“Complex Langevin analysis of 2D U(1) gauge theory on a torus with a θ term”

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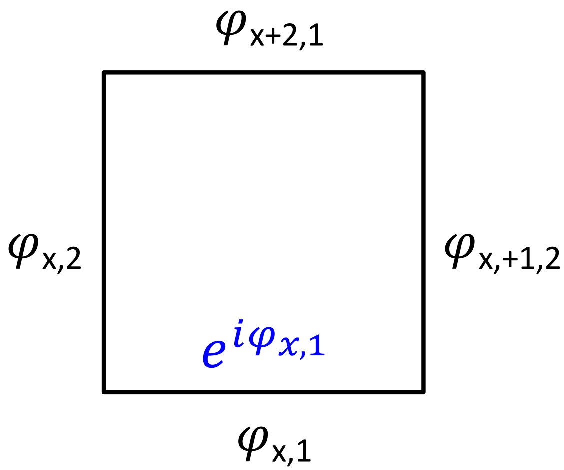
2D pure U(1) gauge theory w/ θ-term

Lattice action of 2D pure U(1) gauge theory w/ θ-term

$$S = -\beta \sum_x \cos p_x - i\theta Q$$

$$p_x = \varphi_{x,1} + \varphi_{x+\hat{1},2} - \varphi_{x+\hat{2},1} - \varphi_{x,2}$$

$$Q = \frac{1}{2\pi}\sum_x q_x, \quad q_x = p_x \ \mathrm{mod}\ 2\pi$$



Periodic boundary condition ⇒ Q is integer

Analytic result

$$Z_{\mathrm{analytic}} = \sum_{Q=-\infty}^{\infty} \left(z_{\mathrm{P}}(\theta + 2\pi Q, \beta)\right)^V,$$

Wiese, NPB318(1989)153

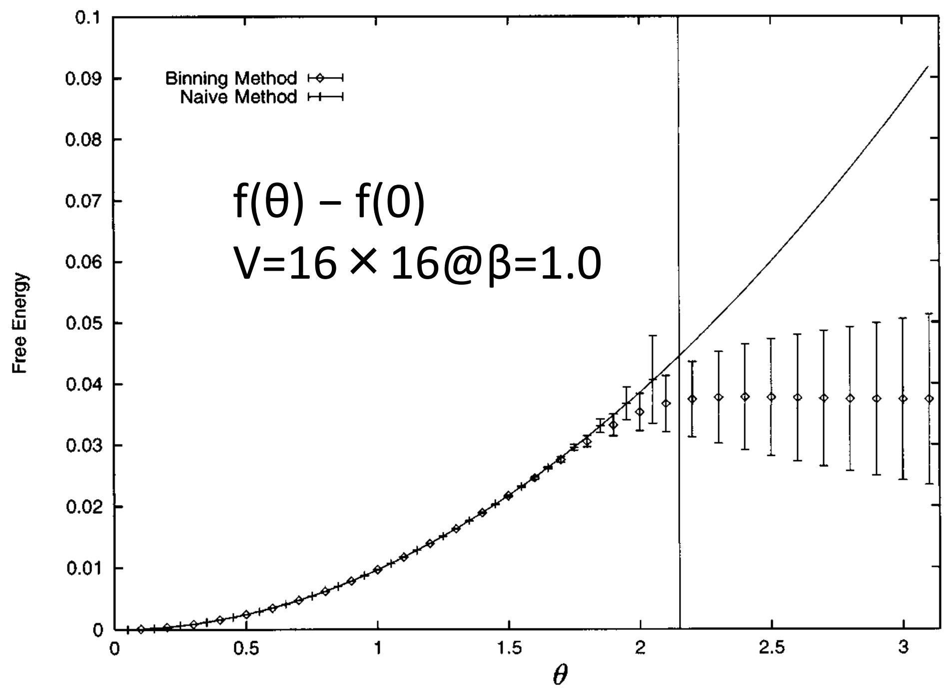
$$z_{\mathrm{P}}(\theta,\beta) = \int_{-\pi}^{\pi} \frac{d\varphi_{\mathrm{P}}}{2\pi} \exp\left(\beta\cos\varphi_{\mathrm{P}} + i\frac{\theta}{2\pi}\varphi_{\mathrm{P}}\right)$$

Predict a first order phase transition at $\theta=\pi$

Sign (complex action) problem

Conventional Monte Carlo method w/ reweighting technique

Plefka-Samuel, PRD56(1997)44



symmetric w.r.t. $\theta=\pi$

Doesn't work for $\theta \gtrsim 0.7\pi$ (difficult near the transition point)

⇒ A typical model with a sign problem in two dimension

Complex Langevin Method (CLM)

Partition function with real variable x_i $(i = 1, \cdots, n)$

$$Z = \int \prod_{i=1}^{n} dx_i \; e^{-S(\{x_i\})}$$

Langevin equation

Parisi-Wu, Sci.Sinica24(1981)483

$$\frac{dx_i(\tau)}{d\tau} = -\frac{\partial S(\{x_i(\tau)\})}{\partial x_i} + \eta_i(\tau)$$

$$\langle \eta_i(\tau)\eta_j(\tau')\rangle = 2\delta_{ij}\delta(\tau - \tau')$$

Long Langevin time gives an expectation value of observable

$$\langle \mathcal{O} \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{\tau_0}^{\tau_0 + T} d\tau \; O(\{x_i(\tau)\})$$

Generalization to complex action

$$x_i \in \mathbb{R} \to z_i \in \mathbb{C}$$

$$\frac{dz_i(\tau)}{d\tau} = -\frac{\partial S(\{z_i(\tau)\})}{\partial z_i} + \eta_i(\tau)$$

2D U(1) gauge theory with CLM

Complex Langevin eq. for link variables

$$\mathcal{U}_{n,\mu}(t+\Delta t) = \mathcal{U}_{n,\mu}(t)\exp\left[i\Big\{-\Delta t\, D_{n,\mu}S + \sqrt{\Delta t}\,\eta_{n,\mu}(t)\Big\}\right]$$

$$D_{n,\mu}S = \lim_{\epsilon\to 0}\frac{S(e^{i\epsilon}U_{n,\mu}) - S(U_{n,\mu})}{\epsilon}$$

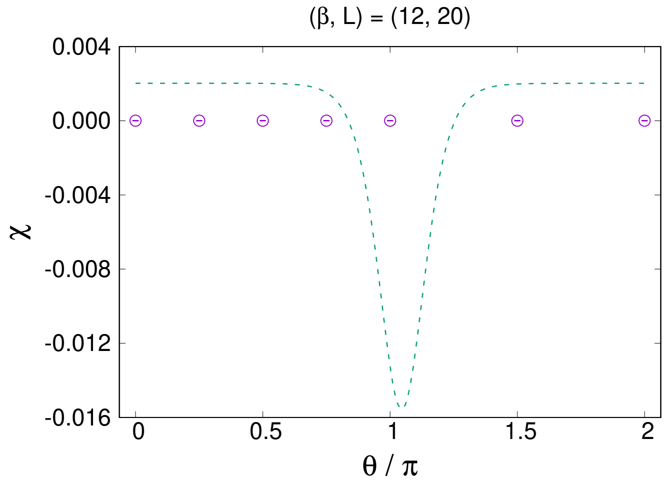
Two problems

(1) Topology freezing problem

(2) Drift term may become singular → wrong convergence problem

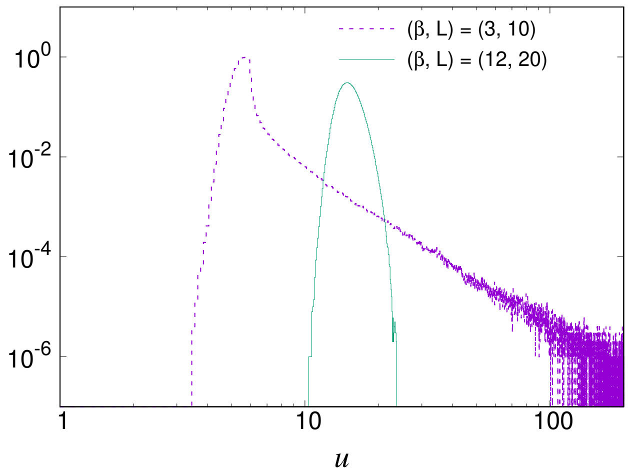
Naïve Implementation on T^2

(1) Topology freezing problem



$$\chi = \frac{1}{V}\left(\langle Q^2\rangle - \langle Q\rangle^2\right) = -\frac{1}{V}\frac{\partial^2}{\partial\theta^2}\log Z$$

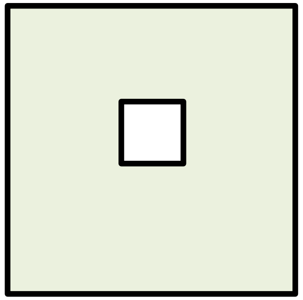
(2) Singular drift term problem



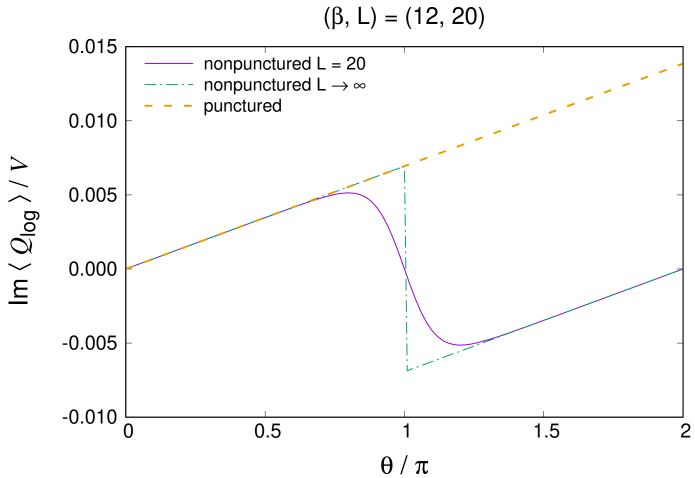
β = 12 is OK

β = 3 is untrustful

Punctured Model on T^2

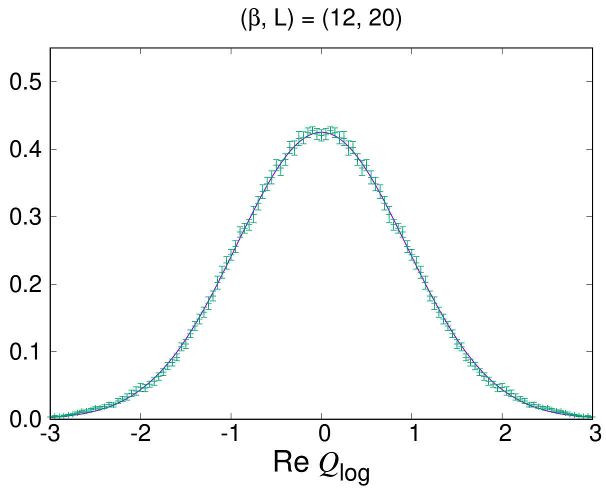


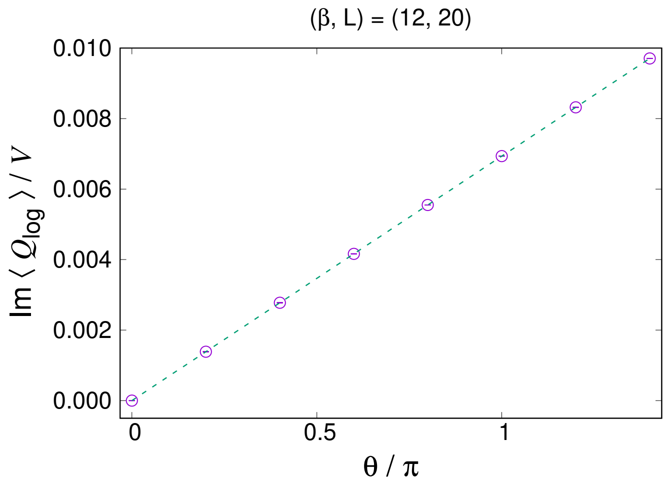
Remove one plaquette from T^2



Analytically, equivalent to T^2 for $0 \leq \theta \leq \pi$

Consistent with exact value → but, give up the first order phase transition





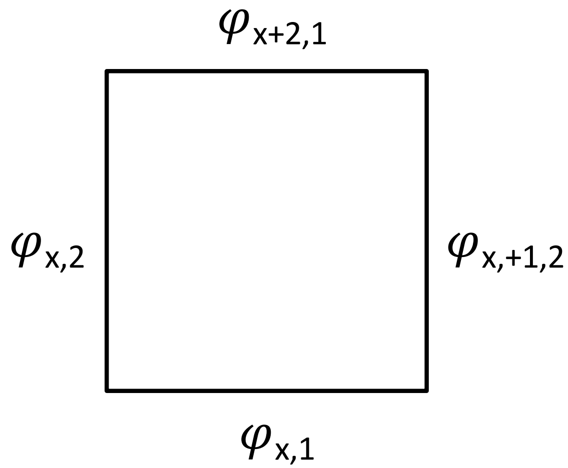
Tensor renormalization group (TRG) method

YK-Yoshimura, JHEP04(2020)089

Partition function w/ continuous indices

$$\mathcal{T}(\varphi_{x,1},\varphi_{x+\hat{1},2},\varphi_{x+\hat{2},1},\varphi_{x,2}) = \exp\left(\beta\cos p_x + i\frac{\theta}{2\pi}q_x\right)$$

$$Z = \left(\prod_{x,\mu}\int_{-\pi}^{\pi}\frac{d\varphi_{x,\mu}}{2\pi}\right)\prod_x \mathcal{T}(\varphi_{x,1},\varphi_{x+\hat{1},2},\varphi_{x+\hat{2},1},\varphi_{x,2}).$$



Discretized partition function w/ Gauss-Legendre quadrature

$$\int d\varphi f(\varphi) \approx \sum_{\alpha=1}^{K} w_\alpha f\left(\varphi^{(\alpha)}\right)$$

K-th order Legendre polynomial

$\varphi^{(\alpha)}$: α-th node, w_α : weight

$$T_{ijkl} = \frac{\sqrt{w_i w_j w_k w_l}}{(2\pi)^2}\mathcal{T}\left(\varphi^{(i)},\varphi^{(j)},\varphi^{(k)},\varphi^{(l)}\right)$$

$$Z \approx \sum_{\{\alpha\}}\prod_x T_{n_{x,1}n_{x+\hat{1},2}n_{x+\hat{2}}n_{x,2}}$$

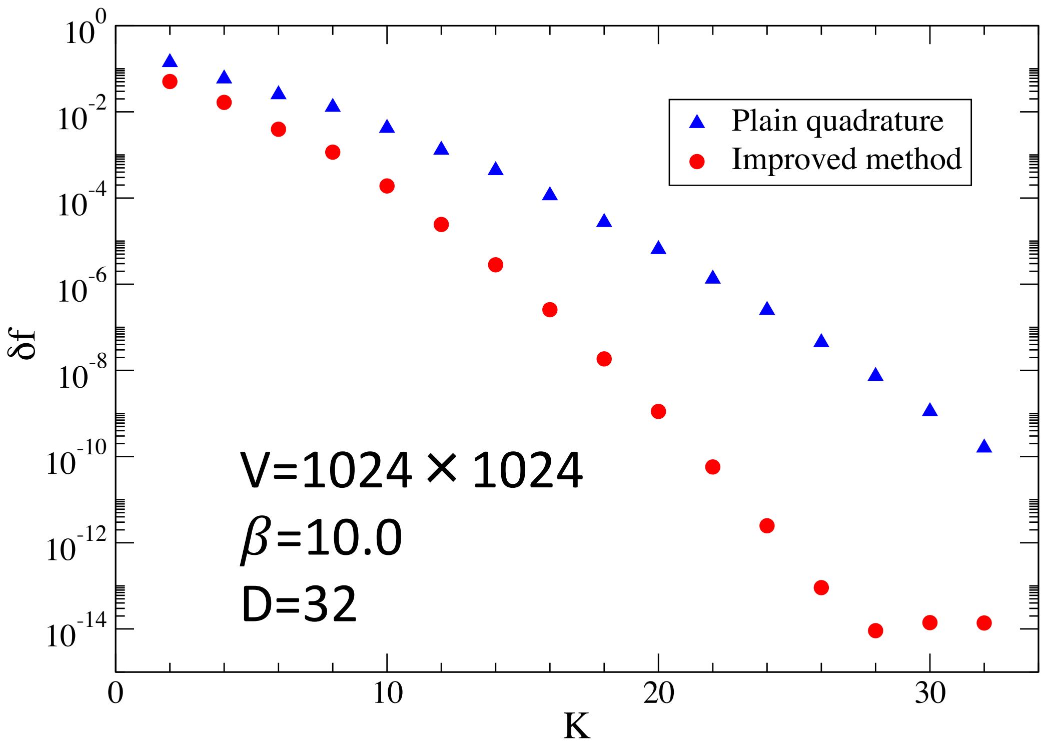
Free energy

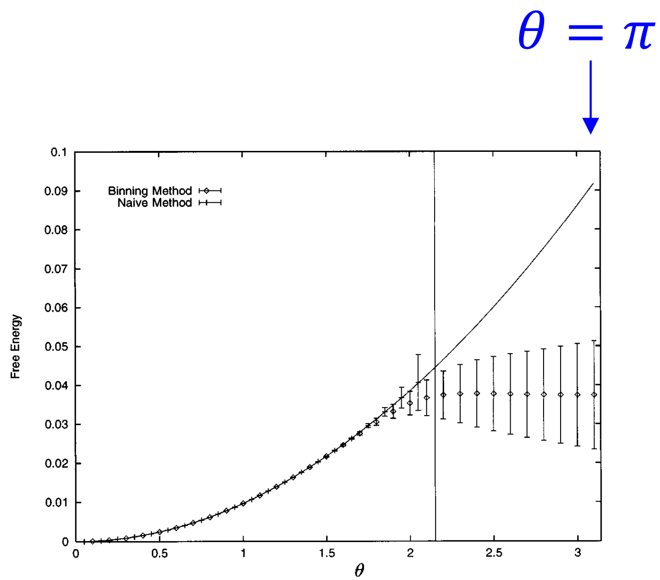
YK-Yoshimura, JHEP04(2020)089

Relative error

$$\delta f = \frac{|\ln Z_{\text{analytic}} - \ln Z(K, D=32)|}{|\ln Z_{\text{analytic}}|}$$

Comparison btw TRG results and analytic ones at $\theta=\pi$ (most difficult point)





$\delta f < 10^{-12}$@(D,K)=(32,32) ⇒ Sufficient precision

Topological charge density

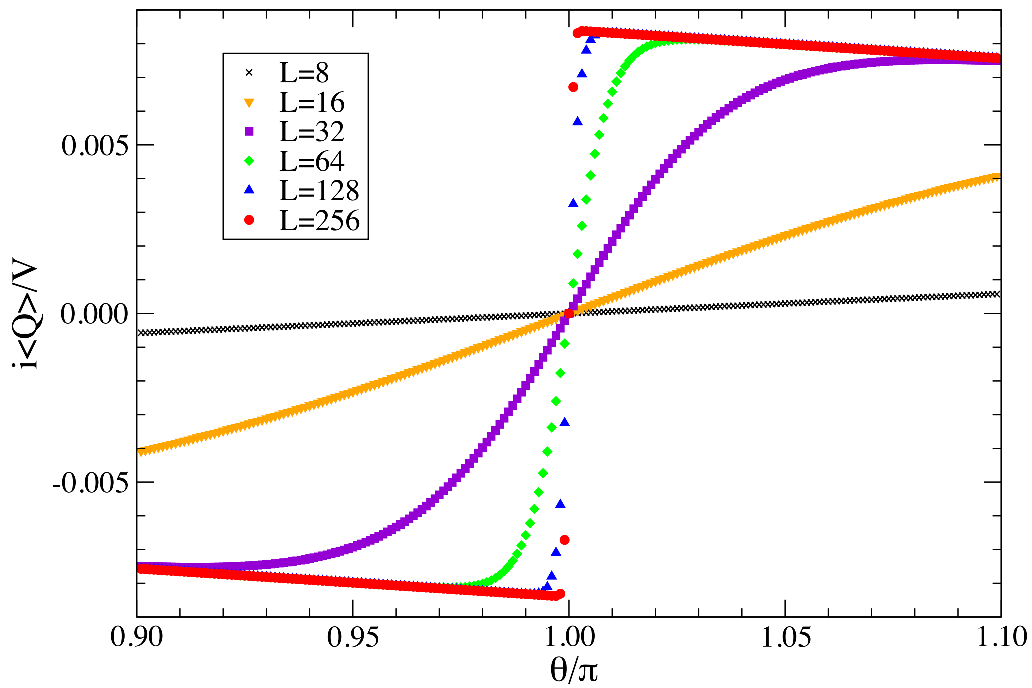
YK-Yoshimura, JHEP04(2020)089

Expectation value of topological charge

$$\langle Q \rangle = -i \frac{\partial \ln Z}{\partial \theta}.$$

Numerical derivative in terms of θ

Volume dependence of topological charge density around $\theta=\pi$



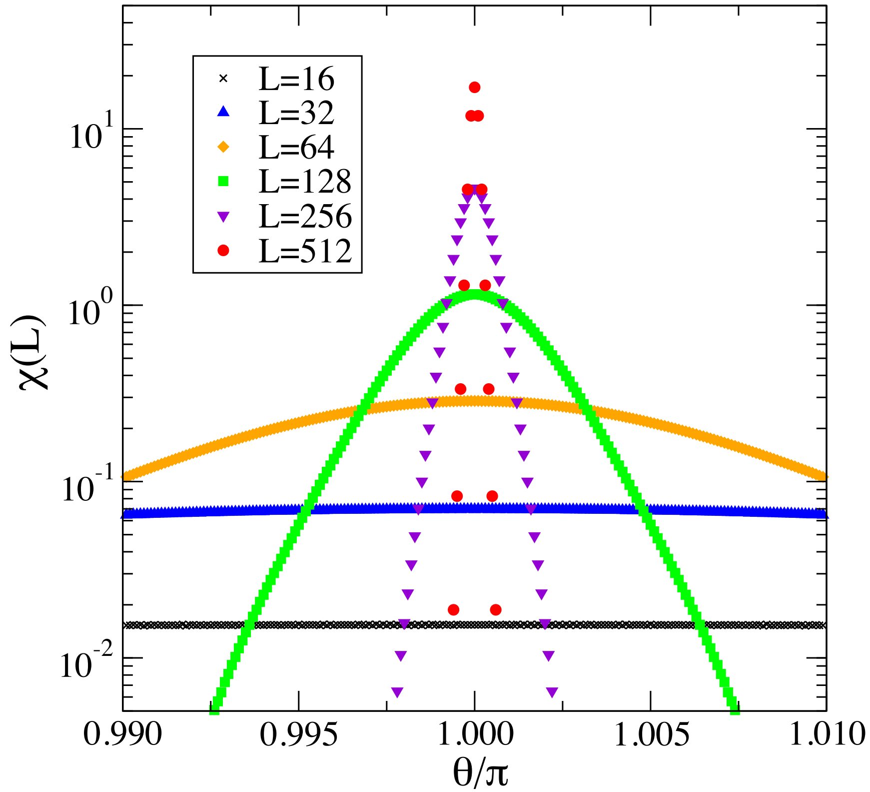
Finite gap is generated toward L→∞ ⇒ 1st order phase transition

Topological susceptibility

YK-Yoshimura, JHEP04(2020)089

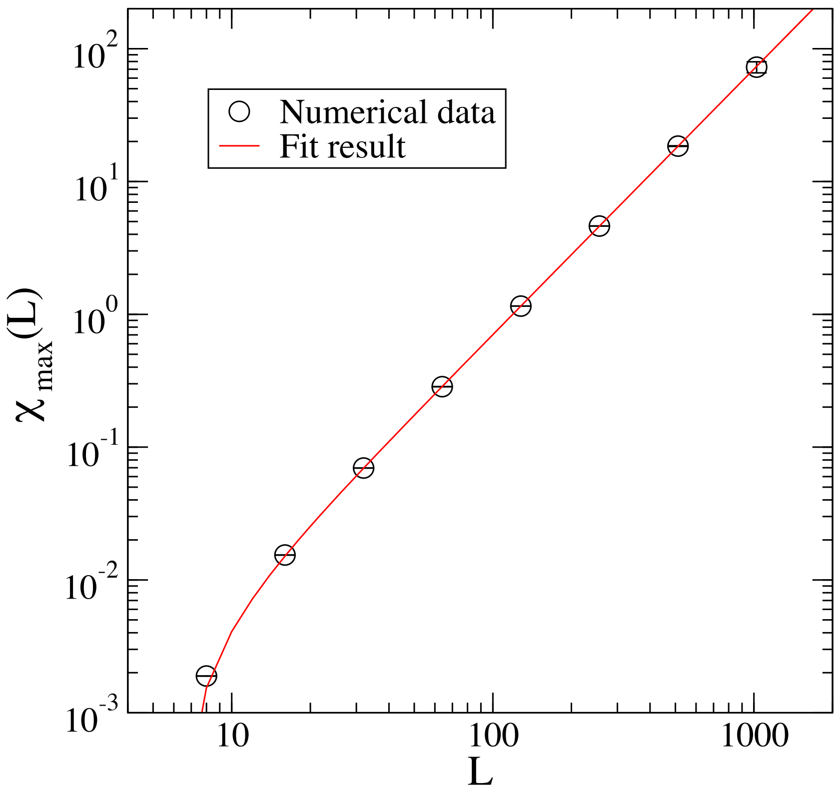
Topological susceptibility

$$\chi(L) = -\frac{1}{V}\frac{\partial^2 \ln Z}{\partial \theta^2}.$$



FSS of peak height

$$\chi_{\max}(L) = A + BL^{\gamma/\nu}$$



Critical exponent γ/ν=1.998(2) ⇒ 1st order phase transition