Journal Club / Dec. 07. 2018 / Naoya Ukita

"Spontaneous symmetry breaking and the Goldstone theorem in non-Hermitian field theories"

Jean Alexandre, John Ellis, Peter Millington, Dries Seynaeve,
Phys. Rev. D 98, 045001 (2018)

1: History of $\mathcal{PT}$ symmetry

2: Symmetry and conservation law in Non-Hermitian, $\mathcal{PT}$-symmetric models

3: Spontaneous symmetry breaking and the Goldstone mode

1. History of $\mathcal{PT}$ symmetry

- Conjecture: non-Hermitian Hamiltonian $H = \hat{p}^2 + i\hat{x}^3$ has only real eigenvalues.
  (Brower, Furman, Moshe 1978, Harms, Jones, Tan 1980, Caliceti, Graffi, Maioli 1980)
- unitarity???
- $\mathcal{PT}$ symmetry (Bender, Boettcher 1998):

$$[H, \mathcal{PT}] = 0,$$

where

$$\mathcal{P} : \begin{cases} \hat{p} \to -\hat{p}, \\ \hat{x} \to -\hat{x}, \end{cases} \qquad \mathcal{T} : \begin{cases} \hat{p} \to -\hat{p}, \\ \hat{x} \to \hat{x}, \end{cases} \qquad \mathcal{T} i = -i\mathcal{T}.$$

- Proof of the conjecture:
  Let H be a $\mathcal{PT}$ symmetric Hamiltonian and $|\psi\rangle$ be a state with an eigenvalue λ.

$$\begin{aligned} & [H, \mathcal{PT}] = 0, \quad H|\psi\rangle = \lambda|\psi\rangle. \\ \Rightarrow \quad & H\mathcal{PT}|\psi\rangle = \lambda^\star \mathcal{PT}|\psi\rangle. \\ \Rightarrow \quad & \begin{cases} \lambda \text{ is real for any } \mathcal{PT} \text{ eigenstate}, \mathcal{PT}|\psi\rangle = |\psi\rangle, \\ (\lambda, \lambda^\star) \text{ complex conjugate pair for others.} \end{cases} \end{aligned}$$

- $\mathcal{PT}$ symmetric Hamiltonian $H = \hat{p}^2 - (i\hat{x})^\epsilon$.
  ex. $-(i\hat{x})^\epsilon = \hat{x}^2, i\hat{x}^3, -\hat{x}^4$ for $\epsilon = 2, 3, 4$.

2. Symmetry and conservation law in Non-Hermitian, $\mathcal{PT}$-symmetric models

- Under the transformation

$$\phi_i \longrightarrow \phi_i + \delta\phi_i \,, \qquad \phi_i^\star \longrightarrow \phi_i^\star + \delta\phi_i^\star \,,$$

the variation of the Lagrangian is

$$\delta\mathcal{L} = \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi_i^\star} - \partial_\nu \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_i^\star)}\right)}_{\equiv \frac{\delta S}{\delta\phi_i^\star}} \delta\phi_i^\star + \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\nu \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_i)}\right)}_{\equiv \frac{\delta S}{\delta\phi_i}} \delta\phi_i + \partial_\nu(\delta j^\nu) \,,$$

where

$$\delta j^\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_i^\star)}\delta\phi_i^\star + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_i)}\delta\phi_i \,.$$

- One cannot simultaneously have $\frac{\delta S}{\delta\phi_i^\star} = 0$ and $\frac{\delta S}{\delta\phi_i} = 0$. $\Rightarrow$ $\frac{\delta S}{\delta\phi_i^\star} = 0$ for EOM and $\frac{\delta S}{\delta\phi_i} \neq 0$.

- Non-Hermitian, $\mathcal{PT}$ symmetric Lagrangian leads

$$\begin{cases} (i)\ \ \delta\mathcal{L} = 0 \quad \& \quad \partial_\nu(\delta j^\nu) \neq 0, \\ (ii)\ \delta\mathcal{L} \neq 0 \quad \& \quad \partial_\nu(\delta j^\nu) = 0. \end{cases}$$

- eg. non-Hermitian, $\mathcal{PT}$ symmetric Lagrangian density

$$\mathcal{L} \;=\; \partial_\mu\phi_1^\star\partial^\mu\phi_1 + \partial_\mu\phi_2^\star\partial^\mu\phi_2 - m_1^2|\phi_1|^2 - m_2^2|\phi_2|^2 - \mu^2\left(\phi_1^\star\phi_2 - \phi_2^\star\phi_1\right)\;.$$

$$\mathcal{PT}:\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \;\rightarrow\; \begin{pmatrix}\phi_1^\star\\ -\phi_2^\star\end{pmatrix}.$$

- Inconsistency condition, $\frac{\delta S}{\delta\phi_1^\star}=0$ and $\frac{\delta S}{\delta\phi_1}=0$

$$\Box\phi_1 + m_1^2\phi_1 + \mu^2\phi_2 = 0,\quad \Box\phi_1^\star + m_1^2\phi_1^\star - \mu^2\phi_2^\star = 0.$$

- $(i)\;\; \delta\mathcal{L}=0 \quad \& \quad \partial_\nu j^\nu \neq 0$

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \;\rightarrow\; e^{i\alpha}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix},$$

$$\delta\mathcal{L}=0,\quad \partial_\nu j^\nu = 2i\mu^2\left(\phi_1^\star\phi_2 - \phi_2^\star\phi_1\right).$$

- $(ii)\;\delta\mathcal{L}\neq 0 \quad \& \quad \partial_\nu j^\nu = 0$

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \;\rightarrow\; \begin{pmatrix}e^{i\alpha}\phi_1\\ e^{-i\alpha}\phi_2\end{pmatrix},$$

$$\delta\mathcal{L} = -\mu^2\left((e^{-2i\alpha}-1)\phi_1^\star\phi_2 - (e^{+2i\alpha}-1)\phi_2^\star\phi_1\right),\quad \partial_\nu j^\mu = 0.$$

3. Spontaneous symmetry breaking and the Goldstone mode

- non-Hermitian, $\mathcal{PT}$-symmetric theory with $\partial_\nu j^\nu = 0$ and $\delta\mathcal{L} \neq 0$

The existence of a non-trivial vacuum $Qv \neq v$. (Spontaneous symmetry breaking) $\Rightarrow$ massless Goldstone mode.

- eg. SSB

$$\mathcal{L} \;=\; \partial_\mu\phi_1^\star\partial^\mu\phi_1 + \partial_\mu\phi_2^\star\partial^\mu\phi_2 + m_1^2|\phi_1|^2 - m_2^2|\phi_2|^2 - \mu^2\left(\phi_1^\star\phi_2 - \phi_2^\star\phi_1\right) - \frac{g}{4}|\phi_1|^4 \;.$$

Using the equations of motion, one find a non-trivial vacuum

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \;=\; \sqrt{2\frac{m_1^2 m_2^2 - \mu^4}{g m_2^2}}\begin{pmatrix} 1 \\ \frac{\mu^2}{m_2^2} \end{pmatrix} \;.$$

Expressing this system in terms of the shifted fields

$$\begin{aligned} \phi_1 &= v_1 + \hat{\phi}_1 \;, \\ \phi_2 &= v_2 + \hat{\phi}_2 \;, \end{aligned}$$

- The mass matrix takes the form

$$M^2 \begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \\ \hat{\phi}_1^\star \\ \hat{\phi}_2^\star \end{pmatrix} = \begin{pmatrix} (gv_1^2 - m_1^2) & \frac{g}{2}v_1^2 & \mu^2 & 0 \\ \frac{g}{2}v_1^2 & (gv_1^2 - m_1^2) & 0 & \mu^2 \\ -\mu^2 & 0 & m_2^2 & 0 \\ 0 & -\mu^2 & 0 & m_2^2 \end{pmatrix} \begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \\ \hat{\phi}_1^\star \\ \hat{\phi}_2^\star \end{pmatrix} ,$$

  and

$$\det M^2 = 0.$$

  The Goldstone mode

$$G_1 = \sqrt{\frac{2m_2^4}{m_2^4 - \mu^4}} \left(\mathrm{Im}\left(\hat{\phi}_1\right) - \frac{\mu^2}{m_2^2} \mathrm{Im}\left(\hat{\phi}_2\right) \right) , \quad \partial_\nu j^\nu \propto v_1 G_1.$$

  The massive modes $G_2 = \cdots , G_3 = \cdots , G_4 = \cdots$.

- At one-loop level, the mass matrix again has determinant zero, showing the existence of the Goldstone mod.

4. Extension to the Higgs mechanism

"Gauge invariance and the Englert-Brout-Higgs mechanism in non-Hermitian field theories"

Jean Alexandre, John Ellis, Peter Millington, Dries Seynaeve,
arXiv:1808.00944 [hep-th]

"Goldstone bosons and the Higgs mechanism in non-Hermitian theories"

Philip D. Mannheim,
arXiv:1808.00437 [hep-th]