the whole amplitudes are represented in this way.
- How can one construct such an SFT?

2. Combinatorial pants decomposition

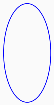


- The critical graphs can be decomposed into three string vertices. (combinatorial pants decomposition)
- We may be able to construct a theory with

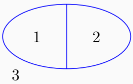


SFT action

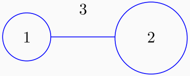
propagator



vertices



θ type
$|L_1 - L_2| < L_3 < L_1 + L_2$



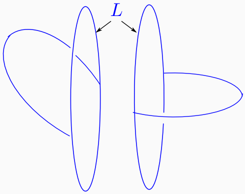
dumbbell type
$L_1 + L_2 < L_3$

- We may be able to construct an SFT action starting from these:

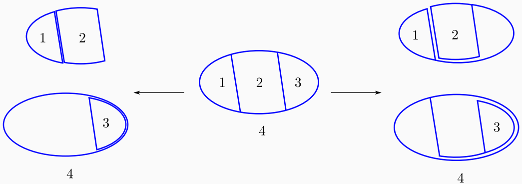
$$
\begin{aligned}
S[\phi] \quad = \quad & \frac{1}{2}\sum_i \int_0^\infty dL \phi^i(L)\phi^i(L) \\
& + \frac{g_s}{6}\sum_{i_1,i_2,i_3}\int d^3L V_{i_1 i_2 i_3}(L_1,L_2,L_3)\phi^{i_1}(L_1)\phi^{i_2}(L_2)\phi^{i_3}(L_3) \\
& + \cdots
\end{aligned}
$$

$\cdots$ are fixed so that the amplitudes are reproduced correctly.

Tree level four point amplitude

$\int_0^\infty dL$ 

- This integral diverges because the moduli space is covered infinitely many times.
  - The pants decomposition of a critical graph is not unique.
  - Different decompositions are transformed to each other by action of the mapping class group.

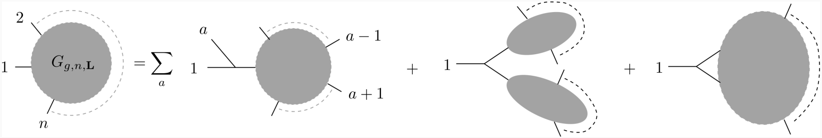


The action is not well-defined

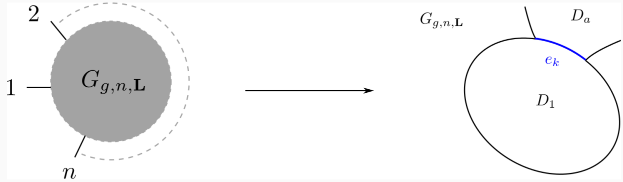
$$\begin{aligned} S[\phi] &= \frac{1}{2}\sum_i \int_0^\infty dL \phi^i(L)\phi^i(L) \\ &\quad + \frac{g_s}{6}\sum_{i_1,i_2,i_3}\int d^3L V_{i_1 i_2 i_3}(L_1,L_2,L_3)\phi^{i_1}(L_1)\phi^{i_2}(L_2)\phi^{i_3}(L_3) \\ &\quad + \cdots \end{aligned}$$

- $\cdots$ should include divergent counter terms to make the amplitude finite.
- This happens for almost all the amplitudes and $S[\phi]$ is not well-defined.
- We need some other formulation to construct the theory.

3. Schwinger-Dyson equation for strings



- Although action is ill-defined, one can derive an SD equation.
  $(3g - 3 + n > 0,\ (g, n) \neq (1, 1))$
  - Given a critical graph $G_{g,n,\mathbf{L}}$, we decompose it into a three string vertex one of whose legs is the first external line, and the rest.
  - In our case, it is impossible to uniquely pin down such a vertex, but it is possible to define a finite set of such vertices canonically.
  - We have a finite set of decompositions and make a weighted sum of them, which gives the right hand side of the above.



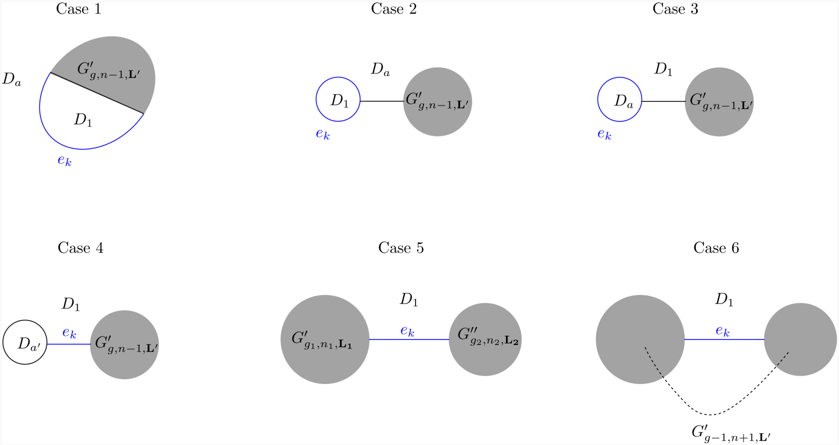
- ∂D_1 consists of edges e_k $(k = 1, \cdots, K)$, whose lengths are denoted by l_k.

$$\sum_{k=1}^{K} l_k = L_1$$

- $\partial D_1, \partial D_a$ and e_k specify a three string vertex uniquely.
- For each k, we get an expression for the amplitude in the form where the three string vertex is connected by propagators to the rest.
- We assign a weight $\frac{l_k}{L_1}$ to the k-th expression and construct a weighted sum of these.

SD equation

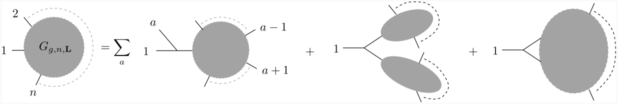
- The configuration falls into one of the following six cases (Bennett et al.)



SD equation

$$
\begin{aligned}
A^{i_1\cdots i_n}_{g,n}(\mathbf{L}) \;=\; & \sum_{a=1}^{n}\varepsilon_a\Bigg[\int_{|L_1-L_a|}^{L_1+L_a}dx\,\frac{L_1+L_a-x}{2L_1}\,V^{i_1 i_a}{}_j(L_1,L_a,x)A^{ji_2\cdots\hat{i_a}\cdots j_n}_{g,n-1}(x,L_2,\cdots,\hat{L_a},\cdots L_n) && \text{Case 1}\\
& +\theta(L_a-L_1)\int_0^{L_a-L_1}dx\,\frac{L_1}{L_1}\,V^{i_1 i_a}{}_j(L_1,L_a,x)A^{ji_2\cdots\hat{i_a}\cdots j_n}_{g,n-1}(x,L_2,\cdots,\hat{L_a},\cdots L_n) && \text{Case 2}\\
& +\theta(L_1-L_a)\int_0^{L_1-L_a}dx\,\frac{L_1-x}{L_1}\,V^{i_1 i_a}{}_j(L_1,L_a,x)A^{ji_2\cdots\hat{i_a}\cdots j_n}_{g,n-1}(x,L_2,\cdots,\hat{L_a},\cdots L_n)\Bigg] && \text{Case 3 + Case 4}\\
& +\frac{1}{2}\sum_{\text{stable}}\frac{\varepsilon_{I_1 I_2}}{(n_1-1)!(n_2-1)!}\int_0^{L_1}dx\int_0^{L_1-x}dy\,\frac{L_1-x-y}{L_1}\,V^{i_1}{}_{jj'}(L_1,x,y)A^{j' i_{I_1}}_{g_1,n_1}(y,L_{I_1})A^{ji_{I_2}}_{g_2,n_2}(x,L_{I_2}) && \text{Case 5}\\
& +\frac{1}{2}\int_0^{L_1}dx\int_0^{L_1-x}dy\,\frac{L_1-x-y}{L_1}\,V^{i_1}{}_{jj'}(L_1,x,y)A^{j'ji_2\cdots j_n}_{g-1,n+1}(y,x,L_2,\cdots,L_n)\,, && \text{Case 6}
\end{aligned}
$$

- We employ the combinatorial Fenchel-Nielsen coordinates $(l_s;\tau_s)$ $(s=1,\cdots,3g-3+n)$ (Andersen et al.) to describe $\mathcal{M}_{g,n}(\mathbf{L})$.
  - l_s: the lengths of the nonperipheral boundaries of the pairs of pants
  - τ_s: twist parameters
- The integrations over the twist parameters make the intermediate states to satisfy the level matching condition.
- We should take care of the b-ghost insertions.



$$A^{I_1\cdots I_n}_{g,n} = \sum_{a=2}^{n} \varepsilon_a B^{I_1 I_a J} G_{JI} A^{I I_2 \cdots \hat{I}_a \cdots I_n}_{g,n-1}$$
$$+ \frac{1}{2} C^{I_1 J' J} G_{JI} G_{J'I'} \left[A^{I I' I_2 \cdots I_n}_{g-1,n+1} + \sum_{\text{stable}} \frac{\varepsilon_{\mathcal{I}_1 \mathcal{I}_2}}{(n_1-1)!(n_2-1)!} A^{I \mathcal{I}_1}_{g_1,n_1} A^{I' \mathcal{I}_2}_{g_2,n_2} \right].$$

$$I \longleftrightarrow (i, \alpha, L)$$
$$X_I Y^I = X^I Y_I = \sum_i \sum_{\alpha=\pm} \int_0^\infty dL X(i,\alpha,L) Y(i,\alpha,L)$$
$$G_{I_1 I_2} \equiv \delta(L_1 - L_2) \delta_{i_1,i_2} \left[\delta_{\alpha_1,+}\delta_{\alpha_2,-} + \delta_{\alpha_1,-}\delta_{\alpha_2,+}(-1)^{|\varphi_{i_1}|} \right]$$

$$A^{I_1\cdots I_n}_{g,n} \equiv 2^{-\delta_{g,1}\delta_{n,1}} (2\pi i)^{-3g+3-n} \int_{\mathcal{M}_{g,n}(\mathbf{L})} \langle G_{g,n,\mathbf{L}} | B_{6g-6+2n} B^1_{\alpha_1} \cdots B^n_{\alpha_n} |\varphi^{\alpha_1}_{i_1}\rangle \cdots |\varphi^{\alpha_n}_{i_n}\rangle$$
$$B^{I_1 I_2 I_3} \equiv \mathsf{B}(L_1,L_2,L_3) \langle G_{0,3,(L_1,L_2,L_3)} | B^1_{\alpha_1} B^2_{\alpha_2} B^3_{\alpha_3} |\varphi^{\alpha_1}_{i_1}\rangle_1 |\varphi^{\alpha_2}_{i_2}\rangle_2 |\varphi^{\alpha_3}_{i_3}\rangle_3$$
$$C^{I_1 I_2 I_3} \equiv \mathsf{C}(L_1,L_2,L_3) \langle G_{0,3,(L_1,L_2,L_3)} | B^1_{\alpha_1} B^2_{\alpha_2} B^3_{\alpha_3} |\varphi^{\alpha_1}_{i_1}\rangle_1 |\varphi^{\alpha_2}_{i_2}\rangle_2 |\varphi^{\alpha_3}_{i_3}\rangle_3$$

$$|\varphi^{\alpha_a}_{i_a}\rangle = \begin{cases} |\varphi_{i_a}\rangle & \alpha_a = + \\ |\varphi^c_{i_a}\rangle & \alpha_a = - \end{cases}$$
$$B^a_{\alpha_a} \equiv \begin{cases} 1 & \alpha_a = + \\ b_0^{-(a)} b_{S^0_a}(\partial_{L_a}) & \alpha_a = - \end{cases}$$

$$\mathsf{B}(L_1,L_2,L_3) = \begin{cases} 0 & L_1 + L_2 \le L_3 \\ \frac{1}{2L_1}(L_1 + L_2 - L_3) & |L_1 - L_2| < L_3 < L_1 + L_2 \\ 1 & L_3 \le L_2 - L_1 \\ \frac{1}{L_1}(L_1 - L_3) & L_3 \le L_1 - L_2 \end{cases}$$
$$\mathsf{C}(L_1,L_2,L_3) = \begin{cases} 0 & L_1 \le L_2 + L_3 \\ \frac{1}{L_1}(L_1 - L_2 - L_3) & L_2 + L_3 < L_1 \end{cases}$$

4. The Fokker-Planck formalism

- In our theory, the Schwinger-Dyson equation can be derived but the action will be ill-defined.
- The Fokker-Planck formalism comes to the rescue.

Euclidean field theory

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \frac{\int [d\phi] e^{-S[\phi]} \phi(x_1) \cdots \phi(x_n)}{\int [d\phi] e^{-S[\phi]}}$$

- the FP formalism

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \lim_{\tau \to \infty} \langle 0 | e^{-\tau \hat{H}_{\mathrm{FP}}} \hat{\phi}(x_1) \cdots \hat{\phi}(x_n) | 0 \rangle$$

$$[\hat{\pi}(x), \hat{\phi}(y)] = \delta(x-y) \, , [\hat{\pi}, \hat{\pi}] = [\hat{\phi}, \hat{\phi}] = 0$$

$$\langle 0 | \hat{\phi}(x) = \hat{\pi}(x) | 0 \rangle = 0$$

$$\hat{H}_{\mathrm{FP}} = - \int dx \left(\hat{\pi}(x) - \frac{\delta S}{\delta \phi(x)} [\hat{\phi}] \right) \hat{\pi}(x)$$

The FP Hamiltonian and SD equation

- SD equation for $e^{W[J]} \equiv \int [d\phi] e^{-S[\phi]+\int dx J(x)\phi(x)}$

$$\begin{aligned} 0 &= \int [d\phi] \frac{\delta}{\delta\phi(x)} \left(e^{-S[\phi]+\int dx J(x)\phi(x)} \right) \\ &= \underbrace{\left(J(x) - \frac{\delta S}{\delta\phi(x)} \left[\frac{\delta}{\delta J(x)} \right] \right)}_{\equiv\, T\left[J(x),\, \frac{\delta}{\delta J(x)}\right]} e^{W[J]} \end{aligned}$$

- The FP Hamiltonian

$$\hat{H}_{\mathrm{FP}} = -\int dx \underbrace{\left(\hat{\pi}(x) - \frac{\delta S}{\delta\phi(x)}[\hat{\phi}] \right)}_{\equiv\, \hat{T}(x)} \hat{\pi}(x)$$

- $\hat{T}(x)$ satisfies

$$\hat{T}(x) e^{\int dx J(x)\hat{\phi}(x)} |0\rangle = T\left[J(x), \frac{\delta}{\delta J(x)} \right] e^{\int dx J(x)\hat{\phi}(x)} |0\rangle$$

This fact gives a quick way to derive FP Hamiltonian from SD equation.

The FP formalism for strings

- The generating functional

$$W[J] \equiv \sum_{g=0}^{\infty}\sum_{n=1}^{\infty} g_{\mathrm{s}}^{2g-2+n}\frac{1}{n!}J_{I_n}\cdots J_{I_1}A_{g,n}^{I_1\cdots I_n}$$

$$A_{0,1}^{I} \equiv 0,\ \ A_{0,2}^{I_1 I_2} \equiv G^{I_1 I_2}$$

- The SD equation for $W[J]$

$$T^I\left[J_K, \frac{\delta}{\delta J_K}\right]e^{W[J]} = 0$$

$$\begin{aligned} T^I\left[J_K, \frac{\delta}{\delta J_K}\right] \equiv\ & L\frac{\delta}{\delta J_I} - LG^{II'}(-1)^{|I'|}J_{I'} \\ & -\frac{1}{2}g_{\mathrm{s}}V^{II'I''}G_{I''K''}G_{I'K'}\frac{\delta^2}{\delta J_{K''}\delta J_{K'}} \\ & -g_{\mathrm{s}}W^{II'I''}G_{I''K''}(-1)^{|I'|}J_{I'}\frac{\delta}{\delta J_{K''}}(-1)^{|I'||I''|} \end{aligned}$$

$$V^{I_1I_2I_3} = \begin{cases}(L_1-L_2-L_3)\langle G_{0,3,(L_1,L_2,L_3)}|B^1_{\alpha_1}B^2_{\alpha_2}B^3_{\alpha_3}|\varphi^{\alpha_1}_{i_1}\rangle_1|\varphi^{\alpha_2}_{i_2}\rangle_2|\varphi^{\alpha_3}_{i_3}\rangle_3 & L_2+L_3<L_1 \\ 0 & L_1<L_2+L_3\end{cases},$$

$$W^{I_1I_2I_3} = \begin{cases}0 & L_1+L_2<L_3 \\ (L_1+L_2-L_3)\langle G_{0,3,(L_1,L_2,L_3)}|B^1_{\alpha_1}B^2_{\alpha_2}B^3_{\alpha_3}|\varphi^{\alpha_1}_{i_1}\rangle_1|\varphi^{\alpha_2}_{i_2}\rangle_2|\varphi^{\alpha_3}_{i_3}\rangle_3 & |L_1-L_2|<L_3<L_1+L_2 \\ \min(L_1,L_2)\langle G_{0,3,(L_1,L_2,L_3)}|B^1_{\alpha_1}B^2_{\alpha_2}B^3_{\alpha_3}|\varphi^{\alpha_1}_{i_1}\rangle_1|\varphi^{\alpha_2}_{i_2}\rangle_2|\varphi^{\alpha_3}_{i_3}\rangle_3 & L_3<|L_1-L_2|\end{cases}.$$

The FP formalism for strings

- Operators and states

$$\begin{aligned}[\hat{\pi}_I, \hat{\phi}^K] &= \delta_I{}^K\,,\\ [\hat{\pi}_I, \hat{\pi}_K] &= [\hat{\phi}^I, \hat{\phi}^K] = 0\,,\\ \langle 0|\hat{\phi}^I &= \hat{\pi}_I|0\rangle = 0\,,\end{aligned}$$

- The FP Hamiltonian

$$\begin{aligned}\hat{H}_{FP} &= \hat{T}^I \hat{\pi}_I\\ &= -L\hat{\pi}_I\hat{\pi}_{I'}G^{I'I} + L\hat{\phi}^I\hat{\pi}_I\\ &\quad -\frac{1}{2}g_{\mathrm{s}}V^{II'I''}G_{I''K''}G_{I'K'}\hat{\phi}^{K''}\hat{\phi}^{K'}\hat{\pi}_I\\ &\quad -g_{\mathrm{s}}W^{II'I''}G_{I''K''}\hat{\phi}^{K''}\hat{\pi}_{I'}\hat{\pi}_I\,,\end{aligned}$$

- One can prove

$$e^{W[J]} = \lim_{\tau\to\infty}\langle 0|e^{-\tau\hat{H}_{\mathrm{FP}}}e^{J_I\hat{\phi}^I}|0\rangle$$

perturbatively using

$$\hat{T}^I e^{J_K\hat{\phi}^K}|0\rangle = T^I\left[J_K, \frac{\delta}{\delta J_K}\right]e^{J_K\hat{\phi}^K}|0\rangle$$

BRST symmetry

- The correlation functions are BRST invariant

$$e^{W[J]} = \lim_{\tau\to\infty} \langle 0| e^{-\tau \hat{H}_{\mathrm{FP}}} e^{J_I \hat{\phi}^I} |0\rangle$$

$$\lim_{\tau\to\infty} \langle 0| e^{-\tau \hat{H}_{\mathrm{FP}}} \hat{Q} = \hat{Q}|0\rangle = 0$$

- $\hat{H}_{\mathrm{FP}}$ itself is not BRST invariant

$$\begin{aligned} \left[\hat{Q}, \hat{H}_{\mathrm{FP}}\right] &= \left[\hat{Q}, \hat{T}^I \hat{\pi}_I\right] \\ &= \left[\hat{Q}, \hat{T}^I\right] \hat{\pi}_I + \hat{T}^I \left[\hat{Q}, \hat{\pi}_I\right] \end{aligned}$$

$$\lim_{\tau\to\infty} \langle 0| e^{-\tau \hat{H}_{\mathrm{FP}}} \left[\hat{Q}, \hat{T}^I\right] = \lim_{\tau\to\infty} \langle 0| e^{-\tau \hat{H}_{\mathrm{FP}}} \hat{T}^I = 0$$

- BRST invariant Hamiltonian can be obtained by introducing auxiliary fields

$$\hat{H}_{\mathrm{FP}} \to \hat{H}_{\mathrm{FP}} + \left[\hat{Q}, \hat{T}^I\right] \lambda_I^Q + \hat{T}^I \lambda_I^T$$

  - This modification does not change the correlation functions.

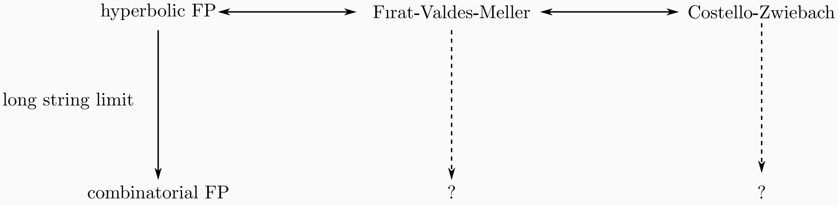
5. Conclusions and outlook

- We have constructed an SFT for closed bosonic strings based on the Strebel differentials via the Fokker-Planck formalism.

$$
\begin{aligned}
\hat{H}_{\mathrm{FP}} \quad = \quad & -L\hat{\pi}_I\hat{\pi}_{I'}G^{I'I} + L\hat{\phi}^I\hat{\pi}_I \\
& -\frac{1}{2}g_{\mathrm{s}}V^{II'I''}G_{I''K''}G_{I'K'}\hat{\phi}^{K''}\hat{\phi}^{K'}\hat{\pi}_I \\
& -g_{\mathrm{s}}W^{II'I''}G_{I''K''}\hat{\phi}^{K''}\hat{\pi}_{I'}\hat{\pi}_I \, ,
\end{aligned}
$$

- Superstrings?
- AdS/CFT?
- Implication for conventional SFT?

Outlook



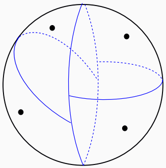
hyperbolic FP
Fırat-Valdes-Meller
Costello-Zwiebach
long string limit
combinatorial FP
?
?

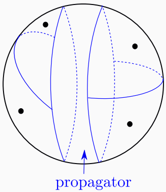
Backup

- Strebel differentials were used to construct the interaction vertices of a closed bosonic string field theory in Saadi-Zwiebach, Kugo-Kunitomo-Suehiro, Kugo-Suehiro.
  - One should consider the critical graphs such that

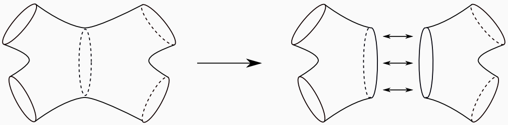
$$\begin{aligned} \text{length of an external string} &= 2\pi \\ \text{length of any closed loop} &\geq 2\pi \end{aligned}$$

  - With such restrictions, a part of the moduli space is covered by graphs with propagators.
  - The SFT reproduces the tree level amplitudes.





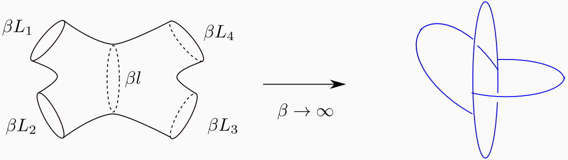
- The combinatorial pants decomposition is analogous to the pants decomposition of hyperbolic surfaces.
  - A Riemann surface with boundaries ($2g - 2 + n > 0$) admits a hyperbolic metric ($R = -2$) such that the boundaries are geodesics.
  - It can be decomposed into pairs of pants whose boundaries are geodesics.



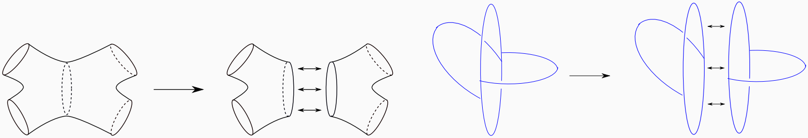
- The combinatorial pants decomposition can be considered as the long string limit of that of hyperbolic surfaces.

Long string limit

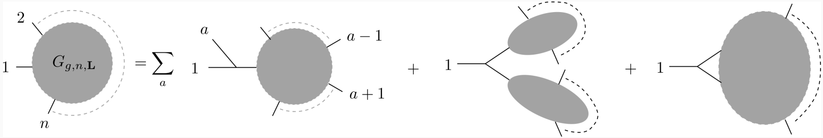
- The critical graphs in the combinatorial moduli space can be regarded as the long string limit of the hyperbolic surfaces. (Mondello, Do)



- The combinatorial pants decomposition can be considered as the long string limit of the hyperbolic pants decomposition.



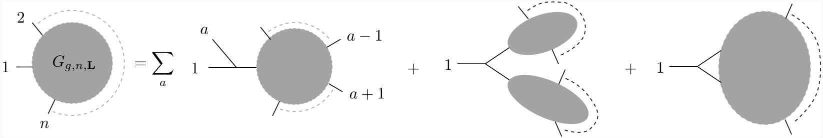
- By attaching semi-infinite cylinders, we get punctured Riemann surfaces.



$$
\begin{aligned}
A_{g,n}^{I_1\cdots I_n} \quad = \quad & \sum_{a=2}^{n} \varepsilon_a B^{I_1 I_a J} G_{JI} A_{g,n-1}^{I I_2 \cdots \hat{I}_a \cdots I_n} \\
& + \frac{1}{2} C^{I_1 J' J} G_{JI} G_{J'I'} \left[A_{g-1,n+1}^{I I' I_2 \cdots I_n} + \sum_{\text{stable}} \frac{\varepsilon_{\mathcal{I}_1 \mathcal{I}_2}}{(n_1-1)!(n_2-1)!} A_{g_1,n_1}^{I \mathcal{I}_1} A_{g_2,n_2}^{I' \mathcal{I}_2} \right]
\end{aligned}
$$

- This equation can be derived from a similar equation based on the hyperbolic pants decomposition by taking the long string limit.

Nonadmissible twists

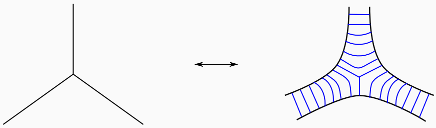


$$A_{g,n}^{I_1\cdots I_n} = \sum_{a=2}^{n} \varepsilon_a B^{I_1 I_a J} G_{JI} A_{g,n-1}^{I I_2\cdots \hat{I}_a \cdots I_n} + \frac{1}{2} C^{I_1 J' J} G_{JI} G_{J'I'} \left[A_{g-1,n+1}^{I I' I_2 \cdots I_n} + \sum_{\text{stable}} \frac{\varepsilon_{\mathcal{I}_1 \mathcal{I}_2}}{(n_1-1)!(n_2-1)!} A_{g_1,n_1}^{I \mathcal{I}_1} A_{g_2,n_2}^{I' \mathcal{I}_2} \right]$$

- Although any critical graph can be decomposed into pairs of pants, a graph made by gluing pairs of pants may not be a critical graph.
  - Some twist parameters do not correspond to critical graphs. (nonadmissible twists)
- Fortunately, nonadmissible twists do not appear on the right hand side. (Andersen et al.)

Nonadmissible twists

- Critical graphs are metric ribbon graphs (~Feynman diagrams of Witten's open SFT)



- For nonadmissible twists, closed string propagators appear