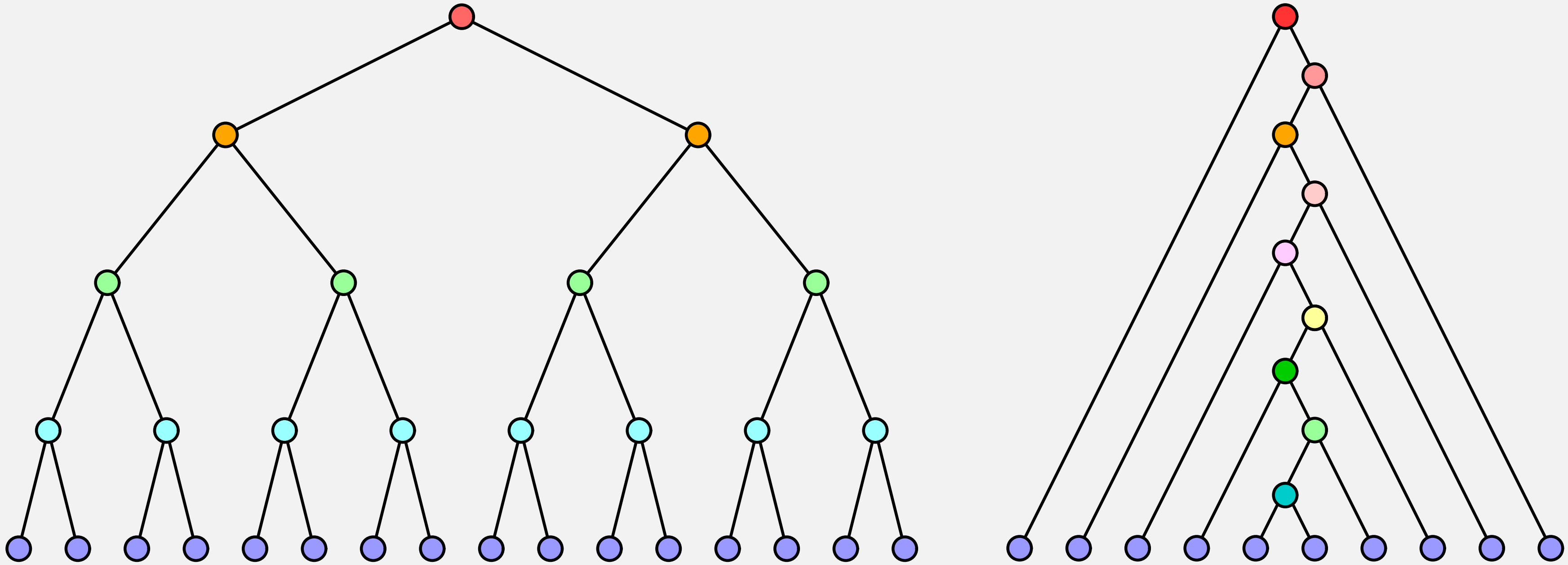
Journal Club 2019/07/26 Akiyama Shinichiro

Tensor Renormalization Group Centered About a Core Tensor

Wangwei Lan and Glen Evenbly

arXiv: 1906.09283 [quant-ph]



Quantum Field Theory

Renormalization Group

(Gell-Mann-Low Eq.)

Wilsonian RG

scale-dependent “effective theory”

Functional RG

Application of
Tensor Network to QFT is
a recent hot topic!

Critical Phenomena

Block Spin Transformation

(Real space RG)

Density Matrix RG

1D quantum systems

Tensor Network Scheme

Attacking higher dimension!

Brief review of TN rep. (1/2)

Ex) 2D classical spin model (with periodic boundary)

$$Z = \sum_{\{\sigma\}} \prod_{\langle ij \rangle} \exp[-\beta K(\sigma_i, \sigma_j)]$$

Decomposition of the transfer matrix element :

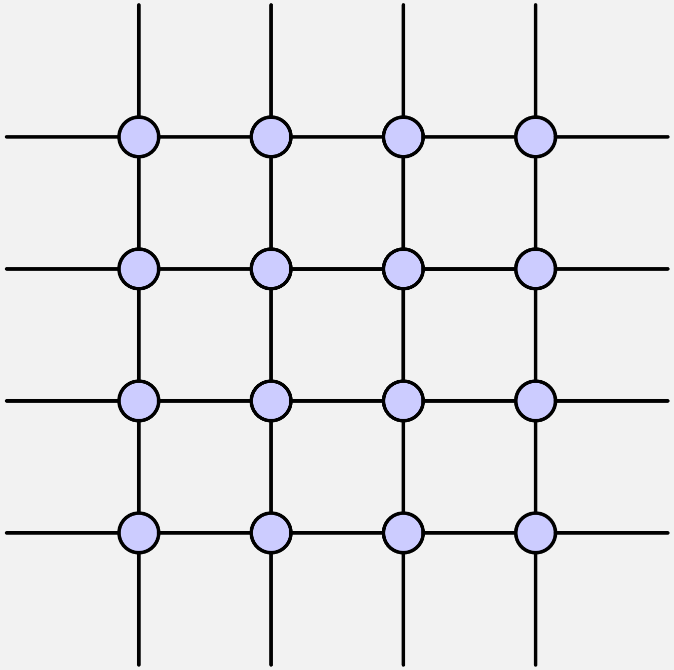
$$\exp[-\beta K(\sigma_i, \sigma_j)] = \sum_{l} W(\sigma_i, l) W(\sigma_j, l)$$

Integrating out σ's, one obtains a local tensor;

$$T_{l_1 l_2 l_3 l_4} \coloneqq \sum_{\sigma_i = \pm 1} W(\sigma_i, l_1) W(\sigma_i, l_2) W(\sigma_i, l_3) W(\sigma_i, l_4)$$

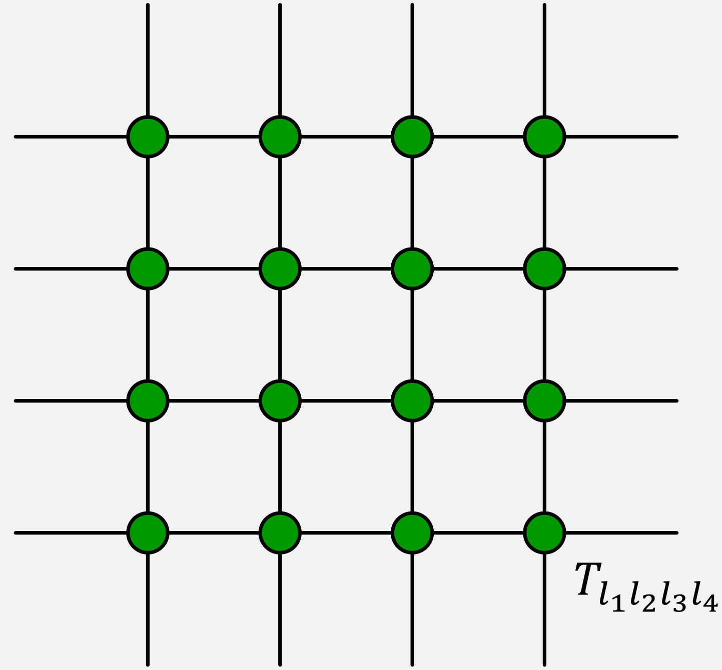
Brief review of TN rep. (2/2)

Real Space



○ : spin variable

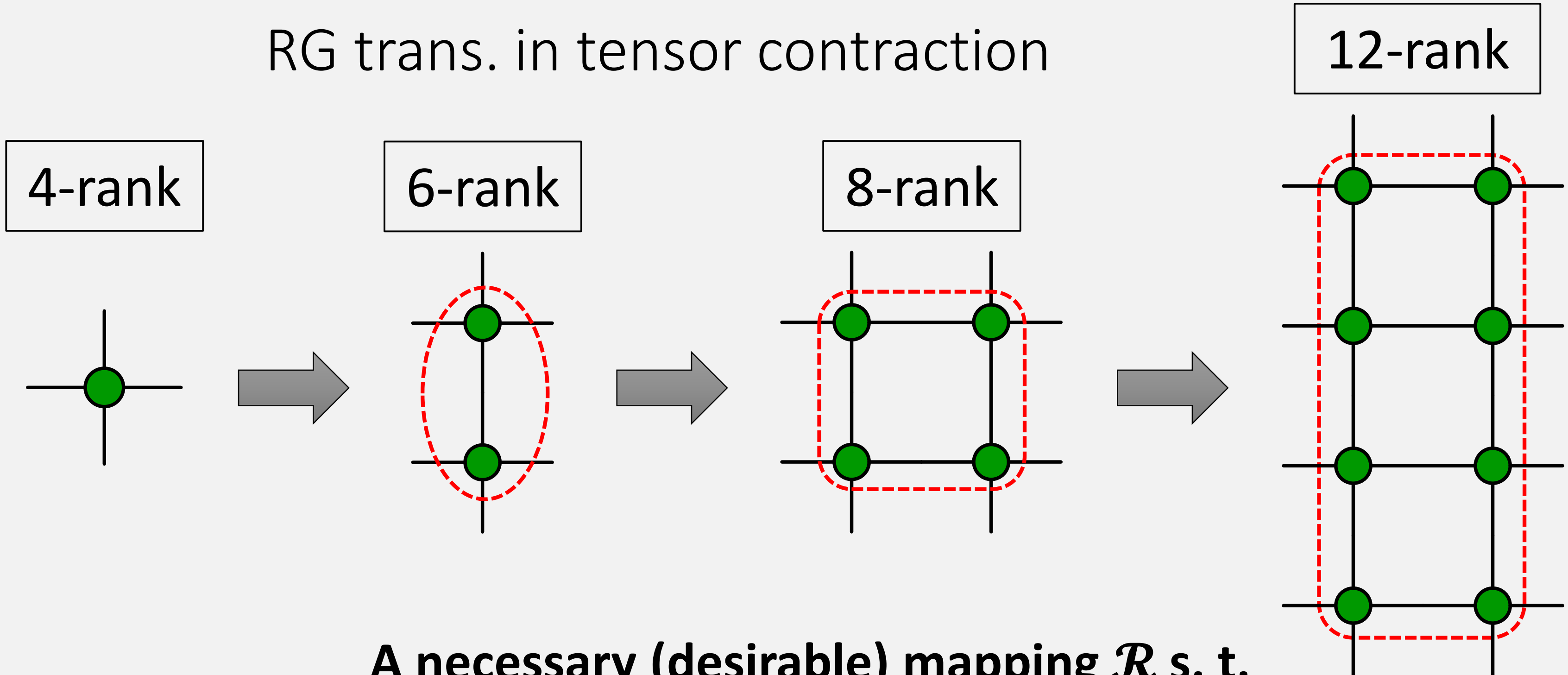
Tensor Network (partition function)



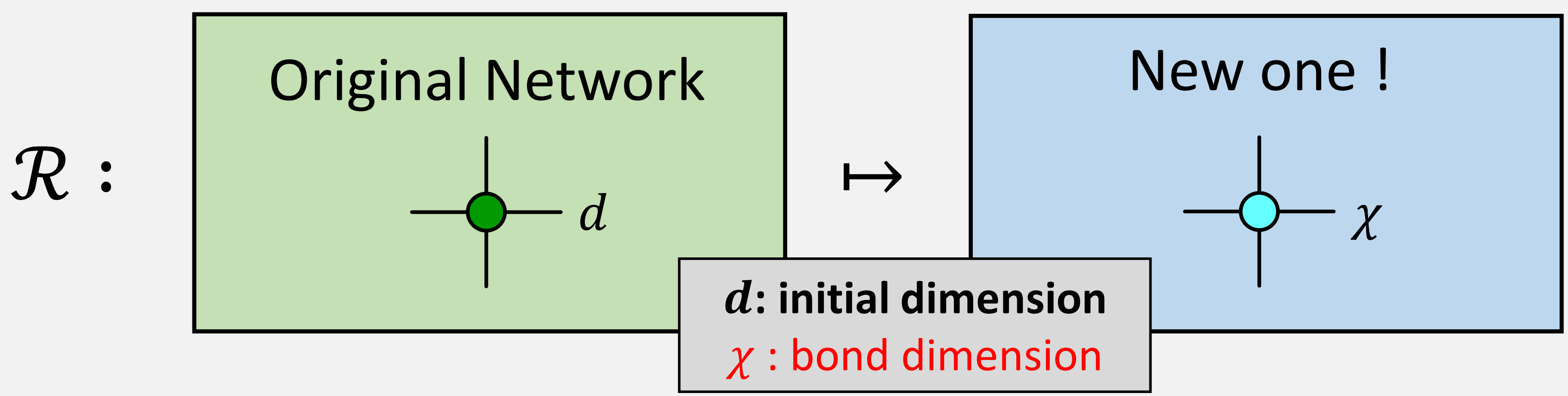
: 4-rank tensor

$$Z = \sum_{\{l\}} \prod_i T_{l_a l_b l_c l_d} =: \mathrm{Tr}\left[\prod_i T_{l_a l_b l_c l_d}\right]$$

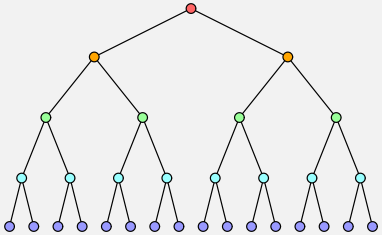
RG trans. in tensor contraction



A necessary (desirable) mapping $\mathcal{R}$ s. t.



Matrix decomposition technique (linear algebra) is very useful



Many types of $\mathcal{R}$enormalization group

χ : bond dimension
L : lattice size

Tensor Renormalization Group (TRG)
$O(\chi^6 \ln L)$ (Levin-Nave, 2007)

Higher dimension

Accuracy

(D-dimensional) Higher-order TRG
$O(\chi^{4D-1} \ln L)$ (Xie et al, 2012)

Tensor Network Renormalization
$O(\chi^7 \ln L)$ (Evenbly-Vidal, 2015)

Cost reduction

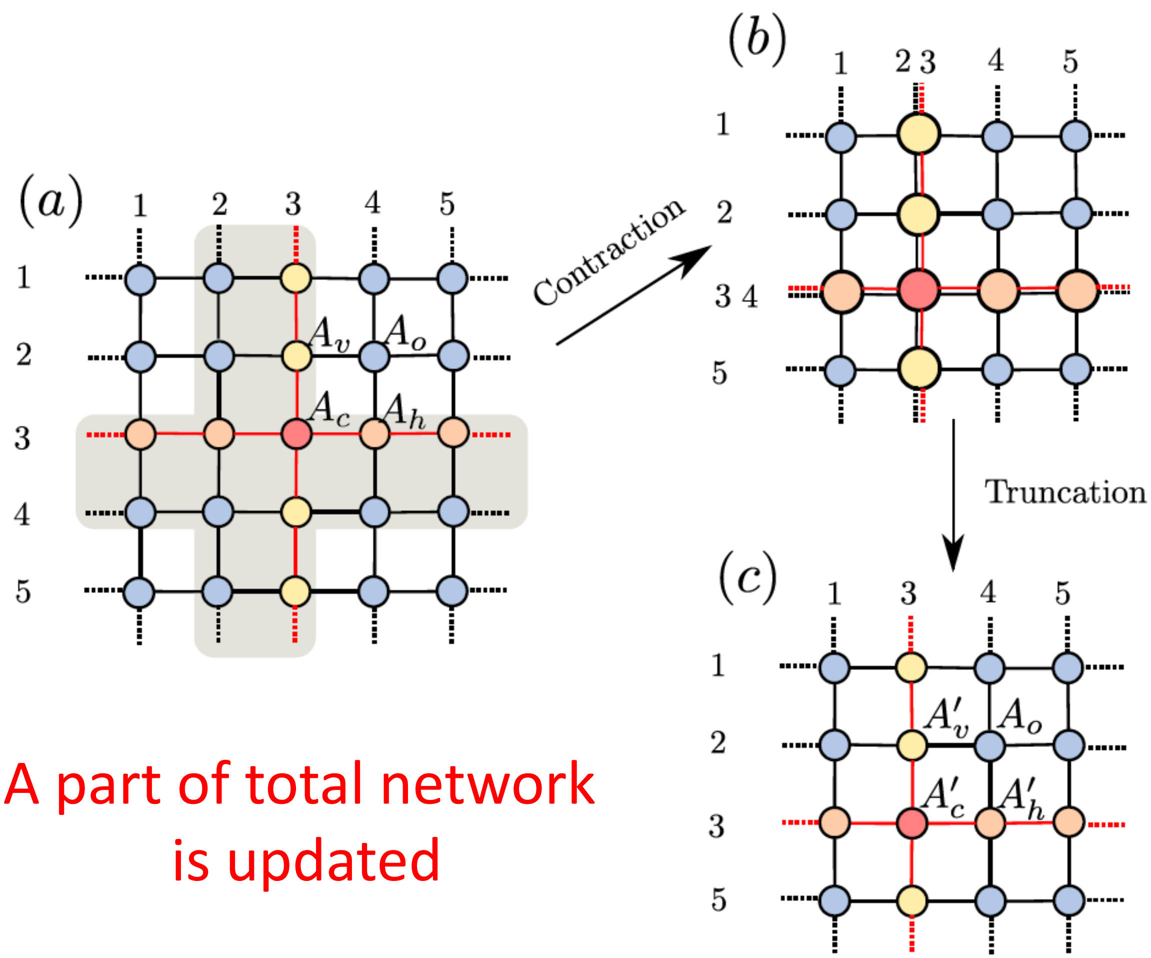
TRG with randomized SVD
$O(\chi^5 \ln L)$ (Morita et al, 2018)

Projectively truncated TRG
$O(\chi^5 \ln L)$ (Nakamura et al, 2019)

(D-dimensional) Anisotropic TRG
$O(\chi^{2D+1} \ln L)$ (Adachi et al, 2019)

Core-Tensor RG
$O(\chi^4 L)$ (Lan-Evenbly, 2019)

Algorithm of CTRG (1/3)

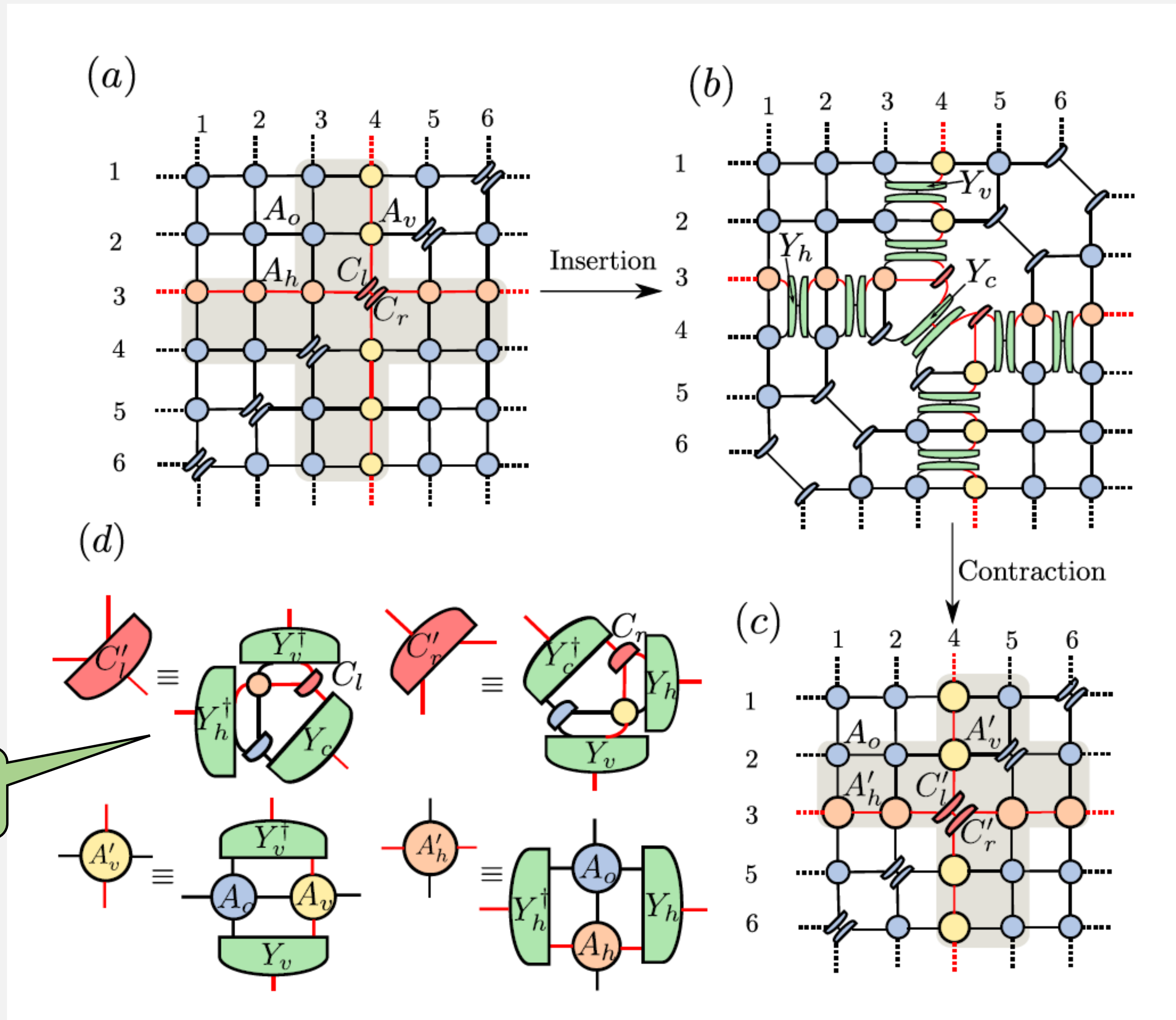


A part of total network
is updated

Algorithm of CTRG (2/3)

d: initial dimension
χ : bond dimension

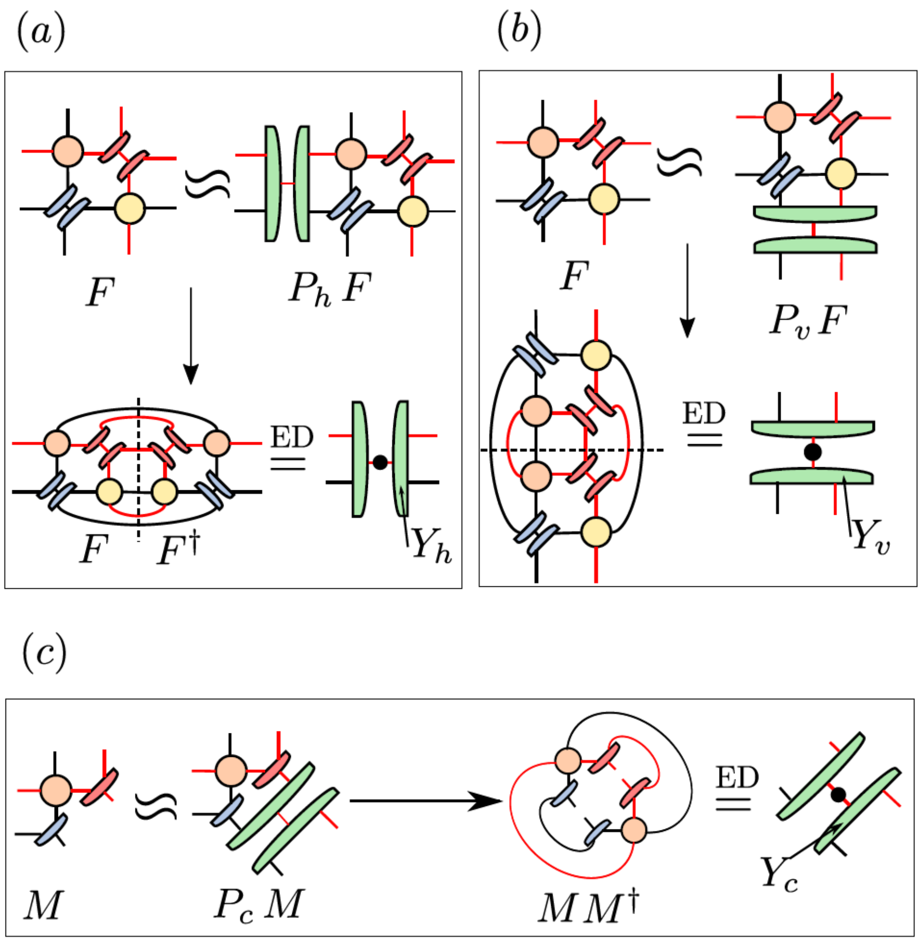
NOTE : Red legs run up to χ, but black ones do up to d ($\chi \gg d$)



$O(d^3\chi^4)$

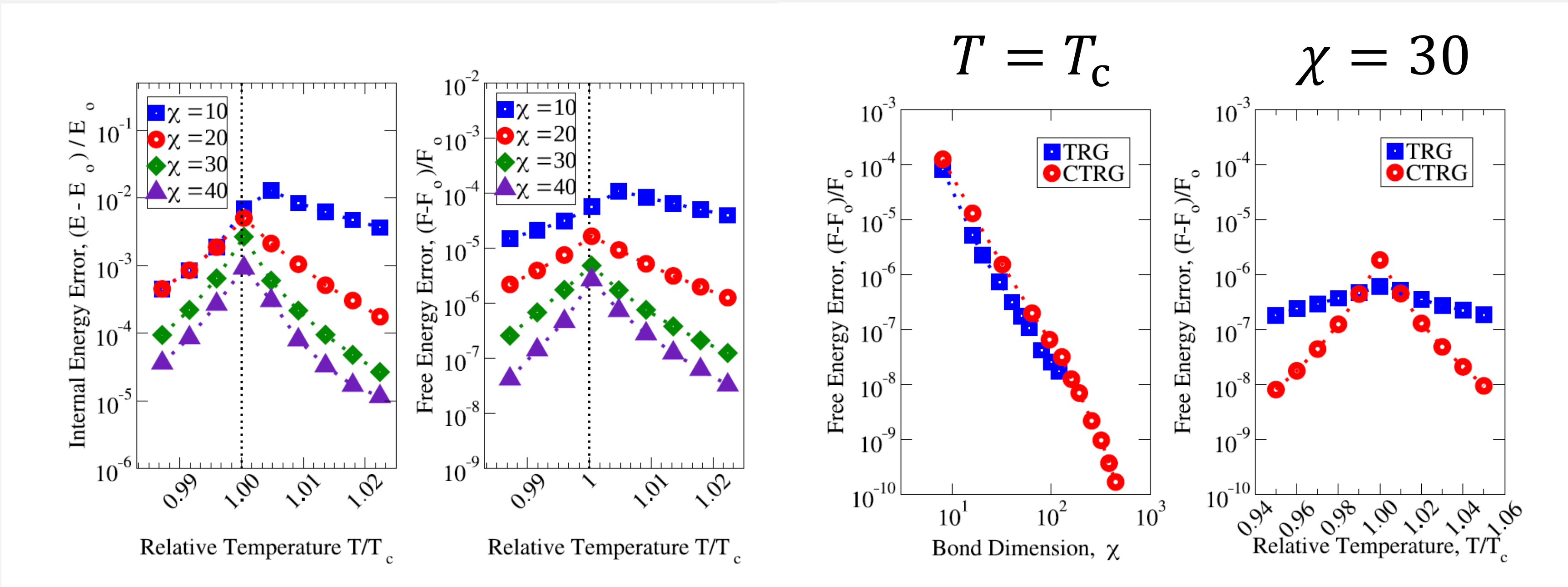
Algorithm of CTRG (3/3)

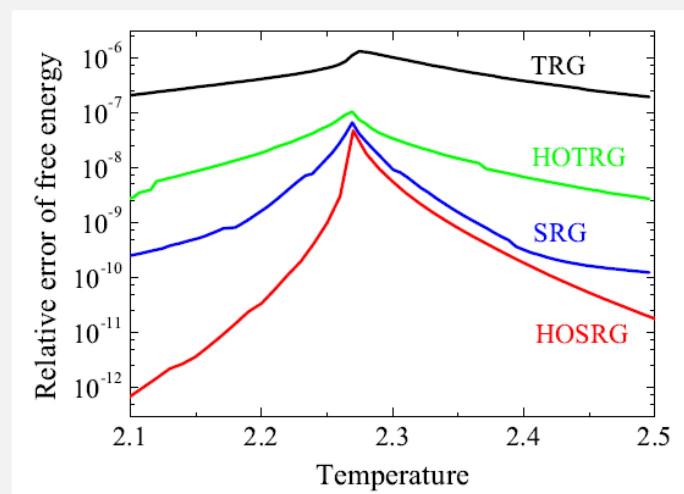
Projectors are directly decided by EVD



Truncation error is minimized!

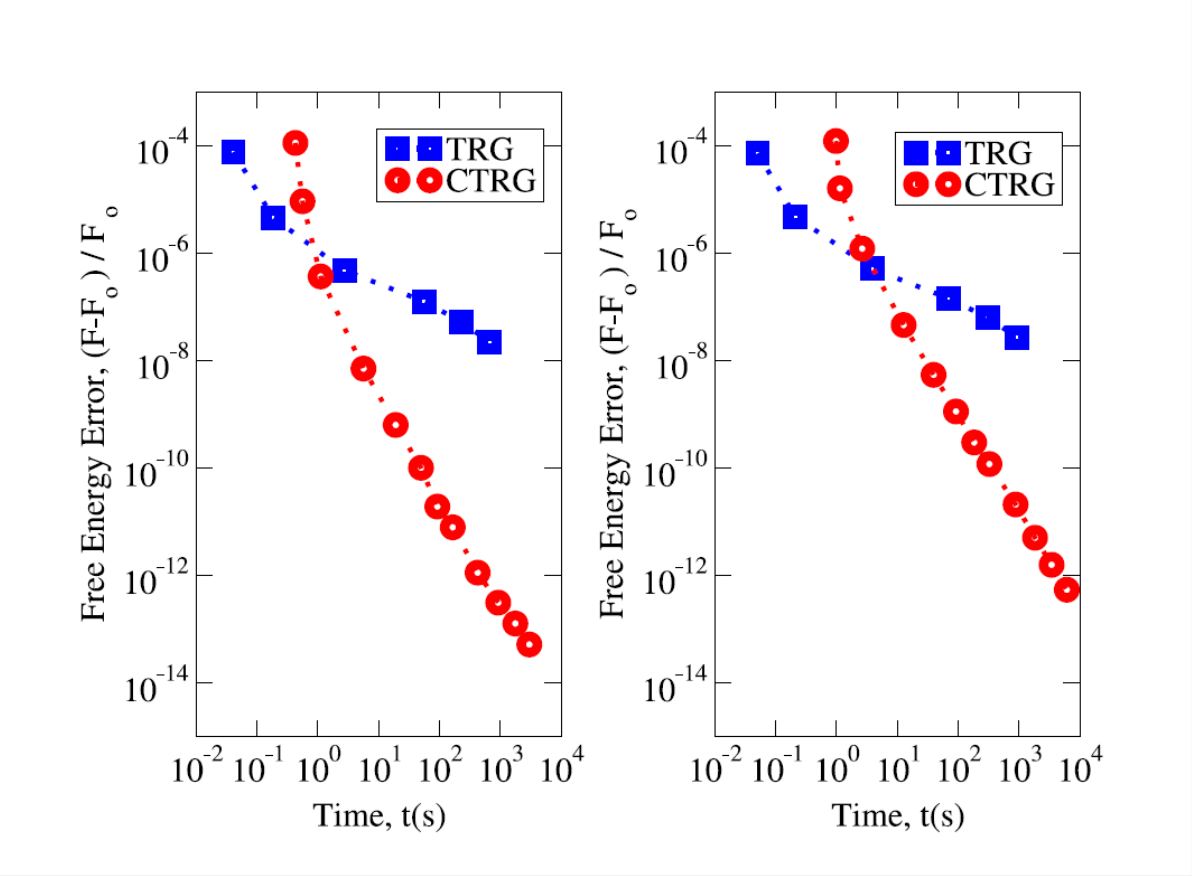
Benchmark Results (2D Ising)





Xie et al, PRB86(2012)045139

Benchmark Results (2D Ising)

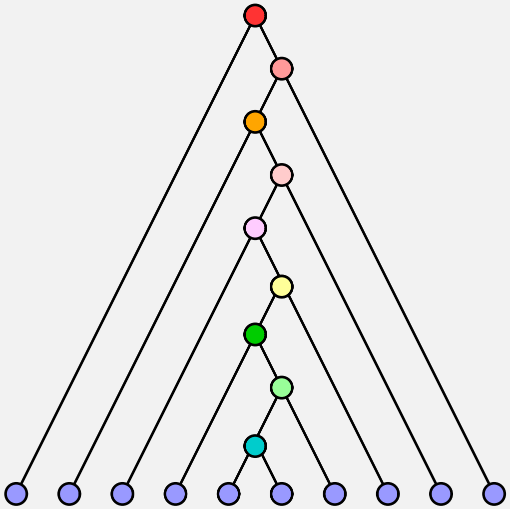


$L = 128$ $\qquad$ $L = 256$

- **2 curves are crossing!** (how about comparison with HOTRG?)

Outlook

"We hypothesize that a version of CTRG generalized for higher spatial dimensions could reproduce results of equivalent accuracy to HOTRG, but with a much lower cost scaling in bond dimension χ."



Comments

- Choice of boundary condition & Choice of $\forall L \in \mathbb{N}$
- Can be seen as a variant of CTMRG (Nishino-Okunishi, 1996), whose cost scales with $O(\chi^3 L)$

http://quattro.phys.sci.kobe-u.ac.jp/dmrg.html

APPENDICES

Singular Value Decomposition (SVD)

For any complex $I_1 \times I_2$ -matrix A can be written as the product

$$A = U^{(1)} S U^{(2)^{\dagger}}$$

where

1. $U^{(1)}$ is an $I_1 \times I_1$ unitary matrix.

2. $U^{(2)}$ is an $I_2 \times I_2$ unitary matrix.

3. S is an $I_1 \times I_2$-matrix such that

   (i) Pseudo-diagonality : $S = \mathrm{diag}\left(\sigma_1, \sigma_2, \cdots, \sigma_{\min(I_1, I_2)}\right)$

   (ii) Ordering : $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min(I_1, I_2)} \geq 0$

σ_i 's are singular values of A and the i-th column vectors of $U^{(1)}$ and $U^{(2)}$ are, resp., i-th left and right singular vector.

Higher-Order Singular Value Decomposition (HOSVD)

Any complex $I_1 \times I_2 \times \cdots \times I_n$-tensor A can be written as the product

$$A_{i_1 i_2 \cdots i_n} = \sum_{j_1 j_2 \cdots j_n} S_{j_1 j_2 \cdots j_n} U^{(1)}_{j_1 i_1} U^{(2)}_{j_2 i_2} \cdots U^{(n)}_{j_n i_n}$$

where

1. $U^{(k)}$ is a unitary $I_k \times I_k$ -matrix.

2. S is a complex $I_1 \times I_2 \times \cdots \times I_n$-tensor such that

(i) Fixing the k-th index of S, say $S_{i_k=\alpha}$, and if $\alpha \neq \beta$, then

$$\sum_{i_1 i_2 \cdots i_n} S_{i_1 i_2 \cdots i_{k-1} \alpha i_{k+1} \cdots i_n} S_{i_1 i_2 \cdots i_{k-1} \beta i_{k+1} \cdots i_n} = 0$$

(ii) Ordering :

$$\left\| S_{i_k=\alpha} \right\| := \sqrt{\sum_{i_1 i_2 \cdots i_n} S_{i_1 i_2 \cdots i_{k-1} \alpha i_{k+1} \cdots i_n} S_{i_1 i_2 \cdots i_{k-1} \alpha i_{k+1} \cdots i_n}}$$

$$\left\| S_{i_k=1} \right\| \geq \left\| S_{i_k=2} \right\| \geq \cdots \geq \left\| S_{i_k=I_k} \right\| \geq 0$$

SVD introduces virtual dof

Consider the system consisting of subsystems X and Y.
Setting the pure state of the total system as

$$|\psi\rangle = \sum_{x \in X} \sum_{y \in Y} \psi(x, y) |x\rangle \otimes |y\rangle$$

If $\psi(x, y) = u(x)v(y)$, then the state is separable. Actually,

$$|\psi\rangle = \left(\sum_{x \in X} u(x) |x\rangle \right) \otimes \left(\sum_{y \in Y} v(y) |y\rangle \right)$$

SVD introduces virtual dof

Regarding $\psi(x, y)$ as a matrix element. By SVD,

$$\psi(x, y) = \sum_{l=1}^{N} u_l(x)\sigma_l v_l(y)$$

If $N > 1$, the state is not pure. However, as a matrix,

$$\psi = U\Sigma V^\dagger = \left(U\Sigma^{1/2}\right)\left(V\Sigma^{1/2}\right)^\dagger =: \tilde{U}\tilde{V}^\dagger$$

This looks very similar with $\psi(x, y) = u(x)v(y)$.