Averaging over Narain Moduli Space

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Bulk/Boundary 対応

$$\text{2D JT Gravity} \iff \text{random ensemble of QMs}$$

$$\text{3D Gravity} \overset{?}{\iff} \text{random ensemble? of 2D CFTs}$$

$$\text{3D (exotic) Gravity} \overset{?}{\iff} \text{averaging 2D CFTs over } \mathcal{M}$$

T^D をターゲットとした sigma model

$$I = \frac{1}{4\pi\alpha'} \int d^2\sigma (G_{pq}\delta^{\alpha\beta} + iB_{pq}\epsilon^{\alpha\beta})\partial_\alpha X^p \partial_\beta X^q, \tag{1}$$

$$X^p \simeq X^p + 2\pi, \quad 1 \le p \le D.$$

central charge $(c_L, c_R) = (D, D)$,

$U(1)^D \times U(1)^D$ current algebra $J^p(z) = i\partial X^p(z), \bar{J}^p(\bar{z}) = i\bar{\partial} X^p(\bar{z})$.

Narain moduli space

$$\{E_{pq} = G_{pq} + B_{pq}\} \simeq SO(D)\times SO(D)\backslash SO(D,D), \tag{2}$$

$$\mathcal{M}_D \simeq SO(D)\times SO(D)\backslash SO(D,D)/SO(D,D;\mathbb{Z}). \tag{3}$$

$$\dim_{\mathbb{R}} \mathcal{M}_D = D^2.$$

Even unimodular (self-dual) lattice in $\mathbb{R}^{D,D}$

$$\begin{pmatrix} p_L^{\hat{a}} \\ p_R^{\hat{a}} \end{pmatrix} = \sum_{p=1}^{D} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\hat{a}p} \\ e^{\hat{a}p} \end{pmatrix} n_p + \sum_{q=1}^{D} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\hat{a}p} \hat{E}_{pq} \\ -e^{\hat{a}p} \hat{E}_{pq}^t \end{pmatrix} w^q, \tag{4}$$

$$e^{\hat{a}p} e^{\hat{a}q} = \hat{G}^{pq}, \quad n_p, w^q \in \mathbb{Z}.$$

genus one partition function $(X : T^2 \to T^D)$

$$Z(m,\tau) = \frac{\Theta(m,\tau)}{|\eta(q)|^{2D}}, \quad m \in \mathcal{M}_D,\ \tau \in \mathcal{H},$$

Dedekind eta function

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2\pi i \tau}.$$

Siegel-Narain theta function

$$\Theta(m,\tau) = \sum_{(p_L;p_R)} q^{p_L^2/2}\bar{q}^{p_R^2/2} = \sum_{\vec{n},\vec{w}\in\mathbb{Z}^D} Q(\vec{n},\vec{w};m,\tau), \tag{5}$$

$$Q(\vec{n},\vec{w};m,\tau) = \exp\left[2\pi i\tau_1 n_p w^p - \pi\tau_2(\alpha' G^{pr} v_p v_r + \frac{G_{qs}}{\alpha'} w^q w^s)\right], \tag{6}$$

$$v_p := n_p + \frac{B_{pq}}{\alpha'} w^q.$$

Upper half plane

$$\mathcal{H} = \{\tau = \tau_1 + i\tau_2 \,|\, \tau_2 > 0\},$$

$$ds^2 = \frac{d\tau_1^2 + d\tau_2^2}{\tau_2^2}, \quad \Delta_{\mathcal{H}} = -\tau_2^2 \left(\frac{\partial^2}{\partial \tau_1^2} + \frac{\partial^2}{\partial \tau_2^2} \right).$$

$PSL(2;\mathbb{Z})$ の作用

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \mapsto \gamma(\tau) = \frac{a\tau + b}{c\tau + d}.$$

Even unimodular lattice $\Rightarrow$ 分配関数の modular 不変性

$$\Theta\left(m, \frac{a\tau + b}{c\tau + d}\right) = |c\tau + d|^D \Theta(m, \tau), \quad Z\left(m, \frac{a\tau + b}{c\tau + d}\right) = Z(m, \tau).$$

$\mathcal{M}_D$ の Zamolodchikov metric

Marginal operator

$$\mathcal{O}(z) = (\delta\hat{G}_{pq} + \delta\hat{B}_{pq})\partial X^p(z)\bar{\partial}X^q(\bar{z}).$$

$$\Rightarrow \ \mathcal{O}(z)\mathcal{O}(w) \sim \frac{1}{(2\pi)^2}\frac{\hat{G}^{pr}\hat{G}^{qs}(\delta\hat{G}_{pq}\delta\hat{G}_{rs} + \delta\hat{B}_{pq}\delta\hat{B}_{rs})}{|z-w|^4} + \cdots,$$

$$\begin{aligned} ds^2 &= \hat{G}^{pr}\hat{G}^{qs}(d\hat{G}_{pq}d\hat{G}_{rs} + d\hat{B}_{pq}d\hat{B}_{rs}) \\ &= \mathrm{tr}(\hat{G}^{-1}d\hat{G}\hat{G}^{-1}d\hat{G} - \hat{G}^{-1}d\hat{B}\hat{G}^{-1}d\hat{B}). \end{aligned} \tag{7}$$

$D = 1$ の場合

$$I = \frac{R^2}{4\pi\alpha'} \int d^2\sigma \partial^\alpha X \partial_\alpha X, \quad G_{11} = R^2.$$

$$Z(R,\tau) = \frac{\Theta(R,\tau)}{|\eta(\tau)|^2}, \quad \Theta(R,\tau) = \sum_{n,w\in\mathbb{Z}} Q(n,w;R,\tau),$$

$$Q(n,w;R,\tau) = \exp\left[2\pi i\tau_1 nw - \pi\tau_2\left(\frac{\alpha' n^2}{R^2} + \frac{R^2 w^2}{\alpha'}\right)\right].$$

$$\mathcal{M}_1 = \{\hat{R} \,|\, \hat{R} \geq 1\}, \quad ds^2 = 4\frac{dR^2}{R^2}, \quad \Delta_{\mathcal{M}_1} = -\frac{1}{4}\left(R\frac{\partial}{\partial R}\right)^2.$$

$$\left[\tau_2^2\left(\frac{\partial^2}{\partial\tau_1^2}+\frac{\partial^2}{\partial\tau_2^2}\right)+\tau_2\frac{\partial}{\partial\tau_2}-\frac{1}{4}\left(R\frac{\partial}{\partial R}\right)^2\right]Q=0,$$

$$\therefore\left[\tau_2^2\left(\frac{\partial^2}{\partial\tau_1^2}+\frac{\partial^2}{\partial\tau_2^2}\right)+\tau_2\frac{\partial}{\partial\tau_2}-\frac{1}{4}\left(R\frac{\partial}{\partial R}\right)^2\right]\Theta(R,\tau)=0.$$

$$\left(\Delta_{\mathcal{H}}-\tau_2\frac{\partial}{\partial\tau_2}-\Delta_{\mathcal{M}_1}\right)\Theta(R,\tau)=0. \tag{8}$$

$Z(R,\tau)$ を $\mathcal{M}_1$ 上で積分したい

$$F_1(\tau) = \int_{\sqrt{\alpha'}}^{\infty} 2\frac{dR}{R}\Theta(R,\tau).$$

$\Theta \sim R$ $(R \to \infty)$ より発散する.

$$\begin{aligned}\left[\tau_2^2\left(\frac{\partial^2}{\partial\tau_1^2}+\frac{\partial^2}{\partial\tau_2^2}\right)+\tau_2\frac{\partial}{\partial\tau_2}\right]F_1(\tau) &= \frac{1}{2}\int_1^{\infty}\frac{d\hat{R}}{\hat{R}}\left(\hat{R}\frac{\partial}{\partial\hat{R}}\right)^2\Theta(R,\tau)\\ &= \frac{1}{2}\int_1^{\infty}d\hat{R}\frac{\partial}{\partial\hat{R}}\left(\hat{R}\frac{\partial}{\partial\hat{R}}\Theta(R,\tau)\right)\end{aligned}$$

$\hat{R}=\infty$ の表面項が残る.

$D=2$ の場合 $(X : T^2 \to T^2)$

$$\begin{pmatrix}\hat{G}_{11} & \hat{G}_{12}\\ \hat{G}_{21} & \hat{G}_{22}\end{pmatrix} = \frac{\rho_2}{\varphi_2}\begin{pmatrix}|\varphi|^2 & \varphi_1\\ \varphi_1 & 1\end{pmatrix}, \quad \begin{pmatrix}\hat{B}_{11} & \hat{B}_{12}\\ \hat{B}_{21} & \hat{B}_{22}\end{pmatrix} = \begin{pmatrix}0 & \rho_1\\ -\rho_1 & 0\end{pmatrix}.$$

$$ds^2 = 2\frac{d\varphi_1^2 + d\varphi_2^2}{\varphi_2^2} + 2\frac{d\rho_1^2 + d\rho_2^2}{\rho_2^2}.$$

$\mathcal{M}_2 = (SL(2;\mathbb{Z})\backslash\mathcal{H})^2/\mathbb{Z}_2\times\mathbb{Z}_2$

$\mathrm{vol}(\mathcal{M}_2) = (\pi/3)^2 < \infty$, しかし $\Theta(\varphi,\rho,\tau)$ の積分は発散:

$$\int_{\mathcal{M}_2} \frac{d\varphi_1 d\varphi_2 d\rho_1 d\rho_2}{\varphi_2^2\rho_2^2}\,\Theta(\varphi,\rho,\tau) = \infty.$$

$Q = q^{p_L^2/2}\bar{q}^{p_R^2/2}$ として

$$\left[-\tau_2^2\left(\frac{\partial^2}{\partial\tau_1^2}+\frac{\partial^2}{\partial\tau_2^2}\right)-2\tau_2\frac{\partial}{\partial\tau_2}+\frac{1}{2}\varphi_2^2\left(\frac{\partial^2}{\partial\varphi_1^2}+\frac{\partial^2}{\partial\varphi_2^2}\right)+\frac{1}{2}\rho_2^2\left(\frac{\partial^2}{\partial\rho_1^2}+\frac{\partial^2}{\partial\rho_2^2}\right)\right]Q=0.$$

つまり

$$\left(\Delta_{\mathcal{H}}-2\tau_2\frac{\partial}{\partial\tau_2}-\Delta_{\mathcal{M}_2}\right)Q=0. \tag{9}$$

$\Theta(\varphi,\rho,\tau)=\sum_{\text{格子点}}Q$ も同じ微分方程式に従う.

$D > 2$ の場合

$$F_D(\tau) = \int_{\mathcal{M}_D} d\mu(m)\Theta(m,\tau) < \infty, \quad \int_{\mathcal{M}_D} d\mu(m) = 1. \tag{10}$$

$$\left(\Delta_{\mathcal{H}} - D\tau_2 \frac{\partial}{\partial \tau_2} - \Delta_{\mathcal{M}_D}\right)\Theta(m,\tau) = 0, \tag{11}$$

$$\Rightarrow \left(\Delta_{\mathcal{H}} - D\tau_2 \frac{\partial}{\partial \tau_2}\right) F_D(\tau) = 0. \tag{12}$$

$$\lim_{\tau_2 \to \infty} F_D(\tau) = 1. \tag{13}$$

$$F_D\left(\frac{a\tau+b}{c\tau+d}\right)=|c\tau+d|^D F_D(\tau),\quad \mathrm{Im}\left(\frac{a\tau+b}{c\tau+d}\right)=\frac{\mathrm{Im}\,\tau}{|c\tau+d|^2}.$$

Modular 不変な組み合わせ

$$W_D(\tau)=\tau_2^{D/2}F_D(\tau),\quad \tau_2=\mathrm{Im}\,\tau.$$

Laplacian の固有関数

$$(\Delta_{\mathcal{H}}+s(s-1))W_D(\tau)=0,\quad s=D/2. \tag{14}$$

$$W_D(\tau)\sim\tau_2^{D/2}+\mathcal{O}(\tau_2^{1-D/2}),\quad \tau_2\to\infty.$$

主張 $W_D(\tau)$ は以下の $E_s(\tau)$, $s = D/2$, に一致する.

$$E_s(\tau) = \sum_{(c,d)=1} \frac{\tau_2^s}{|c\tau + d|^{2s}}. \tag{15}$$

$(c,d) = 1$ ならば ある (a,b) があって

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{Z}), \quad \frac{\tau_2^s}{|c\tau + d|^{2s}} = (\mathrm{Im}(\gamma\tau))^s.$$

(a,b) の不定性は $(a,b) \mapsto (a,b) + n(c,d)$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a + nc & b + nd \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \mapsto \frac{a\tau + b}{c\tau + d} + n$$

$\operatorname{Im}\tau (= \tau_2)$ への作用は (a, b) のとり方に依らず, (c, d) だけで決まる.

$$E_s(\tau) = \sum_{\gamma \in P \backslash SL(2;\mathbb{Z})} (\operatorname{Im}(\gamma\tau))^s, \quad P = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right\} \subset SL(2;\mathbb{Z}).$$

$E_{s=D/2}(\tau)$ と $W_D(\tau)$ との共通点

- $(\operatorname{Im}\tau)^s$ の modular-images の和だから, $E_s(\tau)$ も modular invariant.
- $(\Delta_{\mathcal{H}} + s(s-1))\tau_2^s = 0 \;\Rightarrow\; (\Delta_{\mathcal{H}} + s(s-1))E_s(\tau) = 0.$ $\Delta_{\mathcal{H}}$ は $SL(2;\mathbb{R})$-不変
- $E_s(\tau) \sim \tau_2^s + \mathcal{O}(\tau_2^{1-s}) \quad (\tau_2 \to \infty).$

$W_D(\tau) - E_{D/2}(\tau)$: $\Delta_{\mathcal{H}}$ 固有値 $-s(s-1) < 0$ 固有関数,

$|W_D(\tau) - E_{D/2}(\tau)| \sim C\tau_2^{1-s}$ $(\tau_2 \to \infty) \Rightarrow SL(2;\mathbb{Z})\backslash\mathcal{H}$ 上 2 乗可積分,

$$\therefore\ W_D(\tau) - E_{D/2}(\tau) = 0. \qquad \square$$

従って CFT 分配関数の $\mathcal{M}_D$ 上平均値は

$$\langle Z(m,\tau)\rangle = \int_{\mathcal{M}_D} d\mu(m) Z(m,\tau) = \frac{E_{D/2}(\tau)}{\tau_2^{D/2}|\eta(\tau)|^{2D}}. \tag{16}$$

主張 bulk 理論は以下の $U(1)^{2D}$ Chern-Simons 理論の作用を持つ重力理論

$$I_{\mathrm{CS}} = \sum_{I,J=1}^{2D} \frac{\Lambda_{I,J}}{2\pi} \int_Y A_I \wedge dA_J = D \text{ copies of } \frac{1}{2\pi} \int_Y A \wedge dB, \tag{17}$$

$$(\Lambda_{I,J}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{(even integral unimodular form).}$$

- 忠実な $U(1)^{2D}$ Chern-Simons 理論の量子化では無い
- $\partial Y^{(3)} = \Sigma^{(2)}$ なる $Y^{(3)}$

Chern-Simons 理論の経路積分

$$I_{\mathrm{CS}}^{U(1)} = \frac{1}{2\pi}\int_Y A\wedge dA \Rightarrow \frac{1}{2\pi}\int_Y A\wedge dA + \frac{1}{2\pi}\int_Y d^3x\sqrt{g}(\phi D_i A^i + \bar{c} D_i D^i c).$$

$L = *d \pm d* : \Omega^1 \oplus \Omega^3 \mapsto \Omega^1 \oplus \Omega^3$ として

$$\begin{aligned}&\frac{1}{2\pi}\int_Y A\wedge dA + \frac{1}{2\pi}\int_Y d^3x\sqrt{g}\phi D_i A^i \\ &= \frac{1}{2\pi}\int_Y d^3x\sqrt{g}\langle A + \tfrac{1}{2}*\phi, L(A+\tfrac{1}{2}*\phi)\rangle_g.\end{aligned}$$

更に

$$L^2 = \Delta_1 \oplus \Delta_3.$$

$$\therefore\ Z_{\mathrm{CS}}^{U(1)} = \frac{\det' \Delta_0}{\sqrt{\det' L}} = \frac{(\det' \Delta_0)^{3/4}}{(\det' \Delta_1)^{1/4}}.$$

$A_\pm = (B \pm A)/\sqrt{2}$ とおくと

$$I_{\mathrm{CS}}^{D=1} = \frac{1}{2\pi}\int_Y A \wedge dB = \frac{1}{2\pi}\int_Y (A_+ \wedge dA_+ - A_- \wedge dA_-),$$

$$\therefore\ Z_{\mathrm{CS}}^{D=1} = \frac{(\det' \Delta_0)^{3/2}}{(\det' \Delta_1)^{1/2}}. \tag{18}$$

右辺は Ray-Singer 解析的 torsion (位相不変量).

$\Delta_{0,1}$ zero modes は考慮しない.

境界 $T^2(\tau)$ $\Rightarrow$ 経路積分で足す Y は双曲的な solid torus.

(pure gravity の saddle points と同じ)

solid torus の構成

- 双曲的空間 (Euclidean AdS_3)

$$\mathbb{H}_3 = \left\{(x, y, z) \in \mathbb{R}^3 \,|\, z > 0\right\}, \quad ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}, \quad R_{ij} = -2\, g_{ij}.$$

座標変換

$$(x, y, z) = \left(e^t \tanh r \cos\phi, e^t \tanh r \sin\phi, \frac{e^t}{\cosh r}\right)$$

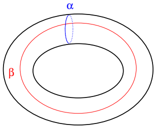
$$ds^2 = \cosh^2 r\, dt^2 + dr^2 + \sinh^2 r\, d\phi^2. \tag{19}$$

- solid torus $= \mathbb{H}_3/\mathbb{Z}$, $\mathbb{Z} \subset PSL(2;\mathbb{C})$ (Möbius 変換群).

solid torus の分類

$$T^2 \cong \boxed{S^1} \times S^1 \ \rightarrow \ \boxed{D^2} \times S^1.$$

- $Y_{0,1}$ (thermal AdS_3): $\phi + it \sim \phi + it + 2\pi \sim \phi + it + 2\pi\tau.$
- $Y_{1,0}$ (BTZ black hole): $\phi + it \sim \phi + it + 2\pi \sim \phi + it - 2\pi\frac{1}{\tau}.$
- $Y_{c,d}$ ($c\beta + d\alpha$ が可縮): $\phi + it \sim \phi + it + 2\pi \sim \phi + it + 2\pi\frac{a\tau+b}{c\tau+d}.$



熱核の方法で $\det{}' \Delta$ を求める.

$$K_t(x,x') = \sum_n e^{-\lambda_n t}\psi_n(x)\psi_n(x'), \quad \Delta\psi_n = \lambda_n\psi_n,$$

$$\frac{\partial}{\partial t}K_t + \Delta K_t = 0, \quad K_0(x,x') = \delta(x,x').$$

$$-\log\det\Delta = \int_0^\infty \frac{dt}{t}\int_Y d^3x\sqrt{g}\,K_t(x,x).$$

$\mathbb{H}_3$, $\Delta = \Delta_0$ の場合

$$K_t^{\mathbb{H}_3}(x,x') = \frac{e^{-t-\frac{d^2}{4t}}}{(4\pi t)^{3/2}}\frac{d}{\sinh d}, \quad d = d(x,x').$$

ハンドル体 $Y = \mathbb{H}_3/G$, Schottky 群 $G \subset PSL(2;\mathbb{C})$ の場合

$$K_t^{\mathbb{H}_3/G}(x,x') = \sum_{\gamma\in G} K_t^{\mathbb{H}_3}(x,\gamma(x')).$$

$$\det{}'\Delta_0 = \exp\left(-\frac{\mathrm{vol}(\mathbb{H}_3/G)}{6\pi}\right)\prod_{\gamma\in\mathcal{P}}\prod_{l,l'=1}^{\infty}(1-q_\gamma^l\bar{q}_\gamma^{l'})^2,$$

$$\det\Delta_1 = \det{}'\Delta_0 \prod_{\gamma\in\mathcal{P}}\prod_{l,l'=0}^{\infty}(1-q_\gamma^l\bar{q}_\gamma^{l'+1})^2(1-q_\gamma^{l+1}\bar{q}_\gamma^{l'})^2.$$

$$\frac{(\det{}'\Delta_0)^{3/2}}{(\det{}'\Delta_1)^{1/2}} = \exp\left(-\frac{\mathrm{vol}(\mathbb{H}_3/G)}{6\pi}\right)\prod_{\gamma\in\mathcal{P}}\prod_{n=1}^{\infty}\frac{1}{|1-q_\gamma^n|^2} \quad (20)$$

特に solid torus $Y_{0,1}$ の場合　$G \cong \mathbb{Z}$: $(\phi, t) \mapsto (\phi + 2\pi\tau_1, t + 2\pi\tau_2)$.

$$\mathrm{vol}(\mathbb{H}_3/\mathbb{Z}) = -\pi^2 \,\mathrm{Im}\,\tau \quad \text{(holographic renorm.)}$$

$$Z_{\mathrm{CS}}^{D=1}(Y_{0,1}) = \frac{(\det{}' \Delta_0)^{3/2}}{(\det{}' \Delta_1)^{1/2}} = |q|^{-1/12} \prod_{n=1}^{\infty} \frac{1}{|1-q^n|^2} = \frac{1}{|\eta(\tau)|^2}. \tag{21}$$

$$\begin{aligned} Z_{\mathrm{bulk}}(\tau) &= \sum_{(c,d)=1} Z_{\mathrm{CS}}^{D}(Y_{c,d}) = \sum_{\gamma \in P \backslash SL(2;\mathbb{Z})} \frac{1}{|\eta(\gamma(\tau))|^{2D}} \\ &= \frac{E_{D/2}(\tau)}{\tau_2^{D/2} |\eta(\tau)|^{2D}} = \langle Z(m,\tau) \rangle_{\mathrm{boundary}}. \end{aligned} \tag{22}$$