Gradient flow for parton distribution functions: first application to the pion

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Motivation

- Parton distribution functions (PDFs) are central to precision QCD phenomenology.
- Proton PDFs remain a dominant source of uncertainty at collider experiments.
- Pion PDFs are much less constrained since pions cannot be used as fixed targets.
- Large-x behavior parameterized as $(1-x)^\beta$ remains debated and motivates new measurements.

Lattice Difficulty

- In Euclidean lattice QCD, the non-local operator defining PDFs becomes point-like ➡ direct computation of x-dependent PDFs is impossible.

- Instead, Mellin moments ($q(x, \mu)$: PDF)

$$\langle x^{n-1} \rangle_q(\mu) = \int dx\, x^{n-1}\, q(x, \mu) \tag{1}$$

can be accessed through local twist-2 operators (r, s: flavor indices):

$$O_n^{rs}(x) \equiv i^{n-1} \bar{\psi}^r \gamma_{\{\mu_1} \overleftrightarrow{D}_{\mu_2} \ldots \overleftrightarrow{D}_{\mu_n\}} \psi^s(x) \tag{2}$$

$\overleftrightarrow{D}_\mu = \frac{1}{2}(D_\mu - \overleftarrow{D}_\mu)$: the forward-backward covariant derivative

- However, lattice discretization breaks $O(4)$ symmetry to $H(4)$ ➡ power-divergent operator mixing among twist-2 operators.

- Mixing grows rapidly with n (for $\langle x^n \rangle$) ➡ previous lattice studies were limited to the lowest moment $\langle x \rangle$.

Key Idea of This Work

- Apply the gradient flow (GF) as an intermediate UV regulator.
- Flowed matrix elements avoid power-divergent mixing: gluonic operators are finite and fermionic ones renormalize multiplicatively.
- Short flow-time expansion (SFTX) links flowed matrix elements to physical ($t = 0$) operators.
- This enables controlled extraction of higher PDF moments on the lattice.

Matching Formula

- Ratios of PDF moments in the $\overline{\mathrm{MS}}$ scheme can be obtained from flowed matrix elements:

$$\frac{\langle x^{n-1}\rangle^{\overline{\mathrm{MS}}}(\mu)}{\langle x^{m-1}\rangle^{\overline{\mathrm{MS}}}(\mu)} = \frac{\zeta_m(t,\mu)}{\zeta_n(t,\mu)} \frac{\langle x^{n-1}\rangle(t)}{\langle x^{m-1}\rangle(t)} + O(t) \tag{3}$$

- $\zeta_n(t,\mu)$ computed at NNLO up to $n = 6$ in this work.

Lattice Setup

- 4 Stabilized Wilson Fermion ensembles from the OpenLat initiative.
- $N_f = 3$ degenerate quarks, tuned to $m_\pi \simeq 411$ MeV.
- Lattice spacings: $a \approx 0.12, 0.094, 0.077, 0.064$ fm.
- t_0 scale used to convert flow time to physical units.

Ground-State Extraction

- Relation between flowed moments and matrix elements:

$$\frac{\langle x^{n-1}\rangle(t)}{\langle x^{m-1}\rangle(t)} = \frac{(-1)^{n-m}}{m_\pi^{n-m}} \frac{\langle \pi(\mathbf{0})|\widehat{O}_n(t)|\pi(\mathbf{0})\rangle}{\langle \pi(\mathbf{0})|\widehat{O}_m(t)|\pi(\mathbf{0})\rangle} \tag{4}$$

- Three-point function used in the measurement:

$$C_n^{\text{3-pt}}(\, x_4 = \tau_s,\, y_4 = \tau_{\mathcal{O}}; t) = a^6 \sum_{\mathbf{x},\mathbf{y}} \langle P^{du}(\mathbf{x}, x_4)\, \widehat{O}_n(\mathbf{y}, y_4; t)\, P^{ud}(0)\rangle_c \tag{5}$$

  where $P^{ud} = \bar{\psi}_u \gamma_5 \psi_d$ is a pseudoscalar interpolator, projected to zero spatial momentum.

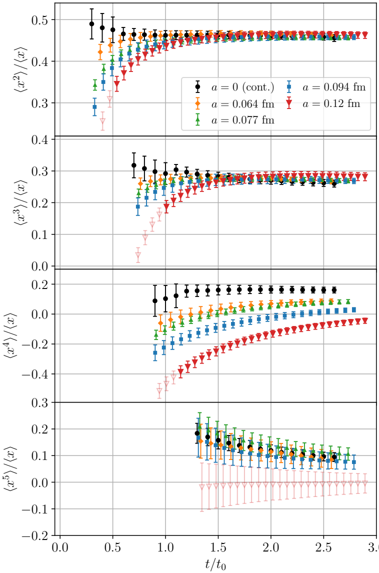
- Ratio of 3pt correlators ($A_n(t) \equiv \langle \pi(\mathbf{0})|\widehat{O}_n(t)|\pi(\mathbf{0})\rangle$):

$$\frac{C_n^{\text{3-pt}}(\tau_s, \tau_{\mathcal{O}}; t)}{C_2^{\text{3-pt}}(\tau_s, \tau_{\mathcal{O}}; t)} \simeq \frac{A_n(t)}{A_2(t)} + \cdots \tag{6}$$

- Plateau search in $\tau_{\mathcal{O}}$ using the window $\frac{\tau_s}{2} - 2 \le \tau_{\mathcal{O}} \le \frac{\tau_s}{2} + 2$.

Continuum Extrapolation (I)

- Continuum limit taken at fixed flow time t.
- Cutoff effects assumed to be $O(a^2)$.
- Results for individual lattice spacings and $a \to 0$ values shown in Figure.



Continuum Extrapolation (II)

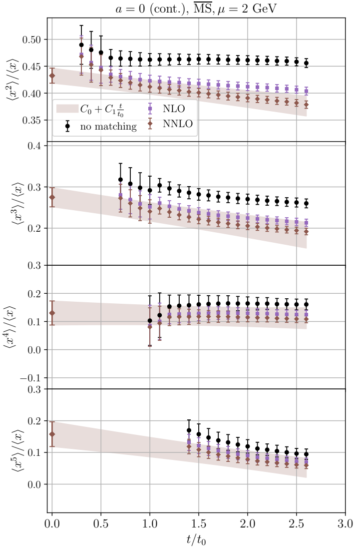
- Continuum-extrapolated ratios plotted vs. flow time t/t_0.

- Black: no matching.
- Purple: NLO matched.
- Brown: NNLO matched ($\overline{\text{MS}}$, $\mu = 2$ GeV).

- Final value from fits of the form:

$$C_0 + C_1 \frac{t}{t_0}$$

  over allowed NNLO ranges.
- Shaded band = envelope of acceptable NNLO fit windows.



Final Moment Ratios

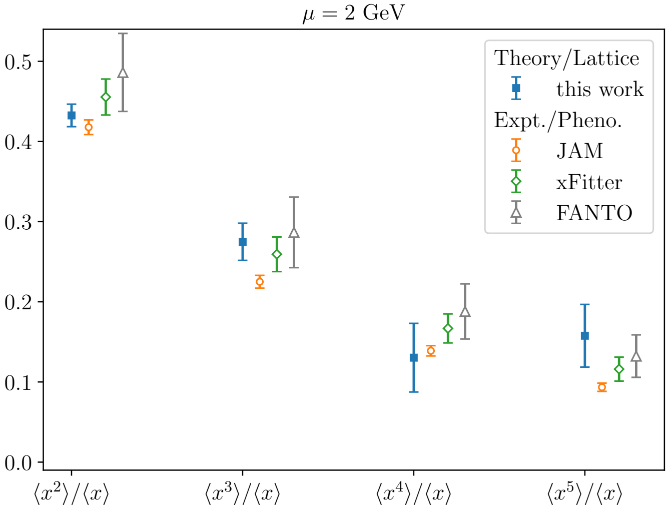
- Final results in $\overline{\text{MS}}$ at $\mu = 2$ GeV:

$$\frac{\langle x^2 \rangle}{\langle x \rangle} = 0.433(14)$$
$$\frac{\langle x^3 \rangle}{\langle x \rangle} = 0.275(23)$$
$$\frac{\langle x^4 \rangle}{\langle x \rangle} = 0.130(43)$$
$$\frac{\langle x^5 \rangle}{\langle x \rangle} = 0.158(39)$$

- Corresponding to $\langle x^{n-1} \rangle / \langle x \rangle$ for $n = 3, 4, 5, 6$.

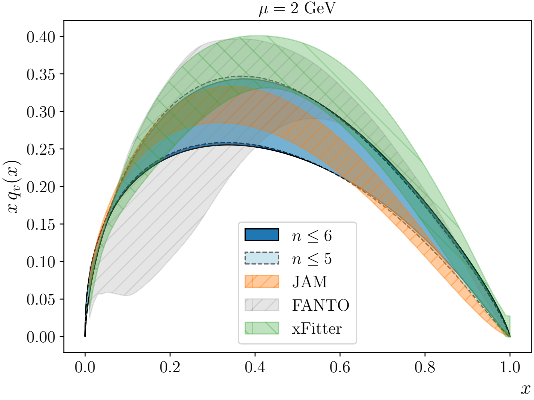
Comparison to phenomenological extractions

- Numerical agreement reported despite $m_\pi \simeq 411$ MeV (heavy).



PDF Reconstruction Test

- Ansatz: $q_v(x) \propto x^\alpha (1-x)^\beta$.
- Fit to analytic dependence of moments on α and β.
- Fit is weakly constrained — prior on α significantly affects result.
- Excluding $n = 6$ produces similar α and β values.
- Preferred $\beta \sim 1$.



Summary and Outlook

- First direct lattice QCD extraction of $\langle x^{n-1}\rangle/\langle x\rangle$ for $n = 3$–6.
- Gradient-flow method validated in practice and yields phenomenology-level precision.
- Future work:
  - Excited-state contamination
  - Cutoff effects and operator discretizations (hypercubic irreps)
  - Higher-order perturbative matching
- Possible extensions: proton PDFs, singlet and gluon PDFs, GPDs.

Thank you very much!