Comments on Summing over Bordisms in TQFT

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A. Banerjee and G. Moore

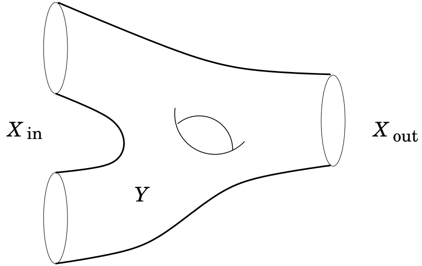
Recent works in QG, motivated by the factorization problem and baby universes, have considered sums over bordisms with fixed boundaries in TQFT. We discuss this construction and observe a curious splitting formula for the total amplitude.

1. TQFT
    1. TQFT as a functor
    2. $d = 1$ TQFT
    3. $d = 2$ TQFT
2. Sums over Bordisms

TQFT as a functor [Atiyah, Segal]

Category $\mathbf{Bord} = \mathbf{Bord}_{\langle d,d-1\rangle}$

- objects: $d-1$ dim. closed manifolds
- morphisms: d dim. $Y \in \mathrm{Hom}_{\mathbf{Bord}}(X_{\mathrm{in}}, X_{\mathrm{out}}) \Rightarrow \partial Y = X_{\mathrm{out}} \amalg X_{\mathrm{in}}$.



Category $\mathbf{Vect} = \mathbf{Vect}_k$

- objects: k-vector spaces
- morphisms: $f \in \mathrm{Hom}_{\mathbf{Vect}}(V, W) \iff f : V \to W$ は k-線形写像

一般の Category $\mathbf{C}$: 頂点と矢印, 矢印の合成

Functor $F : \mathbf{C} \to \mathbf{D}$ とは

- $\mathbf{C}$ の頂点を $\mathbf{D}$ の頂点に, $\mathbf{C}$ の矢印を $\mathbf{D}$ の矢印に写す
- 矢印の合成を保つ: $F(g \circ f) = F(g) \circ F(f)$

(素朴な) d dim. TQFT とは以下の functor である:

$$\mathcal{Z} : \mathbf{Bord} \to \mathbf{Vect}. \tag{1}$$

- $X \in \mathrm{obj}(\mathbf{Bord})$ に対して $\mathcal{Z}(X)$ は vector 空間
- $Y : X_{\mathrm{in}} \to X_{\mathrm{out}}$ に対して $\mathcal{Z}(Y) : \mathcal{Z}(X_{\mathrm{in}}) \to \mathcal{Z}(X_{\mathrm{out}})$ は k-線形写像
- morphisms の合成に対して

$Y_1 : X_0 \to X_1, \quad Y_2 : X_1 \to X_2,$ target と source が一致する bordisms

$$Y_2 \circ Y_1 = Y_2 \amalg_{X_1} Y_1 : X_0 \to X_2,$$

$$\Rightarrow \mathcal{Z}(Y_2 \circ Y_1) = \mathcal{Z}(Y_2) \circ \mathcal{Z}(Y_1) : \mathcal{Z}(X_0) \to \mathcal{Z}(X_2).$$

Symmetric monoidal functor (disjoint union を tensor 積に)

$$\mathcal{Z}(X_1 \amalg X_2 \amalg \cdots \amalg X_n) = \mathcal{Z}(X_1) \otimes_k \mathcal{Z}(X_2) \otimes_k \cdots \otimes_k \mathcal{Z}(X_n).$$

Bord の object として $\emptyset$ もある: $\mathcal{Z}(\emptyset) = k$.

- Y closed d-manifold $\Rightarrow Y \in \mathrm{Hom}_{\mathbf{Bord}}(\emptyset, \emptyset)$

$$\Rightarrow \mathcal{Z}(Y) \in \mathrm{Hom}_{\mathbf{Vect}}(k, k) \cong k$$

この値を分配関数と言う.

- $Y = X \times [0,1] \Rightarrow Y \in \mathrm{Hom}_{\mathbf{Bord}}(X, X)$

$$\Rightarrow \mathcal{Z}(Y) = \mathrm{id}_{\mathcal{Z}(X)} : \mathcal{Z}(X) \to \mathcal{Z}(X).$$

系の Hamiltonian$= 0$. 状態の変化は専ら空間の topology change で起こる.

- $\partial Y = X_{\mathrm{out}} \Rightarrow Y \in \mathrm{Hom}_{\mathbf{Bord}}(\emptyset, X_{\mathrm{out}})$

$$\Rightarrow \mathcal{Z}(Y) \in \mathrm{Hom}_{\mathbf{Vect}}(k, \mathcal{Z}(X_{\mathrm{out}})) \cong \mathcal{Z}(X_{\mathrm{out}})$$ [Hartle-Hawking].

- $\partial Y = X_{\mathrm{in}} \Rightarrow Y \in \mathrm{Hom}_{\mathbf{Bord}}(X_{\mathrm{in}}, \emptyset)$

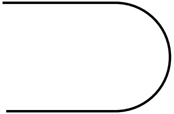
$$\Rightarrow \mathcal{Z}(Y) \in \mathrm{Hom}_{\mathbf{Vect}}(\mathcal{Z}(X_{\mathrm{in}}), k) = \mathcal{Z}(X_{\mathrm{in}})^{\vee}$$ 相関関数.

Claim $d = 1$ unoriented TQFT $\Leftrightarrow (V, b)$ $\quad \dim_k V < \infty$, b non-degenerate form.

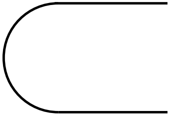
proof $\mathcal{Z}(\mathrm{pt}) = V$ とする.

bordisms

$\mathrm{id} : V \to V$



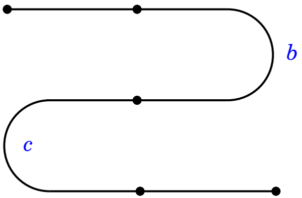
$b : V \otimes V \to k$



$c : k \to V \otimes V$

$$c(1)=\sum_{\alpha=1}^{n} x^\alpha \otimes x_\alpha \in V\otimes V.$$

S diagram



$\mathcal{Z}$(S diagram) :

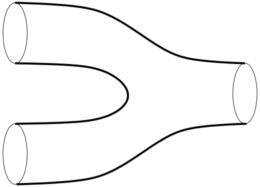
$v \mapsto \sum_\alpha b(v\otimes x^\alpha)x_\alpha = v$

$\{x^\alpha\}$, $\{x_\beta\}$ は V の互いに双対な基底で, $b(x^\alpha\otimes x_\beta)=\delta^\alpha_\beta$.

系 $\mathcal{Z}(S^1)=\dim_k V$.

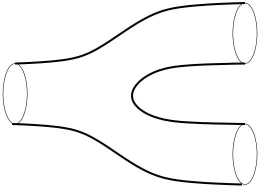
Claim $d = 2$ oriented TQFT $\Leftrightarrow (\mathcal{C} := \mathcal{Z}(S^1), m, \theta)$ 可換 Frobenius algebra

- mult.



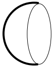
$m : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$

- comult.



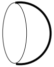
$\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$

- unit



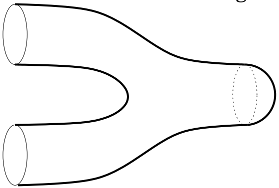
$k \to \mathcal{C}$

- trace

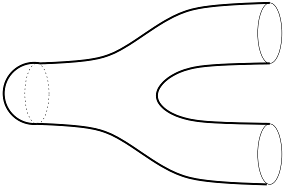


$\theta : \mathcal{C} \to k$

non–degenerate form



$$\mathcal{C} \otimes \mathcal{C} \xrightarrow{m} \mathcal{C} \xrightarrow{\theta} k$$

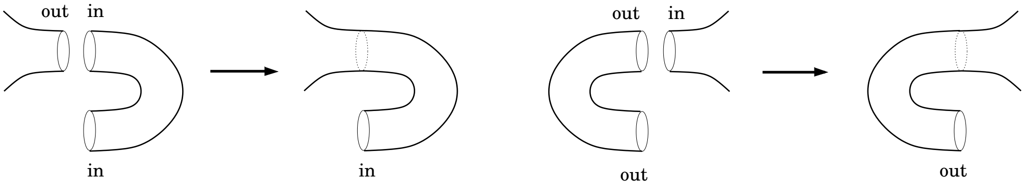


$$k \to C \xrightarrow{\Delta} \mathcal{C} \otimes \mathcal{C}$$

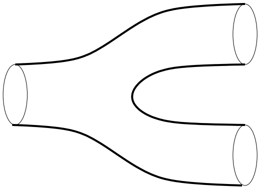
$$1 \mapsto \sum_\mu \phi^\mu \otimes \phi_\mu$$

$b := \theta \circ m$ は $\mathcal{C}$ の非退化 2 次形式で, $\{\phi^\mu\}$, $\{\phi_\nu\}$ は互いに双対な基底: $b(\phi^\mu \otimes \phi_\nu) = \delta^\mu_\nu$.

- input/output boundaries の交換



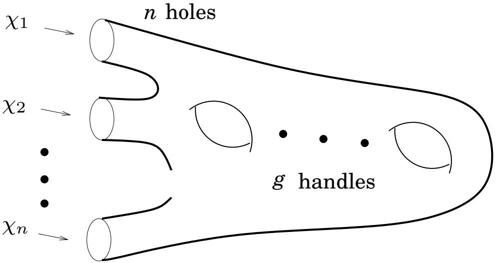
coproduct



$\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \otimes \mathcal{C}$:

$$\phi \mapsto \sum_\mu \phi \cdot \phi^\mu \otimes \phi_\mu$$
$$= \sum_\mu \phi^\mu \otimes \phi_\mu \cdot \phi$$

handle state $\quad h := \sum_\mu \phi^\mu \cdot \phi_\mu \in \mathcal{C}.$

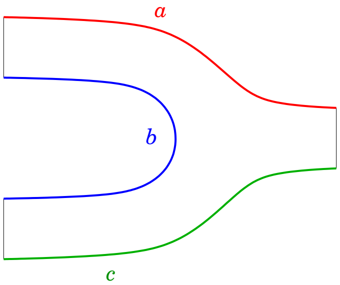


$$\chi_1 \otimes \cdots \otimes \chi_n \mapsto \theta(\chi_1 \cdots \chi_n \cdot h^g)$$

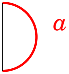
$d = 2$ open-closed TQFT [Moore-Segal, Costello]

Open sector $\mathcal{O}$ category $\mathbf{O}$:

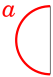
- objects = D-branes: $\mathrm{obj}(\mathbf{O}) = \{a, b, c, \ldots\}$
- morphisms: $\mathrm{Hom}_{\mathbf{O}}(b, a) = \mathcal{O}_{ab}$.



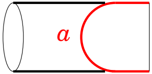
$$\mathcal{O}_{ab} \otimes \mathcal{O}_{bc} \to \mathcal{O}_{ac}$$

- trace 

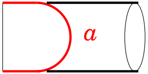
$\theta_a : \mathcal{O}_{aa} \to k$

- unit 

$k \to \mathcal{O}_{aa}$

- closed to open 

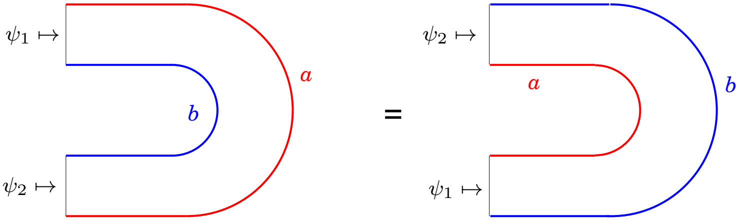
$\iota_a : \mathcal{C} \to \mathcal{O}_{aa}$

- open to closed 

$\iota^a : \mathcal{O}_{aa} \to \mathcal{C}$

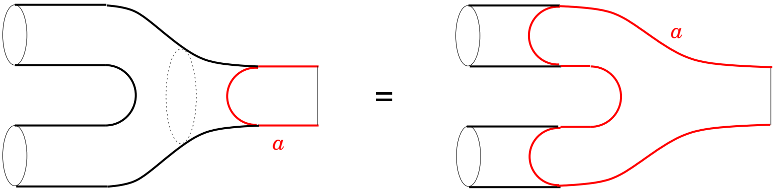
- $(\mathcal{O}_{aa}, \theta_a)$ $\Rightarrow$ 非可換 Frobenius algebra
- $\mathcal{O}_{ab}$ $\Rightarrow$ $\mathcal{O}_{aa}$-$\mathcal{O}_{bb}$ bi-module

Open 状態 $\psi_1 \in \mathcal{O}_{ab}$, $\psi_2 \in \mathcal{O}_{ba}$ の traces



$$\theta_a(\psi_1 \cdot \psi_2) = \theta_b(\psi_2 \cdot \psi_1).$$

- 写像 $\iota_a : \mathcal{C} \to \mathcal{O}_{aa}$ が algebras の準同型であること

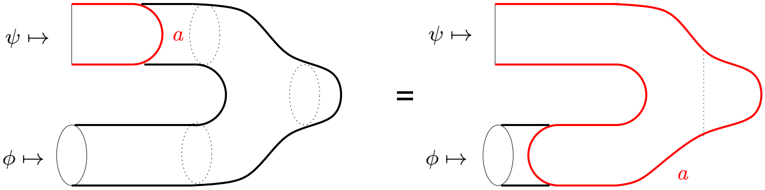


$$\iota_a(\phi_1 \cdot \phi_2) = \iota_a(\phi_1) \cdot \iota_a(\phi_2).$$

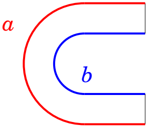
実は ι_a の像は非可換代数 $\mathcal{O}_{aa}$ の中心に入る: $\mathrm{Im}(\iota_a) \subset Z(\mathcal{O}_{aa})$.

- 写像 ι_a と ι^a が互いに随伴であること:

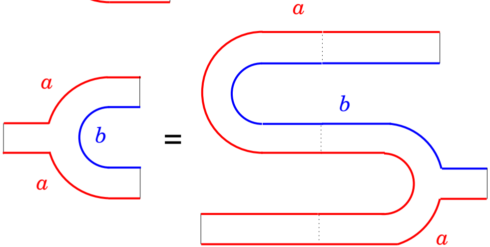
$\psi \in \mathcal{O}_{aa}$, $\phi \in \mathcal{C}$ とする.



$$\theta(\iota^a(\psi) \cdot \phi) = \theta_a(\psi \cdot \iota_a(\phi)).$$



$\sum_\mu \psi^\mu \otimes \psi_\mu \in \mathcal{O}_{ab} \otimes \mathcal{O}_{ba}$

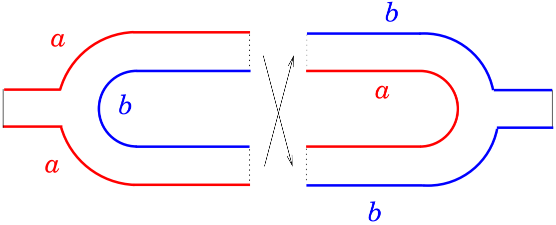


$\therefore$ coproduct

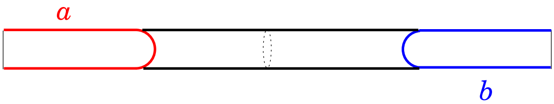
$$\mathcal{O}_{aa} \ni \psi \mapsto \sum_\mu \psi^\mu \otimes (\psi_\mu \cdot \psi).$$

Cardy condition $\quad \pi_b^a = \iota_b \circ \iota^a : \mathcal{O}_{aa} \to \mathcal{O}_{bb}$

π_b^a



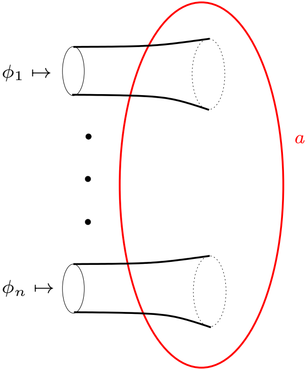
$\iota_b \circ \iota^a$



$$\pi_b^a : \mathcal{O}_{aa} \xrightarrow{\text{coprod.}} \mathcal{O}_{ab} \otimes \mathcal{O}_{ba} \xrightarrow{\text{ex.}} \mathcal{O}_{ba} \otimes \mathcal{O}_{ab} \xrightarrow{\text{prod.}} \mathcal{O}_{bb}$$

$$\psi \mapsto \sum_\mu \psi^\mu \otimes \psi_\mu \cdot \psi \mapsto \sum_\mu \psi_\mu \cdot \psi \otimes \psi^\mu \mapsto \sum_\mu \psi_\mu \cdot \psi \cdot \psi^\mu$$

- D-brane が ∂Y で閉曲線を成す場合

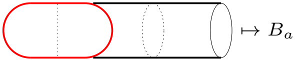


$$\theta_a(\iota_a(\phi_1)\cdots\iota_a(\phi_n))$$

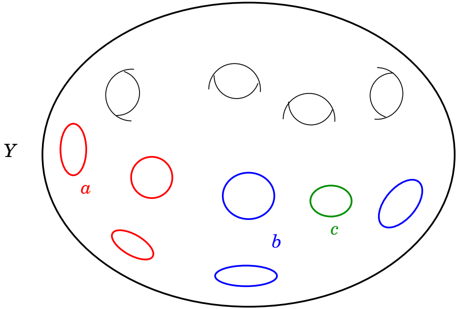
$$= \theta_a(1_{\mathcal{O}_{aa}}\cdot\iota_a(\phi_1\cdots\phi_n))$$

$$= \theta(\iota^a(1_{\mathcal{O}_{aa}})\cdot\phi_1\cdots\phi_n)$$

Cardy state: $B_a = \iota^a(1_{\mathcal{O}_{aa}}) \in \mathcal{C}$

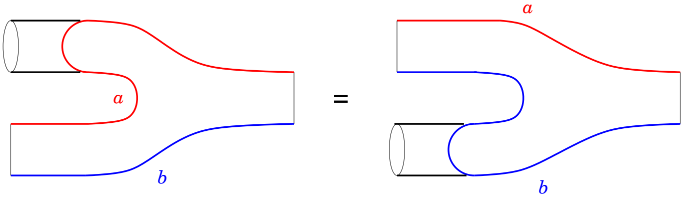


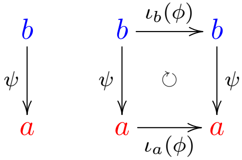
Bordism $Y : \emptyset \to \emptyset$ が handles g 個, holes $\sum_{a \in \mathrm{obj}(\mathbf{O})} m_a$ 個を持つ面とせよ:



$$\mathcal{Z}(Y) = \theta(h^g \prod_{a \in \mathrm{obj}(\mathbf{O})} (B_a)^{m_a}).$$

$\phi \in \mathcal{O}$ と $\psi \in \mathcal{O}_{ab}$ との積　$\iota_a(\phi) \cdot \psi = \psi \cdot \iota_b(\phi) \in \mathcal{O}_{ab}$:





各 $\phi \in \mathcal{C}$ は $\mathbf{O}$ の恒等関手間の自然変換　$\mathrm{Id}_{\mathbf{O}} \stackrel{\phi}{\Longrightarrow} \mathrm{Id}_{\mathbf{O}}$ を与える.

Sum over bordisms in $d = 2$ closed TQFT

$$\mathrm{obj}(\mathbf{Bord}) \ni \underbrace{S^1 \amalg \cdots \amalg S^1}_{n} \Leftrightarrow n.$$

$$\bar{\mathcal{A}}(m,n) = \sum_{Y:m\to n} \frac{1}{|\operatorname{Aut}(Y)|} \mathcal{Z}(m,n) \in \operatorname{Hom}_k(S^m\mathcal{C}, S^n\mathcal{C}).$$

- $\emptyset \to \emptyset$ amplitude

$$\bar{\mathcal{A}}^{\text{connected}}(0,0) = \theta(1) + \theta(h) + \theta(h^2) + \cdots = \theta\left(\frac{1}{1-h}\right),$$
$$\bar{\mathcal{A}}(0,0) = \exp\left[\theta\left(\frac{1}{1-h}\right)\right].$$

$\theta : \mathcal{C} \to k$ は trace map, h は handle state.

可換 Frobenius 代数の分類

(あ) semi-simple

(い) semi-simple でない

Semi-simple 代数

$$\mathcal{C} = \bigoplus_{x \in X} k\, \epsilon_x, \quad \epsilon_x \cdot \epsilon_y = \begin{cases} \epsilon_x & y = x \\ 0 & y \neq x \end{cases},$$

ここで $X = \mathrm{Spec}(\mathcal{C})$ [Gelfand] は有限集合.

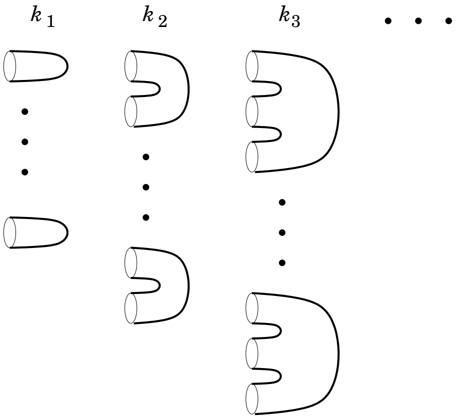
$$\text{handle state} \quad h = \sum_{x \in X} \theta_x^{-1} \epsilon_x, \quad \theta_x = \theta(\epsilon_x) \in k^*.$$

特に $\dim_k \mathcal{C} = 1$ の場合 [Marolf-Maxfield]

$$\mathcal{C} = k\,\epsilon_x \quad (X = \operatorname{Spec}\mathcal{C} = \{x\})$$

$$\epsilon_x \cdot \epsilon_x = \epsilon_x, \quad \theta(\epsilon_x) = \theta_x, \quad h = \theta_x^{-1}\epsilon_x.$$

$$\begin{aligned}
&\bar{\mathcal{A}}^{\text{connected}}(n,0)(\epsilon_x, \cdots, \epsilon_x) \\
&= \sum_{g=0}^{\infty} \theta(h^g \epsilon_x^n) = \sum_{g=0}^{\infty} \theta_x^{-g}\theta(\epsilon_x) = \sum_{g=0}^{\infty} \theta_x^{1-g} \\
&= \frac{\theta_x}{1-\theta_x^{-1}} =: \lambda.
\end{aligned}$$



Bell polynomial

$$B_n(x_1,\ldots,x_n) = \sum_{\boldsymbol{k}} a_{\boldsymbol{k}}\, x_1^{k_1}\cdots x_n^{k_n}$$

$$\bar{A}(n,0)(\epsilon_x,\cdots,\epsilon_x) = e^{\lambda} B_n(\lambda) = \sum_{d=0}^{\infty} \frac{\lambda^d}{d!} d^n$$

Note $\quad e^{\lambda} = \bar{A}(0,0)$

U 字パイプ b で input の一部を output に替える:

$$\bar{\mathcal{A}}(m+n,0)=e^{\lambda}B_{m+n}(\lambda)(\epsilon_x^{\vee})^{m+n}\Rightarrow\bar{\mathcal{A}}(m,n)=e^{\lambda}B_{m+n}(\lambda)(\epsilon_x^{\vee})^{m}\left(\frac{\epsilon_x}{\theta_x}\right)^{n}.$$

Total amplitude と splitting formula

$$\begin{aligned}\bar{\boldsymbol{\mathcal{A}}}:&=\sum_{m,n=0}^{\infty}\bar{\mathcal{A}}(m,n)=\sum_{m,n=0}^{\infty}(\epsilon_x^{\vee})^{m}\left(\sum_{d=0}^{\infty}\frac{\lambda^d}{d!}d^{m+n}\right)\left(\frac{\epsilon_x}{\theta_x}\right)^{n}\\&=\Phi\cdot\tilde{\Phi},\end{aligned}$$

$$\Phi=\sum_{n,d=0}^{\infty}\sqrt{\frac{\lambda^d}{d!}}d^n(\epsilon_x^{\vee})^n\left(\frac{\epsilon_x}{\sqrt{\theta_x}}\right)^{d},\quad\tilde{\Phi}=\sum_{n,d=0}^{\infty}\sqrt{\frac{\lambda^d}{d!}}d^n(\sqrt{\theta_x}\epsilon_x^{\vee})^{d}\left(\frac{\epsilon_x}{\theta_x}\right)^{n}.$$

dualization $\epsilon_x\in\mathcal{C}\leftrightarrow\theta_x\epsilon_x^{\vee}\in\mathcal{C}^{\vee}$.