Recent developments in the construction of superstring field theory I

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February 22, 2016

「弦の場の理論 16」

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1. Introduction

String field theory is one approach to nonperturbative formulations of string theory which plays a role complementary to other approaches such as the AdS/CFT correspondence, matrix models, and so on.

Since the bosonic string contains tachyons both in the open-string and closed-string channels, we need to formulate superstring field theory if we are interested in quantum aspects.

However, construction of an action including the Ramond sector has not been successful for about thirty years. About half a year ago, we finally succeeded in constructing a gauge-invariant action for open superstring field theory including both the Neveu-Schwarz and Ramond sectors. This is the first construction of a complete formulation of superstring field theory.

Kunitomo and Okawa, arXiv:1508.00366

In this talk, we will review recent developments in the construction of superstring field theory focusing on open superstring field theory.

See the following talk by Kunitomo-san for details of the construction of the complete action and developments in heterotic string field theory.

The plan of the talk

- 1. Introduction
- 2. A_∞ structure
- 3. The NS sector: an old story
- 4. The NS sector: a new story
- 5. The Ramond: a new story about an old story
- 6. Future directions

2. A_∞ structure

Open bosonic string field theory

Witten, Nucl. Phys. B268 (1986) 253

$$S = -\frac{1}{2}\langle\,\Psi, Q\Psi\,\rangle - \frac{g}{3}\langle\,\Psi, \Psi * \Psi\,\rangle\,.$$

- Ψ: the open bosonic string field
- Q: the BRST operator
- g: the open string coupling constant
- $\langle\,A, B\,\rangle$: the BPZ inner product
- $A * B$: star product

  noncommutative $A * B \neq B * A$

  but associative $(A * B) * C = A * (B * C)$

The action is invariant under the gauge transformation given by

$$\delta_\Lambda \Psi = Q\Lambda + \Psi * \Lambda - \Lambda * \Psi \,,$$

where Λ is the gauge parameter

The invariance can be shown only from

$$\begin{aligned}
\langle\, B, A \,\rangle &= (-1)^{AB} \,\langle\, A, B \,\rangle \,, \\
Q^2 &= 0 \,, \\
\langle\, QA, B \,\rangle &= \, - (-1)^{A} \langle\, A, QB \,\rangle \,, \\
\langle\, A, B * C \,\rangle &= \langle\, A * B, C \,\rangle \,, \\
(A * B) * C &= A * (B * C) \,, \\
Q\,(A * B) &= QA * B + (-1)^{A} A * QB \,.
\end{aligned}$$

Actually, we can construct a gauge-invariant action based on a string product without associativity. Consider an action in the following form:

$$S = -\frac{1}{2}\langle\, \Psi, Q\Psi \,\rangle - \frac{g}{3}\langle\, \Psi, V_2(\Psi,\Psi) \,\rangle - \frac{g^2}{4}\langle\, \Psi, V_3(\Psi,\Psi,\Psi) \,\rangle + O(g^3)\,.$$

The BRST operator Q can be thought of as a one-string product with the cyclic property

$$\langle\, A_1, QA_2 \,\rangle = -(-1)^{A_1}\langle\, QA_1, A_2 \,\rangle\,.$$

$V_2(A_1,A_2)$: two-string product with the cyclic property

$$\langle\, A_1, V_2(A_2,A_3) \,\rangle = \langle\, V_2(A_1,A_2), A_3 \,\rangle\,.$$

$V_3(A_1,A_2,A_3)$: three-string product with the cyclic property

$$\langle\, A_1, V_3(A_2,A_3,A_4) \,\rangle = -(-1)^{A_1}\langle\, V_3(A_1,A_2,A_3), A_4 \,\rangle\,.$$

The action is invariant up to $O(g^3)$,

$$\delta_\Lambda S = O(g^3)\,,$$

under the gauge transformation in the form

$$\begin{aligned}\delta_\Lambda \Psi = Q\Lambda &+ g\,\Big(\,V_2(\Psi,\Lambda) - V_2(\Lambda,\Psi)\,\Big)\\ &+ g^2\,\Big(\,V_3(\Psi,\Psi,\Lambda) - V_3(\Psi,\Lambda,\Psi) + V_3(\Lambda,\Psi,\Psi)\,\Big) + O(g^3)\end{aligned}$$

if Q, V_2, and V_3 satisfy

$$\begin{aligned}&Q^2 A_1 = 0\,,\\ &QV_2(A_1,A_2) - V_2(QA_1,A_2) - (-1)^{A_1} V_2(A_1,QA_2) = 0\,,\\ &QV_3(A_1,A_2,A_3) - V_2(V_2(A_1,A_2),A_3) + V_2(A_1,V_2(A_2,A_3))\\ &\;+ V_3(QA_1,A_2,A_3) + (-1)^{A_1} V_3(A_1,QA_2,A_3)\\ &\;+ (-1)^{A_1+A_2} V_3(A_1,A_2,QA_3) = 0\,.\end{aligned}$$

- These relations of multi-string products are extended to higher orders, and a set of these relations is called an A_∞ structure.

- This A_∞ structure is closely related to the decomposition of the moduli space of Riemann surfaces.

- The quantization of string field theory based on the Batalin-Vilkovisky formalism is straightforward if the theory has an A_∞ structure.

The A_∞ structure can be thought of as a key to reproducing the world-sheet picture from string field theory which is based on gauge invariance as a spacetime theory.

3. The NS sector: an old story

In the Ramond-Neveu-Schwarz formalism of the superstring, there are infinitely many ways to describe each physical state, and they are labeled by a quantum number called *picture*.

In tree-level scattering amplitudes of the open superstring, the sum of the picture numbers of external states has to be -2.

e.g. four-point amplitudes of bosons
We can choose two states to be in the -1 picture and two states to be in the 0 picture.

On-shell scattering amplitudes do not depend on a choice of pictures, but how should we deal with the picture in string field theory?

Physical states in different pictures are mapped by the picture-changing operator $X(z)$:

$$\Psi^{(0)}(w) = \lim_{z \to w} X(z)\, \Psi^{(-1)}(w)\,,$$

where

$$[\, Q, X(z)\,] = 0\,.$$

e.g. four-point amplitudes
We can choose all the four states to be in the -1 picture and insert two picture-changing operators.

The Witten formulation — Nucl. Phys. B276 (1986) 291

The open superstring field is in the -1 picture.

Choose the two-string product to be

$$V_2(A_1, A_2) = X_{\rm mid}\,(A_1 * A_2)\,.$$

$X_{\rm mid}$: the picture-changing operator inserted at the open-string midpoint.

The operator product expansion of two picture-changing operators is singular.

$\Rightarrow$ divergences in the gauge variation of the action and in four-point amplitudes

The Berkovits formulation   hep-th/9503099

open superstring field:
a state in the matter + bc ghost + superconformal ghost CFT

the superconformal ghost sector
$\beta(z)\,\gamma(z) \quad \rightarrow \quad \xi(z),\ \eta(z), \phi(z)$

The Hilbert space for $\xi(z),\ \eta(z), \phi(z)$ is larger and is called the large Hilbert space.

The Hilbert space we usually use for $\beta\gamma$ ghosts is called the small Hilbert space.

$$\Psi \in \text{the small Hilbert space} \iff \eta\Psi = 0$$

η: the zero mode of $\eta(z)$

Algebraic relations in the large Hilbert space

$$\langle B, A \rangle = (-1)^{AB} \langle A, B \rangle \,,$$

$$Q^2 = 0 \,, \quad \eta^2 = 0 \,, \quad \{ Q, \eta \} = 0 \,,$$

$$\langle QA, B \rangle = -(-1)^{A} \langle A, QB \rangle \,,$$

$$\langle \eta A, B \rangle = -(-1)^{A} \langle A, \eta B \rangle \,,$$

$$\langle A, B * C \rangle = \langle A * B, C \rangle \,,$$

$$(A * B) * C = A * (B * C) \,,$$

$$Q \,(A * B) = QA * B + (-1)^{A} A * QB \,,$$

$$\eta \,(A * B) = \eta A * B + (-1)^{A} A * \eta B \,.$$

How large is the large Hilbert space?

$$\eta^2 = 0 \quad \text{and} \quad {}^{\exists}\xi \quad \text{satisfying} \quad \{\, \eta, \xi \,\} = 1\,.$$

A state Φ in the large Hilbert space can be decomposed as follows:

$$\Phi = \eta\xi\Phi + \xi\eta\Phi = \Psi_1 + \xi\Psi_2\,.$$

$$\Psi_1, \Psi_2 \in \text{the small Hilbert space}$$

We could say that the large Hilbert space is twice as large as the small Hilbert space.

We can realize ξ by a line integral of $\xi(z)$, and we assume that ξ obeys $\xi^2 = 0$ and $\langle\, A, \xi B \,\rangle = (-1)^A \langle\, \xi A, B \,\rangle$. We can choose, for example, ξ to be the zero mode ξ_0 of $\xi(z)$.

The BPZ inner products

For a pair of string fields A and B in the small Hilbert space, we define $\langle\!\langle\, A, B \,\rangle\!\rangle$ by

$$\langle\!\langle\, A, B \,\rangle\!\rangle = \langle\, \xi_0 A, B \,\rangle \,.$$

We can use ξ to relate the two BPZ inner products as

$$\langle\!\langle\, A, B \,\rangle\!\rangle = \langle\, \xi A, B \,\rangle \,.$$

The action of the free theory in the Berkovits formulation:

$$S = -\frac{1}{2}\langle\, \Phi, Q\eta\Phi \,\rangle\,.$$

Φ: the open superstring field in the large Hilbert space

The equation of motion: $Q\eta\Phi = 0\,.$

The gauge transformations: $\delta\Phi = Q\Lambda + \eta\Omega\,.$

Λ, Ω: gauge parameters in the large Hilbert space

By the gauge transformation $\delta\Phi = \eta\Omega\,,$

$$\Phi = \eta\xi\Phi + \xi\eta\Phi \quad \Longrightarrow \quad \Phi = \xi\Psi$$

$\Psi \in$ the small Hilbert space

The equation of motion reduces to

$$Q\eta\Phi = Q\eta\xi\Psi = Q\,\{\,\eta, \xi\}\,\Psi = Q\Psi = 0\,.$$

These gauge transformations can be nonlinearly extended without using the picture-changing operator in the action. The action and the gauge transformations up to $O(g^2)$ are

$$\begin{aligned} S = & -\frac{1}{2}\langle\,\Phi, Q\eta\Phi\,\rangle - \frac{g}{6}\langle\,\Phi, Q\,[\,\Phi, \eta\Phi\,]\,\rangle \\ & -\frac{g^2}{24}\langle\,\Phi, Q\,[\,\Phi, [\,\Phi, \eta\Phi\,]\,]\,\rangle + O(g^3)\,, \\ \delta\Phi = & \; Q\Lambda + \eta\Omega - \frac{g}{2}[\,\Phi, Q\Lambda\,] + \frac{g}{2}[\,\Phi, \eta\Omega\,] \\ & + \frac{g^2}{12}[\,\Phi, [\,\Phi, Q\Lambda\,]\,] + \frac{g^2}{12}[\,\Phi, [\,\Phi, \eta\Omega\,]\,] + O(g^3)\,. \end{aligned}$$

(All products of string fields are defined by the star product and here and in what follows we suppress the star symbol.)

Note that we do not see an A_∞ structure in the Berkovits formulation.

The action to all orders in the coupling constant takes the Wess-Zumino-Witten-like (WZW-like) form: Berkovits, hep-th/9503099

$$
\begin{aligned}
S = & \frac{1}{2} \langle\, e^{-\Phi} Q e^{\Phi}, e^{-\Phi} \eta e^{\Phi} \,\rangle \\
& - \frac{1}{2} \int_0^1 dt \, \langle\, e^{-\Phi(t)} \partial_t e^{\Phi(t)}, \{\, e^{-\Phi(t)} Q e^{\Phi(t)}, e^{-\Phi(t)} \eta e^{\Phi(t)} \,\} \,\rangle \,,
\end{aligned}
$$

where we set $g = 1$ and Φ is the value of $\Phi(t)$ at $t = 1$. The action can also be written as

$$
S = - \int_0^1 dt \, \langle\, A_t(t), Q A_\eta(t) \,\rangle
$$

with

$$
A_\eta(t) = (\, \eta e^{\Phi(t)} \,)\, e^{-\Phi(t)} \,, \qquad A_t(t) = (\, \partial_t e^{\Phi(t)} \,)\, e^{-\Phi(t)} \,.
$$

The t dependence is topological, and the action is a functional of Φ.

The action is invariant under the gauge transformations given by

$$A_\delta = Q\Lambda + D_\eta \Omega \,,$$

where

$$A_\delta = (\, \delta e^{\Phi} \,)\, e^{-\Phi} \,,$$

and D_η is the "covariant derivative" with respect to the gauge transformation generated by η:

$$D_\eta A = \eta A - A_\eta \, A + (-1)^A \, A \, A_\eta \quad \text{with} \quad A_\eta = (\, \eta e^{\Phi} \,)\, e^{-\Phi} \,.$$

The topological t-dependence and the gauge invariance follow from

$$\begin{aligned} \eta A_\eta(t) - A_\eta(t) A_\eta(t) &= 0 \,, \\ \partial_t A_\eta(t) &= \eta A_t(t) - A_\eta(t) A_t(t) + A_t(t) A_\eta(t) \end{aligned}$$

together with the fact that the cohomology of η is trivial.

4. The NS sector: a new story

It had long been thoguht that construction based on the small Hilbert space would be necessarily singular, but it was demonstrated that a regular formulation based on the small Hilbert space can be obtained from the Berkovits formulation by partial gauge fixing.

Iimori, Noumi, Okawa and Torii, arXiv:1312.1677

New ingredient: an operator ξ satisfying $\{\,\eta, \xi\,\} = 1$. Such an operator can be realized by a line integral of the superconformal ghost $\xi(z)$.

- The partial gauge fixing guarantees that the resulting theory is gauge invariant.
- The BRST transformation of ξ yields a line integral $\mathcal{X}$ of the picture-changing operator, $\mathcal{X} = \{\,Q, \xi\,\}$, and singularities associated with local picture-changing operators are avoided in this approach.
- However, it turned out that the resulting theory does not exhibit an A_∞ structure.

In the paper, the four-point amplitude was also calculated in a class of gauges where the relation to the world-sheet calculation can be manifestly seen.

In open bosonic string field theory, it is known that the moduli space of disks with four punctures is covered by Feynman diagrams with two Witten cubic vertices and one propagator when we combine contributions from the s channel and the t channel.

In the four-point amplitude of the Berkovits formulation, the assignment of picture-changing operators is different in the s channel and in the t channel of Feynman diagrams with two vertices and one propagator. This different behavior is adjusted by the contribution from the quartic interaction.

This different behavior persists in the limit where the picture-changing operator localizes at the open-string midpoint. This way the difficulty in the Witten formulation can be understood in the context of the covering of the supermoduli space of super-Riemann surfaces.

We have also seen that the quartic interaction of the Berkovits formulation implements the “vertical integration” in the approach by Sen and Witten. We have obtained preliminary insight into the understanding of why the Berkovits formulation based on the large Hilbert space is successful from the context of the supermoduli space of super-Riemann surfaces.

While Q and η are derivations of the star product, line integrals of $\xi(z)$ do not have simple properties under conformal transformations and were not considered as ingredients of a gauge-invariant action. With hindsight, they are like b-ghost insertions for integrals of bosonic moduli. (In particular, $\partial\xi(z)$ is the ghost insertion associated with the "vertical integration.")

Once we recognize that ξ can be used in constructing a guage-invariant action, we do not have to start from the Berkovits formulation.

Erler, Konopka and Sachs constructed an action with an A_∞ structure for the Nevue-Schwarz sector of open superstring field theory based on the small Hilbert space using ξ as a new ingredient.

Erler, Konopka and Sachs, arXiv:1312.2948

- Because of the A_∞ structure, the Batalin-Vilkovisky quantization is straightforward.

- The constructfion was further generalized to the NS sector of heterotic string field theory and the NS-NS sector of type II superstring field theory.

  Erler, Konopka and Sachs, arXiv:1403.0940

- The constructfion was also generalized to the equations of motion including the Ramond sector.

  Erler, Konopka and Sachs, arXiv:1506.05774

Open superstring field theory with the A_∞ structure

Erler, Konopka and Sachs, arXiv:1312.2948

The two-string product

$$V_2(A_1, A_2) = \frac{1}{3}\Big[\, \mathcal{X}(A_1 A_2) + (\mathcal{X} A_1) A_2 + A_1 (\mathcal{X} A_2) \Big]$$

satisfies

$$\langle\, A_1, V_2(A_2, A_3)\,\rangle = \langle\, V_2(A_1, A_2), A_3)\,\rangle$$

and

$$Q V_2(A_1, A_2) - V_2(Q A_1, A_2) - (-1)^{A_1} V_2(A_1, Q A_2) = 0\,,$$

but V_2 is not associative. We need a three-string product V_3 satisfying

$$\begin{aligned}
& Q V_3(A_1, A_2, A_3) - V_2(V_2(A_1, A_2), A_3) + V_2(A_1, V_2(A_2, A_3)) \\
& + V_3(Q A_1, A_2, A_3) + (-1)^{A_1} V_3(A_1, Q A_2, A_3) \\
& + (-1)^{A_1 + A_2} V_3(A_1, A_2, Q A_3) = 0\,.
\end{aligned}$$

We can construct V_3 using the star product with insertions of ξ and $\mathcal{X}$, but its form is fairly complicated. The strategy of Erler, Konopka and Sachs is based on the observation that the action

$$S = -\frac{1}{2}\langle\!\langle\, \Psi, Q\Psi \,\rangle\!\rangle - \frac{g}{3}\langle\!\langle\, \Psi, V_2(\Psi,\Psi) \,\rangle\!\rangle + O(g^2)$$

can be generated from the free theory

$$S = -\frac{1}{2}\langle\!\langle\, \widetilde{\Psi}, Q\widetilde{\Psi} \,\rangle\!\rangle$$

by the following field redefinition:

$$\widetilde{\Psi} = \Psi + \frac{g}{3}\Big[\,\xi(\Psi\Psi) + (\xi\Psi)\Psi - \Psi(\xi\Psi)\,\Big] + O(g^2)\,.$$

If $\widetilde{\Psi}$ were in the small Hilbert space, the resulting theory would be free. However, $\widetilde{\Psi}$ is not in the small Hilbert space,

$$\eta\widetilde{\Psi} \neq 0\,,$$

so the field redefinition is “illegal” and the resulting theory is interacting.

While the field redefinition is not in the small Hilbert space,

$$\eta\widetilde{\Psi} \neq 0\,,$$

but the multi-string products $V_n(A_1, A_2, \ldots, A_n)$ have to be in the small Hilbert space. It turned out that this can be achieved if the field redefinition satisifes the condition:

$$\eta\widetilde{\Psi} - \widetilde{\Psi}^2 = 0\,,$$

This is the same as the relation

$$\eta A_\eta(t) - A_\eta(t)^2 = 0$$

in the Berkovits formulation!

In fact, we can bring the action of the theory by Erler, Konopka and Sachs to the WZW-like form, where $A_\eta(t)$ and $A_t(t)$ satisfying

$$\eta A_\eta(t) - A_\eta(t) A_\eta(t) = 0\,,$$
$$\partial_t A_\eta(t) = \eta A_t(t) - A_\eta(t) A_t(t) + A_t(t) A_\eta(t)$$

are parameterized in a different way in terms of a one-parameter family of string fields in the small Hilbert space.

In the Berkovits formulation, $A_\eta(t)$ and $A_t(t)$ are parameterized as

$$A_\eta(t) = (\,\eta e^{\Phi(t)}\,)\, e^{-\Phi(t)}\,, \qquad A_t(t) = (\,\partial_t e^{\Phi(t)}\,)\, e^{-\Phi(t)}\,.$$

The action is written in a closed form, but it does not exhibit an A_∞ structure.

In the theory constructed by Erler, Konopka and Sachs, $A_\eta(t)$ and $A_t(t)$ are parameterized in a complicated way and we do not have a closed-form expression for the action, but it exhibits an A_∞ structure.

It was shown that the two theories are related by field redefinition and partial gauge fixing, where the field redefinition is given by equating $A_\eta(t)$ at $t = 1$ on both sides.

Erler, Okawa and Takezaki, arXiv:1505.01659
Erler, arXiv:1505.02069
Erler, arXiv:1510.00364

Construction of the NS sector of open superstring field theory has thus been successful. What is the difficulty in incorporating the Ramond sector?

5. The Ramond sector:
a new story about an old story

Unintegrated vertex operators of the open bosonic string: cV^{matter}
$\to$ the open bosonic string field Ψ: ghost number 1

The BPZ inner product in the open string is defined by a correlation function of the boundary CFT on a disk, and the total ghost number has to be 3 for the BPZ inner product to be nonvanishing.

$$S = -\frac{1}{2}\langle\, \Psi, Q\Psi \,\rangle \quad \to \quad 1+1+1=3 \quad \text{OK!}$$

As we mentioned earlier, there are infinitely many vertex operators labeled by the picture number to describe each physical state for the open superstring. In formulating string field theory, we need to choose the picture number of the open superstring field.

The choice of the picture number is related to the choice of the vacuum of the superconformal ghost sector with the commutation relation

$$[\, \gamma_n, \beta_m \,] = \delta_{n+m,0}\,.$$

A natural choice in the Neveu-Schwarz sector is -1 picture.

$$\begin{aligned} \text{annihilation operators:} \quad & \gamma_{1/2}, \ \beta_{1/2}, \ \gamma_{3/2}, \ \beta_{3/2}, \ \ldots \\ \text{creation operators:} \quad & \gamma_{-1/2}, \ \beta_{-1/2}, \ \gamma_{-3/2}, \ \beta_{-3/2}, \ \ldots \end{aligned}$$

the open superstring field Ψ: ghost number 1, picture number -1

$$S = -\frac{1}{2} \langle\langle \Psi, Q\Psi \rangle\rangle$$

$$\to \begin{cases} \text{the ghost number:} \quad 1+1+1=3 \quad \text{OK!} \\ \text{the picture number:} \quad (-1)+0+(-1)=-2 \quad \text{OK!} \end{cases}$$

Natural choices in the Ramond sector would be
$-1/2$ picture:

$$\text{annihilation operators:} \quad \beta_0, \; \gamma_1, \; \beta_1, \; \gamma_2, \; \beta_2, \; \ldots$$
$$\text{creation operators:} \quad \gamma_0, \; \gamma_{-1}, \; \beta_{-1}, \; \gamma_{-2}, \; \beta_{-2}, \; \ldots$$

and $-3/2$ picture:

$$\text{annihilation operators:} \quad \gamma_0, \; \gamma_1, \; \beta_1, \; \gamma_2, \; \beta_2, \; \ldots$$
$$\text{creation operators:} \quad \beta_0, \; \gamma_{-1}, \; \beta_{-1}, \; \gamma_{-2}, \; \beta_{-2}, \; \ldots$$

Suppose that we choose the string field Ψ in the $-1/2$ picture.

$$S = -\frac{1}{2} \langle\!\langle \, \Psi, Q\Psi \, \rangle\!\rangle$$
$$\rightarrow \begin{cases} \text{the ghost number:} \quad 1+1+1=3 \quad \text{OK!} \\ \text{the picture number:} \quad (-1/2)+0+(-1/2) \neq -2 \end{cases}$$

The picture number does not work out for any choice in the Ramond sector.

Actually, there is a similar problem in the closed bosonic string.

Unintegrated vertex operators of the closed bosonic string: $c\tilde{c}\,V^{\text{matter}}$
$\longrightarrow$ the closed bosonic string field Ψ: ghost number 2

$$S = -\frac{1}{2}\langle\,\Psi, Q\Psi\,\rangle \quad \longrightarrow \quad \text{the ghost number:} \quad 2+1+2 \neq 6$$

Let us explain the solution to this problem by viewing surfaces for string propagators as Riemann surfaces.

The propagator strip in the open bosonic string can be generated by L_0 as e^{-tL_0}, where t is the modulus corresponding to the length of the strip.

In open bosonic string field theory, the integration over this modulus is implemented by the propagator in Siegel gauge as

$$\frac{b_0}{L_0} = \int_0^\infty dt\, b_0\, e^{-tL_0}\,.$$

The propagator surface in the closed bosonic string can be generated by $L_0 + \widetilde{L}_0$ and $i\,(L_0 - \widetilde{L}_0)$ as $e^{-t(L_0+\widetilde{L}_0)+i\theta(L_0-\widetilde{L}_0)}$, where t and θ are moduli.

In closed bosonic string field theory, the integration over t is implemented by the propagator in Siegel gauge as in the open bosonic string:

$$\frac{b_0^+}{L_0^+} = \int_0^\infty dt\, b_0^+\, e^{-t\,L_0^+}\,,$$

where

$$L_0^+ = L_0 + \widetilde{L}_0\,, \qquad b_0^+ = b_0 + \widetilde{b}_0\,.$$

On the other hand, the integration over θ is implemented as a constraint on the space of string fields. The integration over θ yields the operator given by

$$B = b_0^- \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{i\theta L_0^-} \,,$$

where

$$L_0^- = L_0 - \widetilde{L}_0 \,, \qquad b_0^- = b_0 - \widetilde{b}_0 \,.$$

Schematically, $B \sim \delta(b_0^-)\,\delta(L_0^-)\,.$

The closed bosonic string field Ψ is constrained to satisfy

$$b_0^- \, \Psi = 0 \,, \qquad L_0^- \, \Psi = 0 \,,$$

and the BRST cohomology on this restricted space is known to give the correct spectrum of the closed bosonic string.

The appropriate inner product of Ψ_1 and Ψ_2 in the restricted space can be written as

$$\langle \Psi_1, c_0^- \Psi_2 \rangle ,$$

where

$$c_0^- = \frac{1}{2} (c_0 - \widetilde{c}_0) .$$

The kinetic term of closed bosonic string field theory is then given by

$$S = -\frac{1}{2} \langle \Psi, c_0^- Q \Psi \rangle .$$

$\rightarrow$ the ghost number: $2 + 1 + 1 + 2 = 6$ OK!

The operator B can also be written as

$$B = -i \int_0^{2\pi} \frac{d\theta}{2\pi} \int d\tilde{\theta}\, e^{i\theta L_0^- + i\tilde{\theta}\, b_0^-}$$

using a Grassmann-odd variable $\tilde{\theta}$.
(Note that the extended BRST transformation introduced by Witten in arXiv:1209.5461 maps θ to $\tilde{\theta}$.)

Since $B\, c_0^-\, B = B$, the operator $B\, c_0^-$ is a projector, and the closed bosonic string field Ψ in the restricted space can be characterized as

$$B\, c_0^-\, \Psi = \Psi\,.$$

The propagator strip for the Ramond sector of the open superstring has a fermionic modulus in addition to the bosonic modulus corresponding to the length of the strip.

The fermionic direction of the moduli space can be parameterized as $e^{\zeta G_0}$, where G_0 is the zero mode of the supercurrent and ζ is the fermionic modulus. The integration over ζ with the associated ghost insertion yields the operator X given by

$$X = \int d\zeta \int d\tilde{\zeta}\, e^{\zeta G_0 - \tilde{\zeta}\,\beta_0}\,,$$

where $\tilde{\zeta}$ is a Grassmann-even variable. (The extended BRST transformation maps ζ to $\tilde{\zeta}$.) If we perform the integration over ζ, we obtain

$$X = -\,\delta(\beta_0)\, G_0 + \delta'(\beta_0)\, b_0\,.$$

We expect that this operator would play a key role in constructing the kinetic term for the Ramond sector.

On the other hand, it was shown a long time ago that the kinetic term for the Ramond sector can be constructed if we restrict the state space appropriately.

The open superstring field Ψ of picture $-1/2$ in the Ramond sector can be expanded as

$$\Psi = \sum_{n=0}^{\infty} (\gamma_0)^n \, (\phi_n + c_0 \, \psi_n) \, ,$$

where

$$b_0 \, \phi_n = 0 \, , \qquad \beta_0 \, \phi_n = 0 \, , \qquad b_0 \, \psi_n = 0 \, , \qquad \beta_0 \, \psi_n = 0 \, .$$

The restricted form of the string field is given by

$$\Psi = \phi - (\, \gamma_0 + c_0 \, G \,) \, \psi \, ,$$

where $G = G_0 + 2 \, b_0 \gamma_0$ and

$$b_0 \, \phi = 0 \, , \qquad \beta_0 \, \phi = 0 \, , \qquad b_0 \, \psi = 0 \, , \qquad \beta_0 \, \psi = 0 \, .$$

The restricted space is preserved by the action of the BRST operator. The BRST cohomology in the restricted space is the same as that in the unrestricted space and reproduces the correct spectrum.

However, this characterization of the restricted space does not seem illuminating...

The important point is that the open superstring field Ψ in the restricted space can be characterized using the operator X as

$$XY\Psi = \Psi \,,$$

where

$$Y = -\, c_0\, \delta'(\gamma_0)\,.$$

Kugo and Terao, Phys. Lett. B 208 (1988) 416

Since $XYX = X$, the operator XY is a projector to the restricted space. This is analogous to

$$B\, c_0^- \,\Psi = \Psi$$

for the closed bosonic string field, and we regard this characterization of the string field in the Ramond sector as fundamental.

The appropriate inner product of Ψ_1 and Ψ_2 in the restricted space can be written as

$$\langle\!\langle\, \Psi_1, Y\Psi_2 \,\rangle\!\rangle\,,$$

and the kinetic term of open superstring field theory for the Ramond sector is given by

$$S = \,-\,\frac{1}{2}\,\langle\!\langle\, \Psi, YQ\Psi \,\rangle\!\rangle\,.$$

$$\rightarrow \begin{cases} \text{the ghost number:} \quad 1+0+1+1=3 \quad \text{OK!} \\ \text{the picture number:} \quad (-1/2)+(-1)+0+(-1/2)=-2 \quad \text{OK!} \end{cases}$$

If we impose an arbitrary constraint on the string field, there will be no hope of constructing a gauge-invariant action. However, this constraint in the Ramond sector has an interpretation in the context of the super-moduli space.

Furthermore, there was another important development last year which indicated that an interacting theory consistent with the projection is possible.

Sen constructed the 1PI effective superstring field theory including the Ramond sector.

Sen, arXiv:1501.00988

The equations of motion in the Ramond sector takes the following form:

$$Q\Psi_{\mathrm{R}} + X_0\, f(\Psi_{\mathrm{NS}}, \Psi_{\mathrm{R}}) = 0\,,$$

where Ψ_{NS} is the string field in the NS sector, Ψ_{R} is the string field in the Ramond sector, and X_0 is the zero mode of the picture-changing operator. The origin of X_0 is the propagator in the Ramond sector, and we can it by X:

$$Q\Psi_{\mathrm{R}} + X\, f(\Psi_{\mathrm{NS}}, \Psi_{\mathrm{R}}) = 0\,.$$

This is reminiscent of the equation of motion of closed bosonic string field theory, where the interaction terms of the equation of motion are multiplied by B, and this structure indicates that the open superstring field for the Ramond sector in the restricted space can be consistently used for the interacting theory. In fact, the equation of motion is consistent with the restriction since $XYX = X$.

Can we can introduce interactions which are consistent with this constraint?

See the talk by Kunitomo-san!

6. Future directions

In a very near future (in about thirty minutes), you will hear in the talk by Kunitomo-san that a gauge-invariant action of open superstring field theory including both the NS sector and the Ramond sector can be indeed constructed.

In a slightly further future (tomorrow), you will hear in the talk by Takezaki-kun that a gauge-invariant action of open superstring field theory including both the NS sector and the Ramond sector with an A_∞ structure can be constructed. This provides the first explicit solution to the classical Batalin-Vilkovisky master equation.

We do not foresee any conceptual problems in the generalization to heterotic string field theory.

Can we construct a solution to the quantum Batalin-Vilkovisky master equation?

In closed bosonic string field theory, a solution to the quantum Batalin-Vilkovisky master equation is constructed, and it is known that we need corrections to the classical action in each order of $\hbar$. We might be able to interpret these corrections as corrections to the measure of the path integral.

In the lattice gauge theory, for example, we need to use the Haar measure for fields taking values in a group manifold. If we expand the fields around the "north pole" of the group manifold and use the path-integral measure for fields taking values in the flat space, the gauge symmetry will be anomalous. The current formulation of string field theory, where the background independence is not manifest, might be in a similar situation.

We hope that the recent developments in superstring field theory will enable us to address the important question of what the space of string fields is and help us unveil the nonperturbative theory underlying the perturbative superstring theory.