# Topological phase transitions in four dimensions

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Abstract

4D sine-Gordon model w/ higher-derivative term may exhibit

a topological phase transition analogous to the KT transition in 2D

# Lifshitz point

Hornreich-Luban-Shtrikman, PRL35(1975)1678

General form of the free energy density ($m$: scalar order parameter)

$$f(m) = a_2 m^2 + a_4 m^4 + a_6 m^6 + \cdots + c_1 (\partial m)^2 + c_2 (\partial^2 m)^2 + \cdots$$

| | |
|---|---|
| Critical point | described just by $(a_2, a_4, c_1)$ |
| Tri-critical point | characterized by the $a_6$ term<br>w/ the vanishing of $a_4$ |
| **Lifshitz point** | **characterized by the $c_2$ term<br>w/ the vanishing of $c_1$** |

-> the Lifshitz point is analogous to a tri-critical point

# Critical dimensions

✓ $d = 2$ is the lower critical dimension of the scalar field theory

-> there is no long-range ordered phase, but the KT transition occurs

✓ For the $d$-dim. scalar field theory,

$d = 4$ is the conjectured lower critical dimension of the Lifshitz point

**Q. Is there any KT(-like) transition in 4D higher-derivative scalar field theory?**

cf. Zappala, PRD98(2018)085005

# Kosterlitz-Thouless transition

| |
|---|
| $T > T_{\mathrm{KT}}$ ⇨ paramagnetic |
| $T < T_{\mathrm{KT}}$ ⇨ vortex & anti-vortex pair |

XY model (@ low $T$)

$$H = -J \sum_{<i,j>} \cos(\theta_i - \theta_j) \approx \frac{J}{2} \int \mathrm{d}\boldsymbol{r}\, (\partial\theta)^2$$

✔ $\langle \cos(\theta_r - \theta_0) \rangle = \left(\frac{r}{a}\right)^{-T/2\pi J}$ @ $T < T_{\mathrm{KT}}$

-> Line of fixed points

-> $\eta = 1/4$ @ $T = T_{\mathrm{KT}} (= \frac{\pi J}{2})$

# KT theory in 2D

**XY model**

$$H = -J \sum_{<i,j>} \cos(\theta_i - \theta_j) \approx \frac{J}{2} \int \mathrm{d}\boldsymbol{r}\, (\partial\theta)^2$$

vortex ⇔ Coulomb charge

Coulomb gas model

$$H = -\pi J \sum_{i \neq j} n_i n_j \log \left| \frac{r_i - r_j}{a} \right|$$

**sine-Gordon model**

$$S[\phi] = \int [(\partial\phi)^2 + g_0(1 - \cos\beta\phi)]$$

KT transition @ $\beta_c^2 = 8\pi$

Coleman, PRD11(1975)2088

# Outline of this study

**Quartic XY model**

$$H[\theta] = \frac{\mathcal{K}}{2}\int \mathrm{d}^4x\,\Delta\theta(x)\Delta\theta(x)$$

↕ (ii) A possible relationship?

**Coulomb gas model**

$$H = -\pi J \sum_{i\neq j} n_i n_j \log\left|\frac{r_i - r_j}{a}\right|$$

**Higher-derivative sine-Gordon model**

$$S[\phi] = \int [(\Delta\phi)^2 + g_0(1-\cos\beta\phi)]$$

-> (i) investigation w/ the FRG

# (i) 4D higher-derivative sine-Gordon model w/ the FRG

Running action

$$\Gamma_k[\varphi] = \int \mathrm{d}^4x \left[\frac{w_k}{2}(\Delta\varphi)^2 + V_k(\varphi)\right]$$

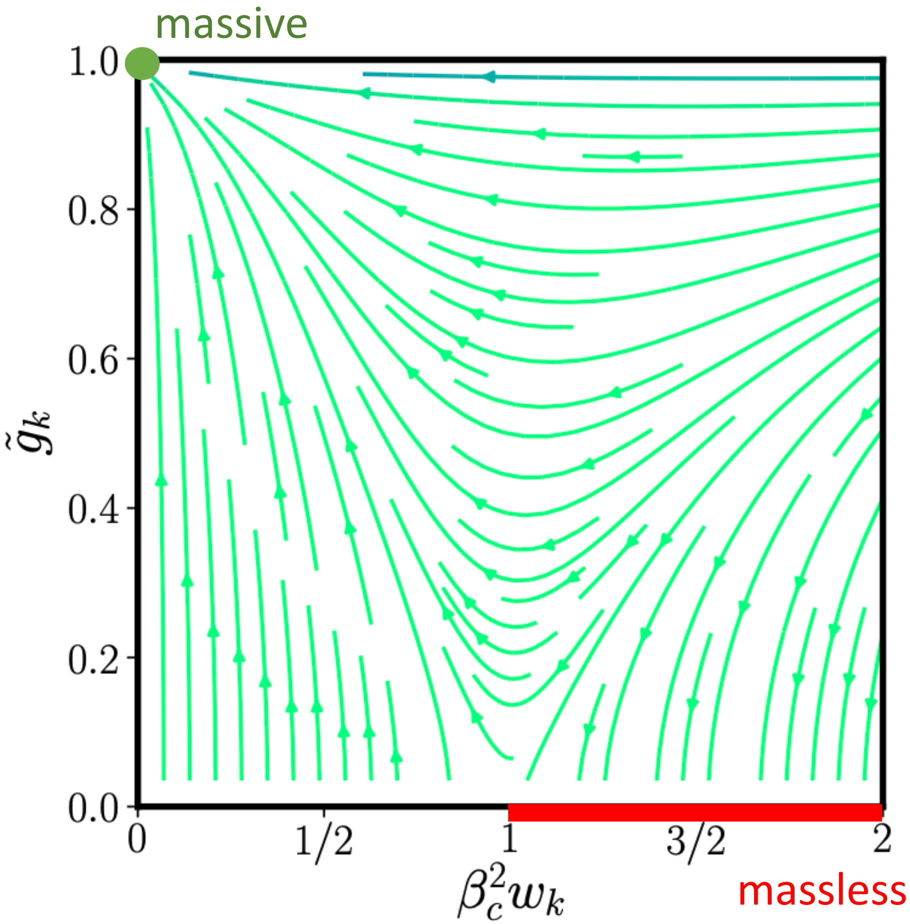
with $V_k(\varphi) = g_k(1-\cos\varphi)$

FRG flow equations

$$\partial_t w_k = -\frac{9}{160\pi^2}\frac{\tilde{g}_k^2}{(1-\tilde{g}_k^2)^{\frac{3}{2}}}$$

$$(4+\partial_t)\tilde{g}_k = \frac{1}{8\pi^2 w_k \tilde{g}_k}\left(1-\sqrt{1-\tilde{g}_k^2}\right)$$

with $t = -\log(k/\Lambda)$ and $\tilde{g}_k = g_k/k^4$




a line of attractive Gaussian fixed points for $\beta_c^2 w_k > 1$ w/ $\tilde{g}_k = 0$

$\beta_c^2 = 64\pi^2$ ( cf. $\beta_c^2 = 8\pi$ in 2D sine-Gordon model )

# (ii) A topological configuration in 4D quartic XY model

$$H[\theta] = \frac{\mathcal{K}}{2}\int \mathrm{d}^4x\, \Delta\theta(x)\Delta\theta(x)$$

Claim: $A_{x'}(x) = \frac{1}{2}\left(\alpha_4 - \frac{\pi}{2}\right)\cot\alpha_4$ gives a specific field configuration

✔ $\Delta_x A_{x'}(x) = -\frac{1}{(x-x')^2}$ ✔ $\Delta_x^2 A_{x'}(x) = (2\pi)^2\delta^{(4)}(x-x')$

✔ $H[A_{x'}] = \mathcal{K}\pi^2 \ln\frac{R}{r_0}$ ( $R$: large dist. cutoff, $r_0$: short dist. cutoff )

✔ $S[A_{x'}] = \ln\left[C\left(\frac{R^4}{r_0^4}\right)\right]$

✔ $\delta F = H - \mathcal{K}S = (\mathcal{K}\pi^2 - 4)\ln\frac{R}{r_0} + \text{const.}$

✔ $\mathcal{K}_c = \frac{4}{\pi^2}$

$$x_1 - x_1' = r\sin\alpha_4\sin\alpha_3\sin\alpha_2$$
$$x_2 - x_2' = r\sin\alpha_4\sin\alpha_3\cos\alpha_2$$
$$x_3 - x_3' = r\sin\alpha_4\cos\alpha_3$$
$$x_4 - x_4' = r\cos\alpha_4$$

# (ii) Relation to the sine-Gordon model

$$H[\mathcal{G}^C] = \frac{\mathcal{K}}{2}\int \mathrm{d}^4x\, \Delta\mathcal{G}^C(x)\, \Delta\mathcal{G}^C(x)$$

$\mathcal{G}^C(x) = \Sigma_i n_i\, G(x - x_i)$ w/ $n_i \in \mathbb{Z}$

$$\Delta_x G(x - x') = -\frac{1}{(x - x')^2}$$

⇓ can be identified as the higher-derivative sine-Gordon model via $Z$

$$Z \sim \int [\mathrm{d}\phi] \exp\left[-\frac{w}{2}\int \mathrm{d}^4r\, \Delta\phi(r)\Delta\phi(r) + 2y\int \mathrm{d}^4r \cos\phi(r)\right]$$

- ✔ $\frac{1}{w} \coloneqq (2\pi)^4\mathcal{K}$
- ✔ $(2\pi)^4\mathcal{K}_c = (2\pi)^4\frac{4}{\pi^2} = 64\pi^2 (= \beta_c^2)$
- ✔ $\eta = \frac{1}{8\pi^2\mathcal{K}} \Rightarrow \eta = \frac{1}{32}$ @ $\mathcal{K} = \mathcal{K}_c$ holds on the line of fixed points

# Summary

- 4D systems might exhibit a topological phase analogous to the KT phase in 2D

| | **4D (conjecture)** | **2D** |
|---|---|---|
| $\beta_c^2$ (sine-Gordon) | $64\pi^2$ | $8\pi$ |
| $\mathcal{K}_c$ (XY) | $4/\pi^2$ | $2/\pi$ |
| $\eta$ | $1/32$ | $1/4$ |

- A delicate point is the following relationship

<u>4D quartic XY model</u>

$$H[\theta] = \frac{\mathcal{K}}{2}\int \mathrm{d}^4x\, \Delta\theta\Delta\theta$$

$\overset{?}{\Longleftrightarrow}$

<u>4D higher-derivative sine-Gordon model</u>

$$S[\phi] = \int \mathrm{d}^4x[(\Delta\phi)^2 + g_0(1-\cos\beta\phi)]$$

# Backup

# Lifshitz point

Hornreich-Luban-Shtrikman, PRL35(1975)1678

<u>Coexistence of three phases</u>

✓ homogeneous phase

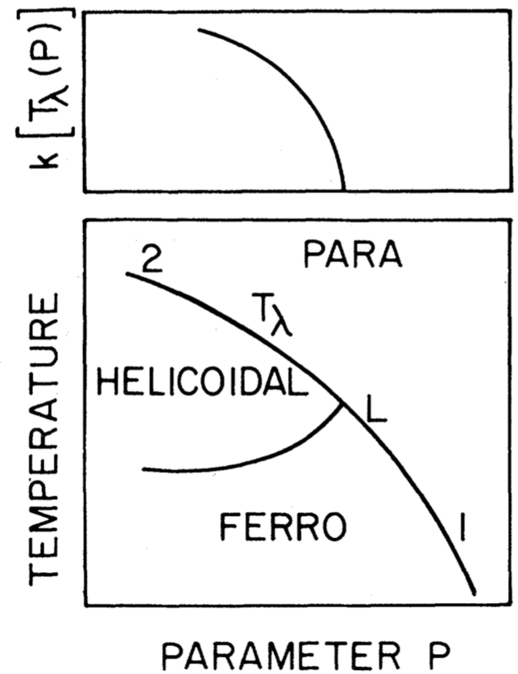
( order parameter is spatially uniform )

✓ disordered phase

( order parameter is zero )

✓ inhomogeneous phase

( order parameter is spatially modulated w/ a finite wave vector )



# FRG flow equations

Running action

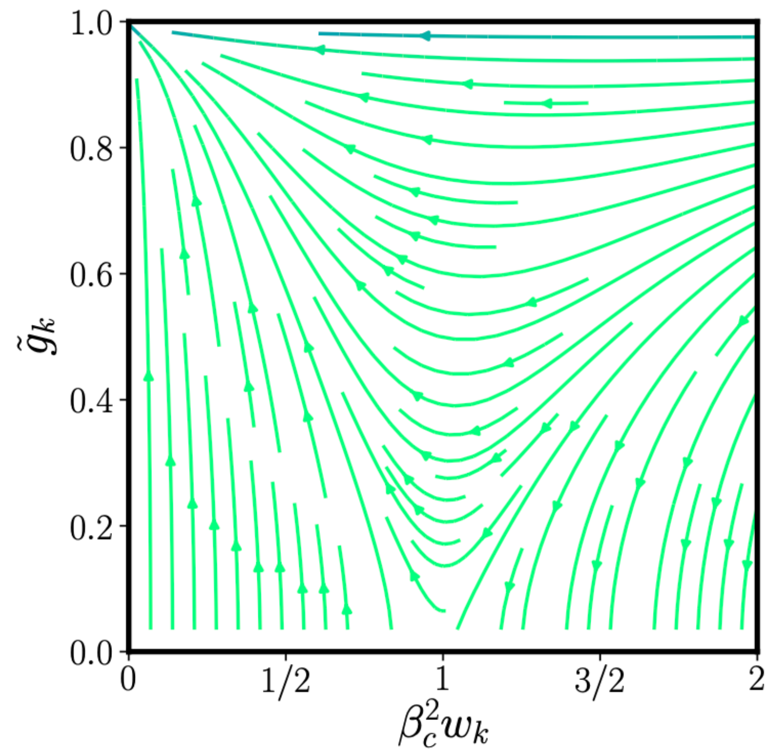
$$\Gamma_k[\varphi] = \int \mathrm{d}^4x \left[\frac{w_k}{2}(\Delta\varphi)^2 + \underbrace{g_k(1-\cos\varphi)}_{=:\,V_k(\varphi)}\right]$$

FRG flow

$$\partial_t V_k(\varphi) = \int \frac{\mathrm{d}^d q}{(2\pi)^d} G(q)\partial_t R_k(q)$$

$$\partial_t w_k = \lim_{p\to 0}\int_{-\pi}^{\pi}\frac{\mathrm{d}\varphi}{2\pi}\int\frac{\mathrm{d}^d q}{(2\pi)^d}\partial_t R_k(q) \times G(q)^2 V_k'''(\varphi)^2 \frac{\mathrm{d}^4}{\mathrm{d}p^4} G(p+q)$$

$$G(q) = \frac{1}{w_k q^4 + V_k''(\varphi) + R_k(q)}$$




with the RG scale $t = -\log\frac{k}{\Lambda}$, the regulator $R_k(q) = k^4$ and taking $d \to 4^+$