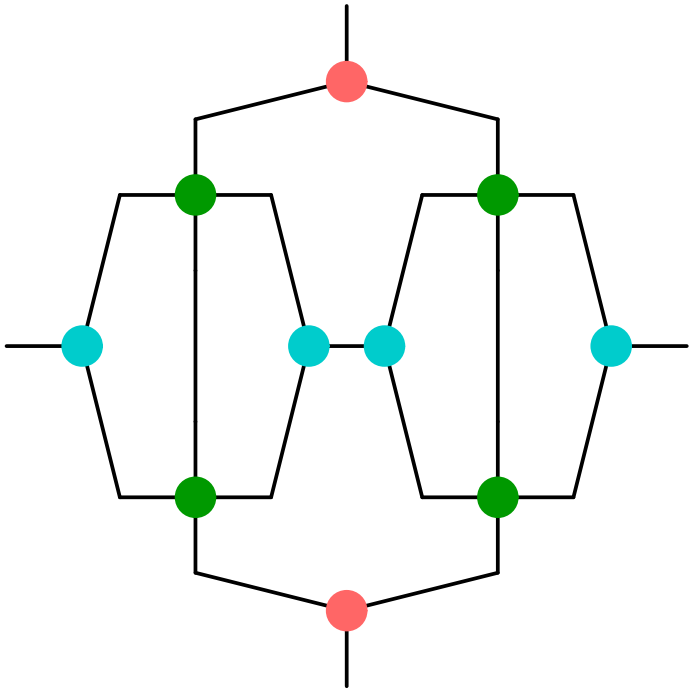
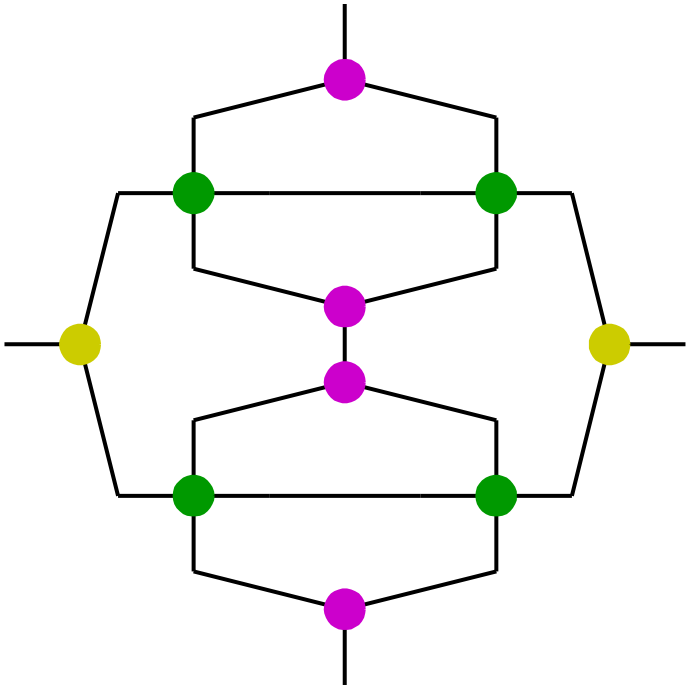
Journal Club 2018/10/5 Akiyama Shinichiro

Coarse-graining renormalization by higher-order singular value decomposition

Z. Y. Xie, J. Chen, M. P. Qin, J. W. Zhu, L. P. Yang, and T. Xiang

Phys. Rev. B 86 045139 (2012)





Contents

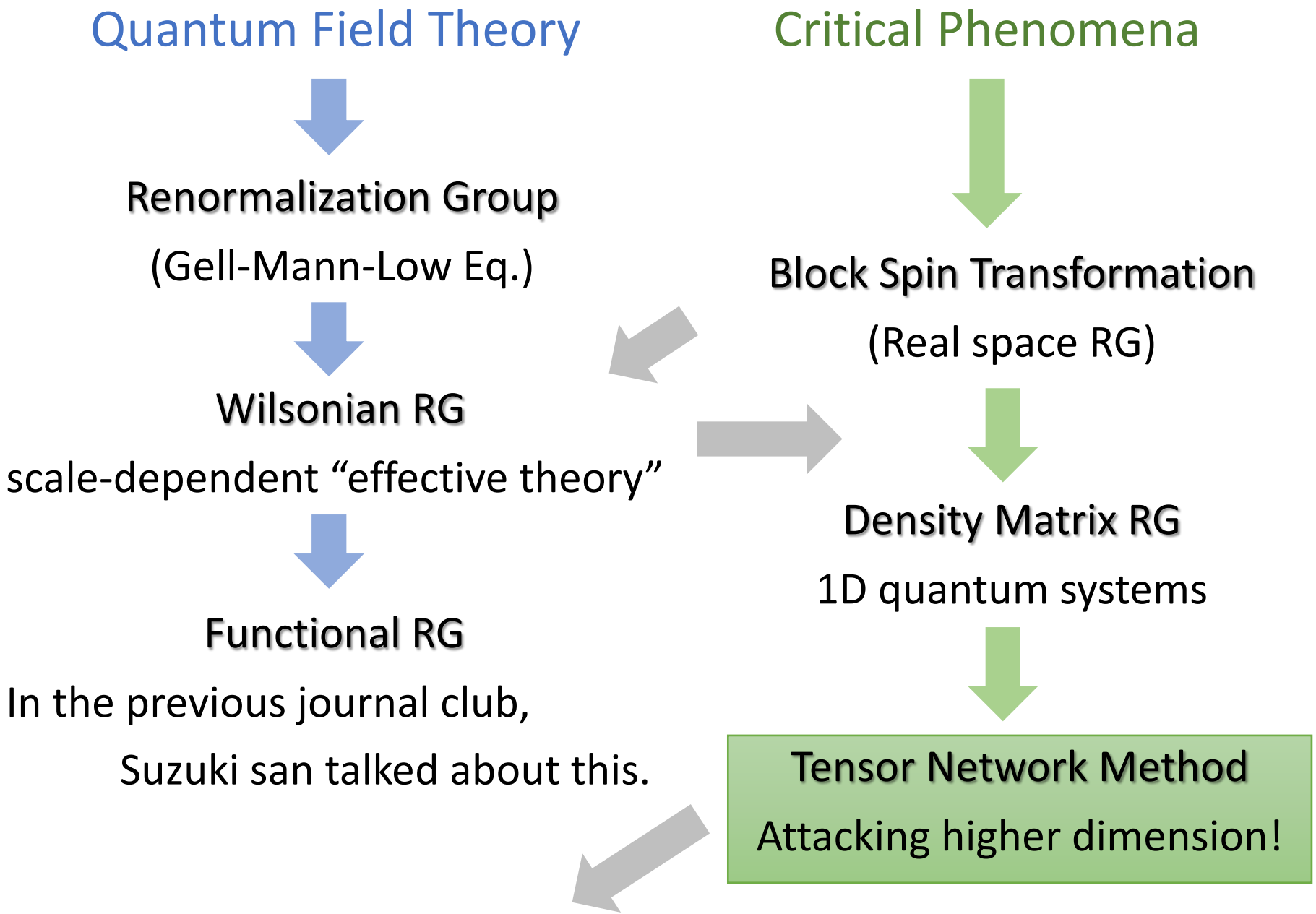
1. Introduction

2. Tensor Network Method (Ex. 2D Ising model)

I) Construction of Tensor Network Representation

II) Higher-Order Tensor Renormalization Group

3. Conclusion


Quantum Field Theory
Critical Phenomena
Renormalization Group
(Gell-Mann-Low Eq.)
Block Spin Transformation
(Real space RG)
Wilsonian RG
scale-dependent “effective theory”
Density Matrix RG
1D quantum systems
Functional RG
In the previous journal club,
Suzuki san talked about this.
Tensor Network Method
Attacking higher dimension!
The application for QFT is a recent hot topic!

The “Tensor Network Method” discussed here consists of 2 steps :

① Convert the system on the real space
into the “virtual” network by Singular Value Decomposition

② Compress the network,
preserving relevant elements in the long-scale physics

Step1 Move on to the TN representation

Ex) 2D classical Ising Model (with periodic boundary): $\sigma_i = \pm 1$

$$H = -\sum_{\langle ij \rangle} \sigma_i \sigma_j =: \sum_{\langle ij \rangle} K(\sigma_i, \sigma_j)$$

$$Z = \sum_{\{\sigma\}} \exp[-\beta H] = \sum_{\{\sigma\}} \prod_{\langle ij \rangle} \exp[-\beta K(\sigma_i, \sigma_j)]$$

Singular value decomposition for the transfer matrix element :

$$\exp[-\beta K(\sigma_i, \sigma_j)] = \sum_l \sqrt{\lambda_l} U(\sigma_i, l) \sqrt{\lambda_l} U(\sigma_j, l) = \sum_l W(\sigma_i, l) W(\sigma_j, l)$$

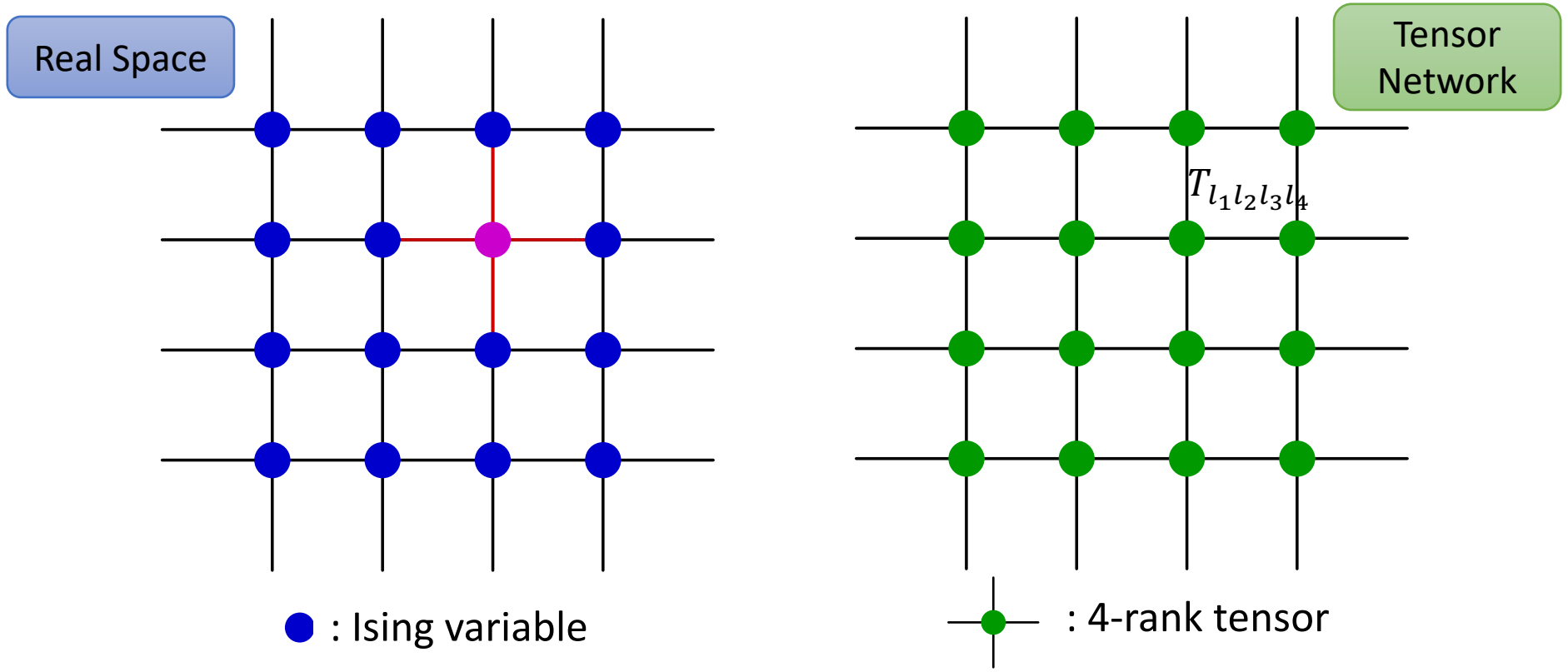
➡ Change of variables from $\{\sigma\}$ to $\{l\}$ (bond d.o.f.).

For each site in the real space, we define the 4-rank tensor :

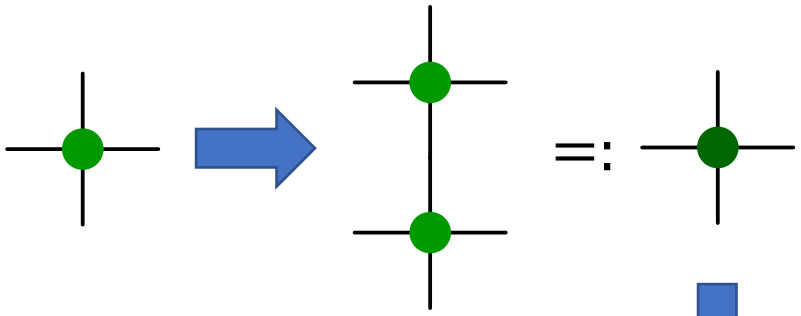
$$T_{l_1 l_2 l_3 l_4} := \sum_{\sigma_i = \pm 1} W(\sigma_i, l_1) W(\sigma_i, l_2) W(\sigma_i, l_3) W(\sigma_i, l_4)$$

One obtains the TN representation of the partition function :

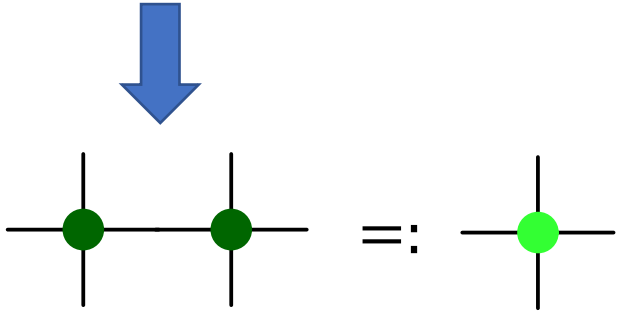
$$Z = \sum_{\{l\}} \prod_i T_{l_a l_b l_c l_d} = \mathrm{Tr}\left[\prod_i T_{l_a l_b l_c l_d} \right]$$



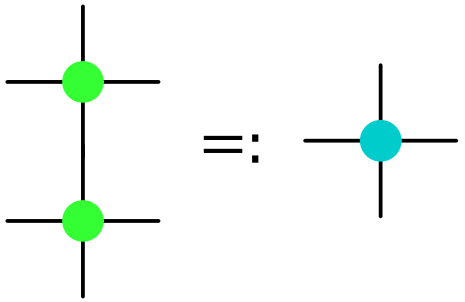
Step2 Higher-Order Tensor Renormalization Group (HOTRG)



A naïve strategy to approach a square network, starting by T.



When we map a 4-rank tensor by ➡, **proper approximation** is necessary.



The basic idea of the transformation is

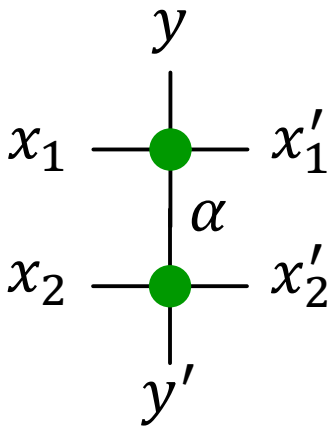
"2 local tensors ⇒ 1 Coarse-grained tensor"

And we want to obtain accurate thermodynamic properties.

(i) Align 2 local tensors vertically and contract them :

n: # of iteration

$$M^{(n)}_{x_1 \otimes x_2 x_1' \otimes x_2' y y'} := \sum_{\alpha} T^{(n)}_{x_1 x_1' y \alpha} T^{(n)}_{x_2 x_2' \alpha y'}$$

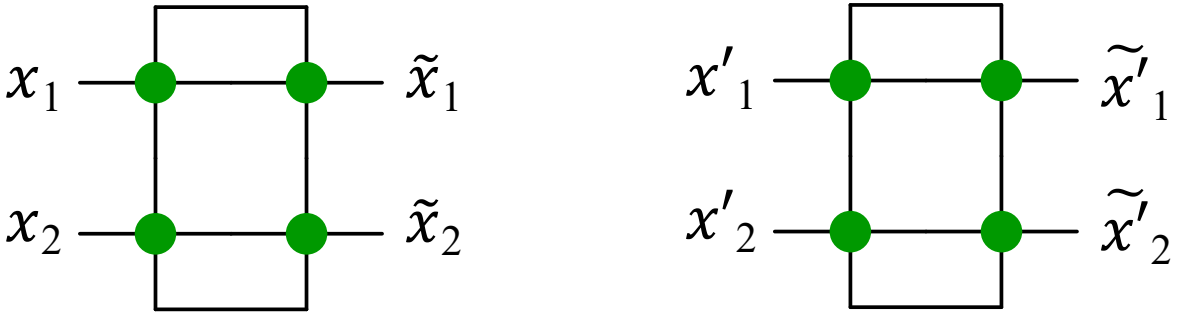


$T^{(n)}$: $2^n \times 2^n$-site lattice

$\Rightarrow M^{(n)}$: $2^n \times 2^{n+1}$-site lattice

(ii) Let us define the block spin transformation!

Unfolding M into matrices in two ways, consider $\mathcal{M}\mathcal{M}^{\dagger}$ such that



and by the eigen value decomposition,

$$\sum_{(jkl)} \mathcal{M}_{x(jkl)} \mathcal{M}^{*}_{\tilde{x}(jkl)} = \sum_{i} U^{\mathrm{L}}_{ix} \Lambda^{\mathrm{L}}_{i} U^{\mathrm{L}}_{i\tilde{x}}$$

$$\sum_{(ikl)} \mathcal{M}_{x'(ikl)} \mathcal{M}^{*}_{\tilde{x}'(ikl)} = \sum_{j} U^{\mathrm{R}}_{jx'} \Lambda^{\mathrm{R}}_{j} U^{\mathrm{R}}_{j\tilde{x}'}$$

Here, diagonal elements of Λ's are arranged in the descending order. Corresponding vectors in U's are also done in the same way.

(iii) Define the block spin transformation.

$$\epsilon_{\mathrm{L(R)}} \coloneqq \sum_{i>D_{\mathrm{cut}}} \Lambda_i^{\mathrm{L(R)}}$$

where D_{cut} is an integer we can choose freely.

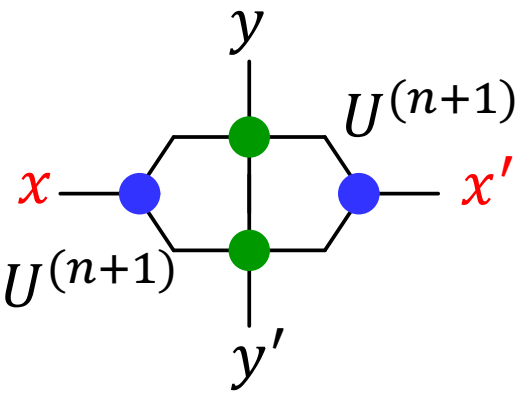
$$\epsilon_{\mathrm{L}} < \epsilon_{\mathrm{R}} \quad \Rightarrow \quad U^{(n+1)} \coloneqq U^{\mathrm{L}}$$

$$\epsilon_{\mathrm{L}} > \epsilon_{\mathrm{R}} \quad \Rightarrow \quad U^{(n+1)} \coloneqq U^{\mathrm{R}}$$

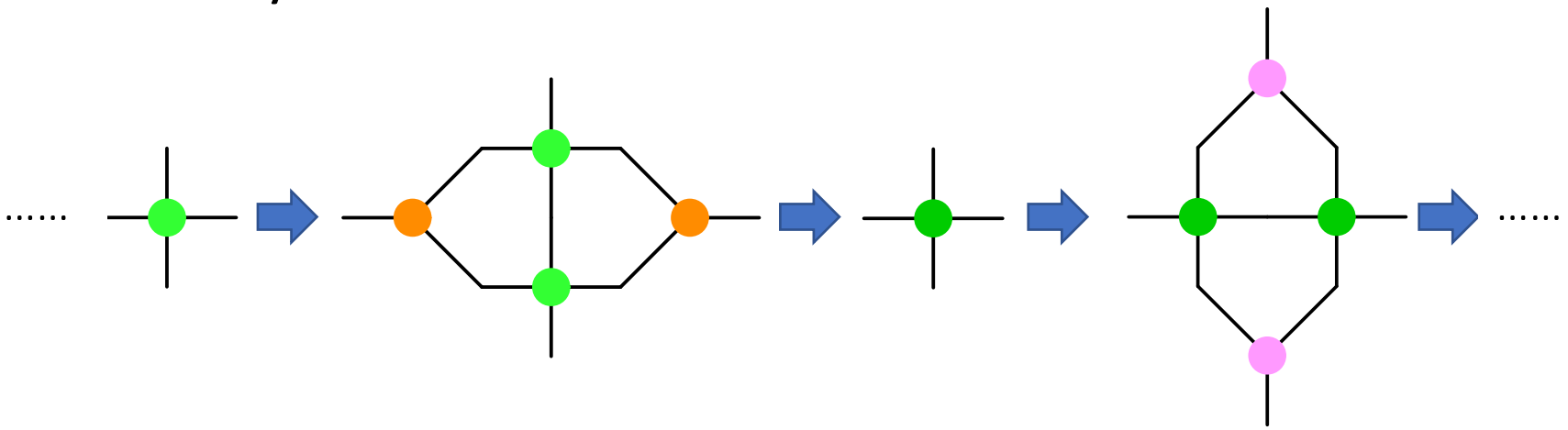
⇒ Using a part of unitary matrix, we define the transformation

$$T^{(n+1)}_{xx'yy'} \coloneqq \sum_{ij} U^{(n+1)}_{ix} M^{(n)}_{ijyy'} U^{(n+1)}_{jx'}$$

where $x, x' \in \{1,2,\cdots,D_{\mathrm{cut}}\}$.



If one aligns 2 local tensors vertically, it is necessary to do horizontally in the next.



In this way, $2N$-times HOTRG calculation gives the partition function on the $2^N \times 2^N$ square lattice by

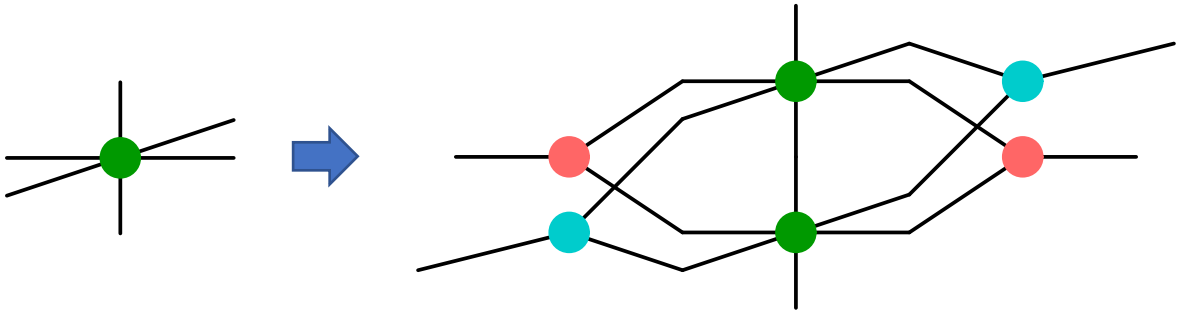
$$Z \approx \mathrm{Tr}\left[T^{(2N)}\right]$$

In the 2D HOTRG calculation,

$$\text{memory} \sim D_{\mathrm{cut}}^4$$

$$\text{computational time} \sim D_{\mathrm{cut}}^7$$

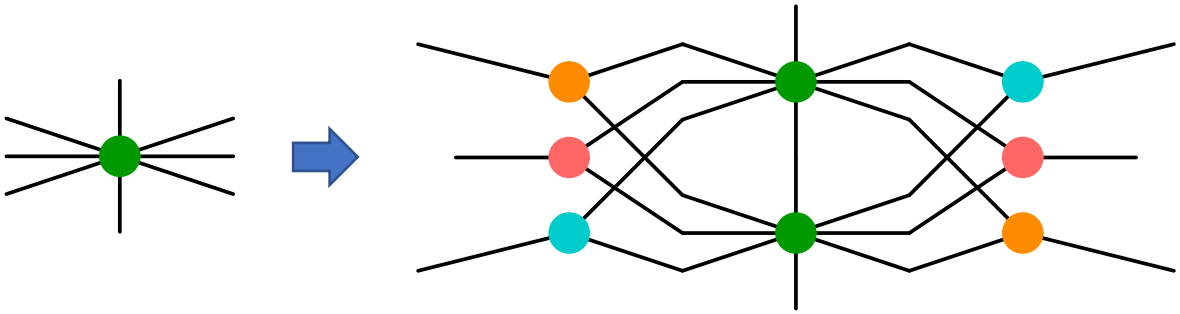
In the 3D HOTRG,



$$Z \approx \mathrm{Tr}\left[T^{(3N)}\right]$$

memory $\sim D_{\mathrm{cut}}^{6}$
computational time $\sim D_{\mathrm{cut}}^{11}$

In the 4D HOTRG,



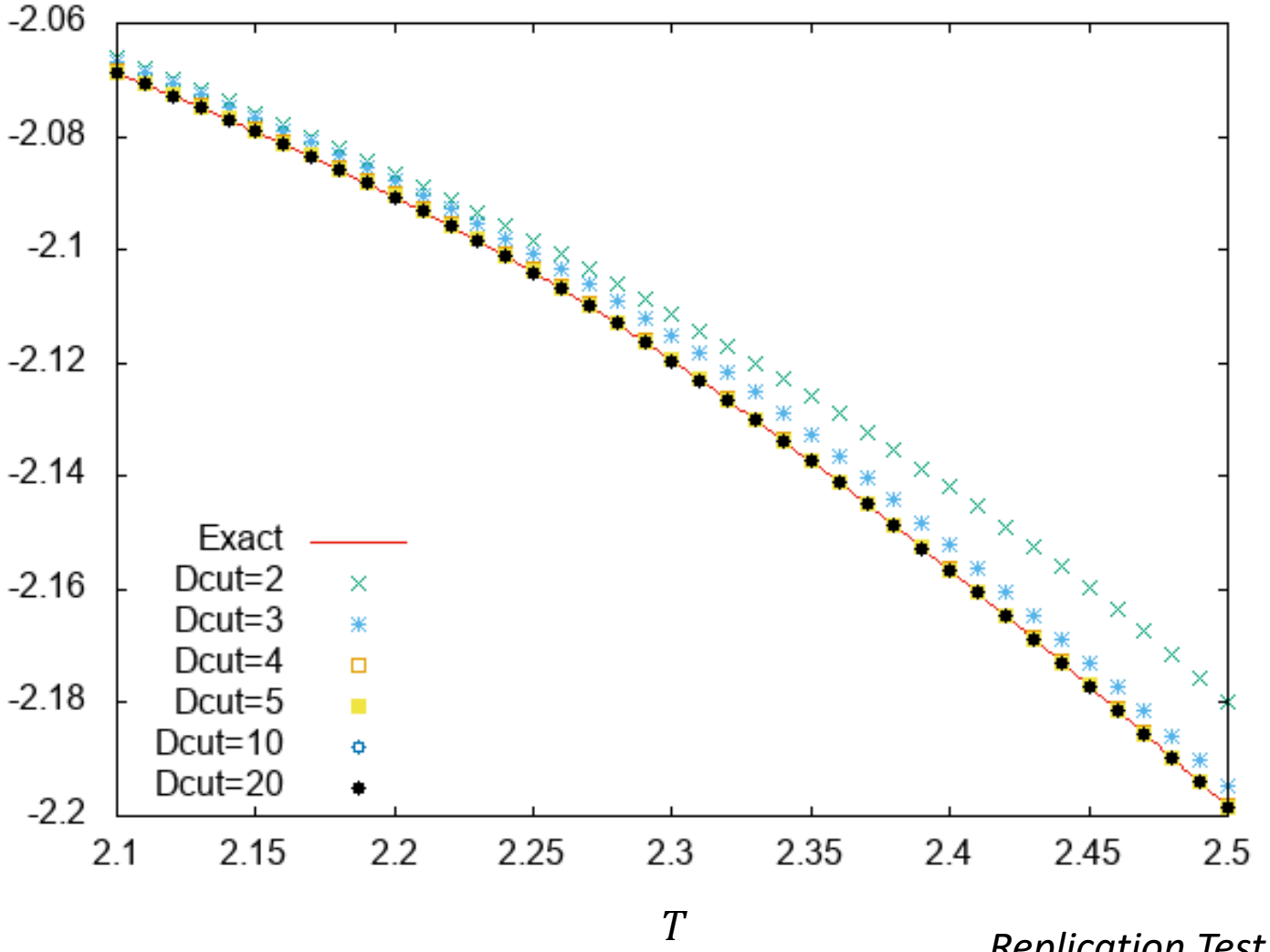
$$Z \approx \mathrm{Tr}\left[T^{(4N)}\right]$$

memory $\sim D_{\mathrm{cut}}^{8}$
computational time $\sim D_{\mathrm{cut}}^{15}$

Exact solution VS HOTRG

Free energy density $f(T, D_{\mathrm{cut}}) = -\frac{T}{V}\ln Z(T, D_{\mathrm{cut}})$

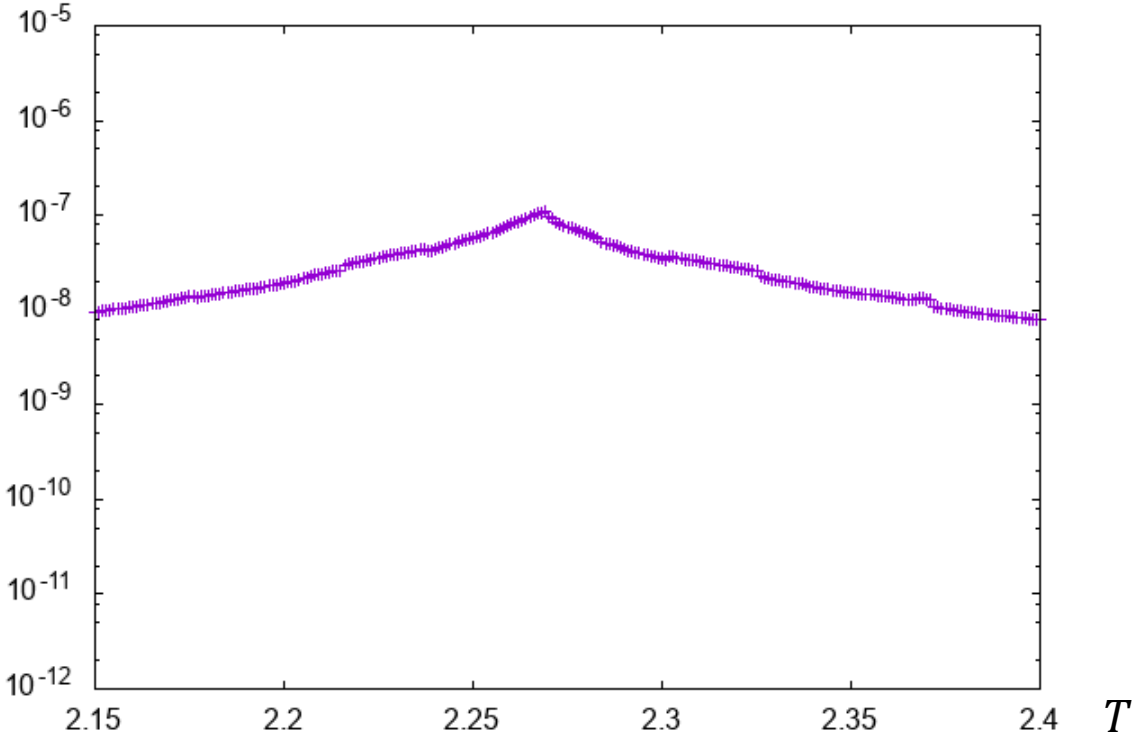
$V = 2^{25}\times 2^{25}$



Replication Test (S.A.)

Exact solution VS HOTRG

Relative error $\delta f(T) = \left| \frac{f(T, D_{\text{cut}} = 24) - f_{\text{exact}}(T)}{f_{\text{exact}}(T)} \right|$



In the vicinity of $T_c \approx 2.269$, eigen values of $\mathcal{M}\mathcal{M}^\dagger$ decreases slowly.

Replication Test (S.A.)

Conclusion

- HOTRG exploits a kind of block-spin transformation and it is easy to reach the thermodynamic limit.
- **In principle, HOTRG can be applied for higher dimensional systems**.
- However, large-D_{cut} calculation in higher dimensional systems is computationally challenging.
- Research for the "finite-D_{cut} scaling" is of great interest!

APPENDICES

- “Phase Transitions of Ferromagnetic Potts Model on the Simple Cubic Lattice”
  S. Wang, Z. Y. Xie, J. Chen, B. Normand, and T. Xiang
  Chin. Phys. Lett. 31, 070503 (2014)
- “A Multilinear Singular Value Decomposition”
  L. de Lathauwer, B. de Moor, and J. Vandewalle
  SIAM J. MATRIX ANAL. APPL. Vol.21, No.4, 1253-1278
- “ The density-matrix renormalization group”
  U. Schollwock
  Rev. Mod. Phys., 77, 259 (2005)
- 「『密度行列繰り込み群』の変分原理」
  西野友年, 奥西巧一, 引原俊哉
  物性研究 (1997), 68(2), 133-155
- Lecture by H. Matsueda @ Chuo Univ. (2018/8/29~31)

q-state Ferromagnetic Potts Model on Cubic Lattice

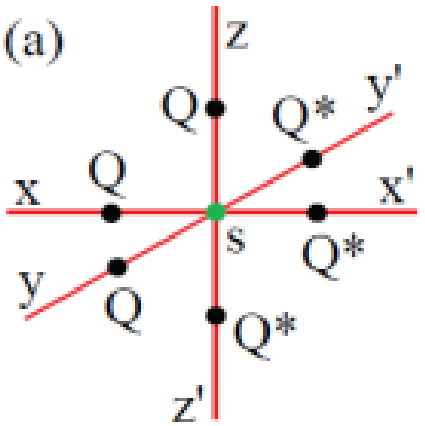
$$H = -\sum_{\langle ij \rangle} \delta_{s_i s_j} \qquad \text{where} \qquad s_i \in \{0,1,\cdots,q-1\}$$

$q = 2 \quad \Rightarrow \quad$ Ising Model

$q = 3 \quad \Rightarrow \quad Z_3$ is the center symmetry of $SU(3)$

The initial local tensor is given by

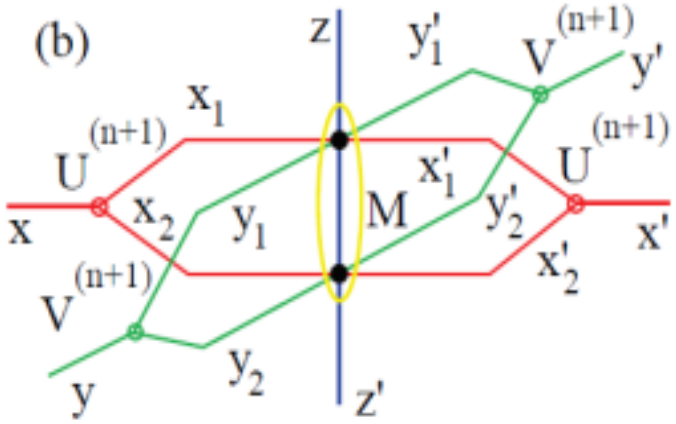
$$Q(s_i, l) = \mathrm{e}^{2\pi \mathrm{i} l s_i / q} \sqrt{\frac{\mathrm{e}^{\beta} - 1 + q\delta_{l,0}}{q}}$$



$$T_{xx'yy'zz'} := \sum_{s_i} Q(s_i, x) Q^*(s_i, x') Q(s_i, y) Q^*(s_i, y') Q(s_i, z) Q^*(s_i, z')$$

HOTRG calculation

Contract 2 tensors vertically and one has to truncate the size of the tensor corresponding to the rest directions :



The partition function on the cube is given by

$$Z \approx \mathrm{Tr}\left[T^{(3N)}\right]$$

After enough times of iteration, consider the $D_{\mathrm{cut}} \times D_{\mathrm{cut}}$ matrix,

$$A_{zz'} \coloneqq \sum_{xy} T^{(n)}_{xxyyzz'}$$

the degeneracy of A plays a good indicator of the transition point

$$X \coloneqq \frac{(\mathrm{Tr}[A])^2}{\mathrm{Tr}[A^2]}$$

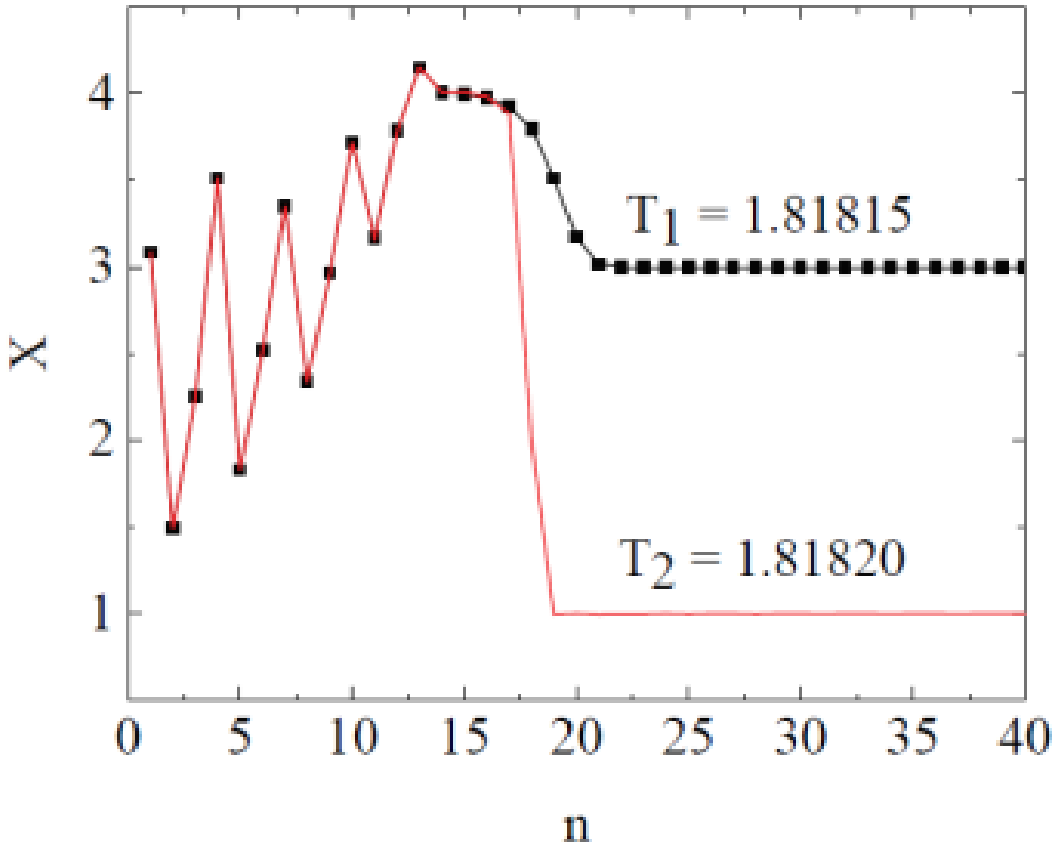
Ordered phase $\Rightarrow$ Z_3 symmetry is broken spontaneously

The largest eigenvalue of A is 3-fold degenerated, $X = 3$

Disordered phase $\Rightarrow$ Z_3 symmetry is preserved

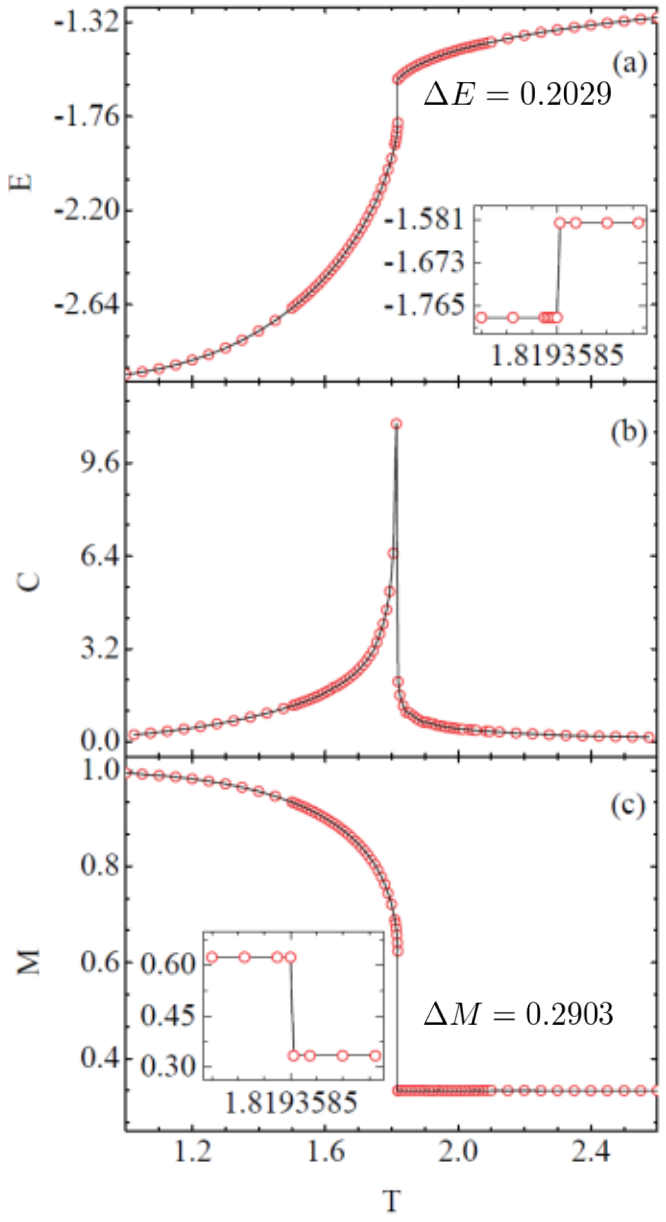
The largest eigenvalue of A is unique, $X = 1$

S. Wang, Z. Y. Xie, J. Chen, B. Normand, and T. Xiang (2014)



⇒ Distinguishable 2 phases are confirmed

$D_{\text{cut}} = 14$



$$E = -\frac{1}{V}\frac{\partial \ln Z}{\partial \beta}$$

$$C = \frac{\beta^2}{V}\frac{\partial^2 \ln Z}{\partial \beta^2}$$

⇒ Numerical derivative

$$M = \frac{1}{V}\sum_i \delta_{s_i,0}$$

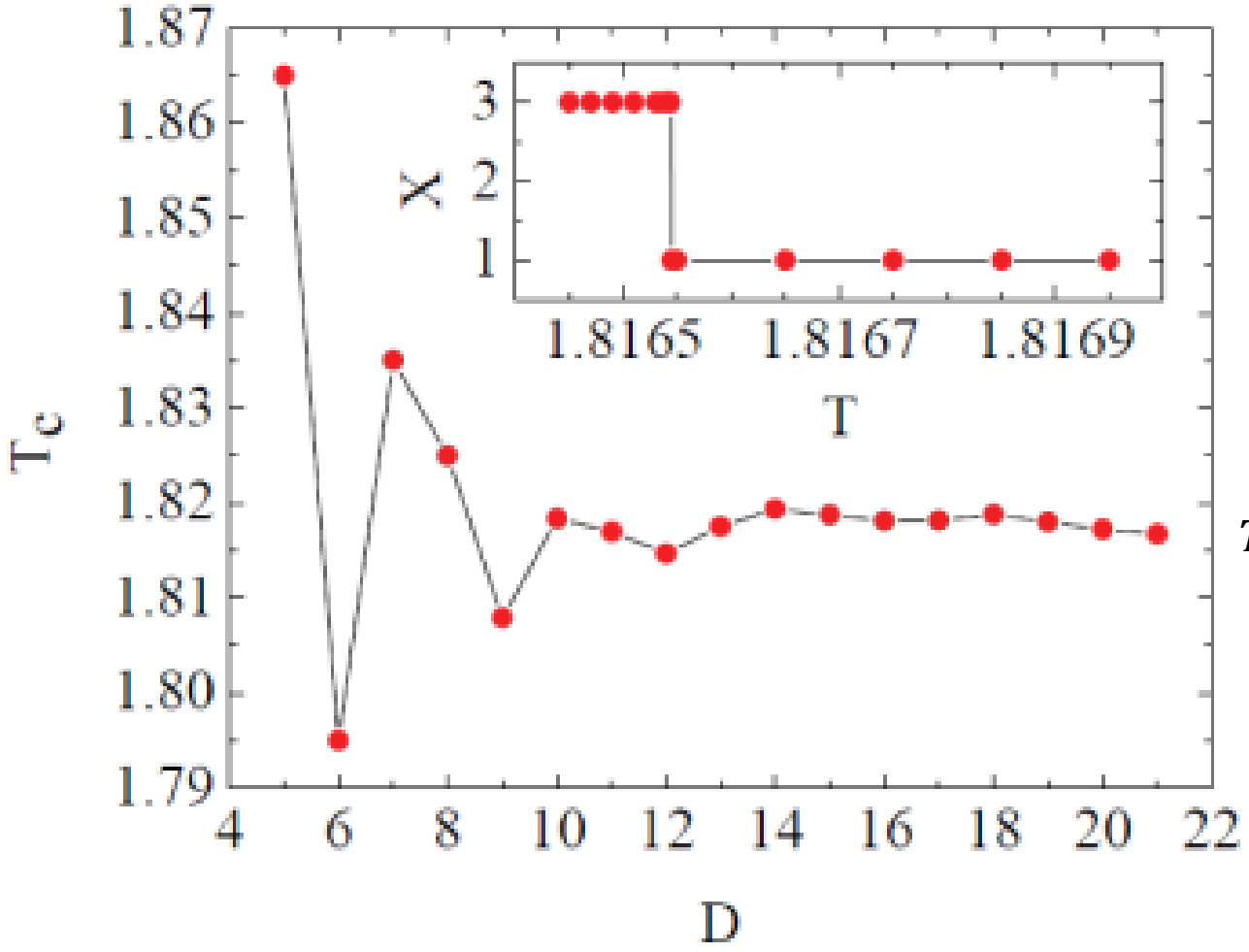
⇒ Adding the source term to the Boltzmann weight, i.e. the initial tensor $T^{(0)}$.

⇒ Numerical derivative w.r.t. the magnetic field.

S. Wang, Z. Y. Xie, J. Chen, B. Normand, and T. Xiang (2014)

Investigation of the D_{cut}–dependence of T_{c}, but ……

the lack of convergence !



$T_{\mathrm{c}} = 1.8166(5)$

S. Wang, Z. Y. Xie, J. Chen, B. Normand, and T. Xiang (2014)

Comparison of the estimated jump of internal energy and transition point

S. Wang, Z. Y. Xie, J. Chen, B. Normand, and T. Xiang (2014)

Method	ΔE	T_c
Series expansion (1979) [48]		1.7289(12)
Monte Carlo RG (1979) [14]		1.818
Monte Carlo (1982, $L = 8$) [45]	0.12	1.81
Pair approximation (1982) [45]	0.123	1.879
Monte Carlo (1987, $L = 16$) [46]	0.2222(7)	1.81618(7)
Monte Carlo (1991, $L = 36$) [47]	0.16062(52)	1.816455(35)
Monte Carlo (1997, $L = 36$) [18]	0.1614(3)	1.816316(33)
Monte Carlo (2007, $L = 50$) [5]	0.1643(8)	1.816315(19)
TPVA (2002) [19]	0.228	1.8195
HOTRG (this work)	0.2029	1.8166
	($D = 14$)	($D = 21$)

[5] A. Bazavov and B. A. Berg, Phys. Rev. D 75, 094506 (2007)
[14] H. W. J. Blote and R. H. Swendsen, Phys. Rev. Lett. 43, 799 (1979)
[18] W. Janke and R. Villanova, Nucl. Phys. B 489, 679 (1997)
[19] T. Nishino, K. Okunishi, Y. Hieida, N. Maeshima, and Y. Akutsu, Nucl. Phys. B 575, 504 (2000); A. Gendiar and T. Nishino, Phys. Rev. E 65, 046702 (2002)
[45] I. Ono and K. Ito, J. Phys. C 15, 4417 (1982)
[46] W. G. Wilson and C. A. Vanse, Phys. Rev. B 36, 587 (1987)
[47] N. A. Alves, B. A. Berg, and R. Villanova, Phys. Rev. B 43, 5846 (1991)
[48] S. Miyashita, D. D. Betts, and C. J. Elliott, J. Phys. A 12, 1605 (1979)

$q = 2$ Potts Model $\Leftrightarrow$ Ising Model

$$H = -J \sum_{\langle ij \rangle} \delta_{s_i, s_j} - h \sum_i \delta_{s_i, 1} \quad \text{where} \quad s_i \in \{0,1\}$$

Regarding $s_i = 0$ as $\sigma_i = -1$ and $s_i = 1$ as $\sigma_i = +1$,

$$\delta_{s_i, s_j} = \frac{1 + \sigma_i \sigma_j}{2} \qquad \delta_{s_i, 1} = \frac{1 - \sigma_i}{2}$$

hold.

That is, the Hamiltonian of 2-state Potts Model can be written as

$$H = -\frac{J}{2} \sum_{\langle ij \rangle} \sigma_i \sigma_j + \frac{h}{2} \sum_i \sigma_i + \text{Const.}$$

This is nothing but the Ising model.

Singular Value Decomposition (SVD)

For any complex $I_1 \times I_2$ -matrix A can be written as the product

$$A = U^{(1)} S U^{(2)\dagger}$$

where

1. $U^{(1)}$ is an $I_1 \times I_1$ unitary matrix.

2. $U^{(2)}$ is an $I_2 \times I_2$ unitary matrix.

3. S is an $I_1 \times I_2$-matrix such that

   (i) Pseudo-diagonality : $S = \mathrm{diag}\left(\sigma_1, \sigma_2, \cdots, \sigma_{\min(I_1, I_2)}\right)$

   (ii) Ordering : $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min(I_1, I_2)} \geq 0$

σ_i 's are singular values of A and the i-th column vectors of $U^{(1)}$ and $U^{(2)}$ are, resp., i-th left and right singular vector.

SVD introduces virtual dof

Consider the system consisting of subsystems X and Y.
Setting the pure state of the total system as

$$|\psi\rangle = \sum_{x\in X}\sum_{y\in Y} \psi(x,y)|x\rangle \otimes |y\rangle$$

If $\psi(x,y) = u(x)v(y)$, then the state is separable. Actually,

$$|\psi\rangle = \left(\sum_{x\in X} u(x)|x\rangle\right) \otimes \left(\sum_{y\in Y} v(y)|y\rangle\right)$$

SVD introduces virtual dof

Regarding $\psi(x,y)$ as a matrix element. By SVD,

$$\psi(x,y) = \sum_{l=1}^{N} u_l(x)\sigma_l v_l(y)$$

If $N > 1$, the state is not pure. However, as a matrix,

$$\psi = U\Sigma V^\dagger = \left(U\Sigma^{1/2}\right)\left(V\Sigma^{1/2}\right)^\dagger =: \tilde{U}\tilde{V}^\dagger$$

This looks very similar with $\psi(x,y) = u(x)v(y)$.

Higher-Order Singular Value Decomposition (HOSVD)

Any complex $I_1 \times I_2 \times \cdots \times I_n$-tensor A can be written as the product

$$A_{i_1 i_2 \cdots i_n} = \sum_{j_1 j_2 \cdots j_n} S_{j_1 j_2 \cdots j_n} U^{(1)}_{j_1 i_1} U^{(2)}_{j_2 i_2} \cdots U^{(n)}_{j_n i_n}$$

where

1. $U^{(k)}$ is a unitary $I_k \times I_k$ -matrix.

2. S is a complex $I_1 \times I_2 \times \cdots \times I_n$-tensor such that

(i) Fixing the k-th index of S, say $S_{i_k = \alpha}$, and if $\alpha \neq \beta$, then

$$\sum_{i_1 i_2 \cdots i_n} S_{i_1 i_2 \cdots i_{k-1} \alpha i_{k+1} \cdots i_n} S_{i_1 i_2 \cdots i_{k-1} \beta i_{k+1} \cdots i_n} = 0$$

(ii) Ordering :

$$\left\| S_{i_k = \alpha} \right\| := \sqrt{\sum_{i_1 i_2 \cdots i_n} S_{i_1 i_2 \cdots i_{k-1} \alpha i_{k+1} \cdots i_n} S_{i_1 i_2 \cdots i_{k-1} \alpha i_{k+1} \cdots i_n}}$$

$$\left\| S_{i_k = 1} \right\| \geq \left\| S_{i_k = 2} \right\| \geq \cdots \geq \left\| S_{i_k = I_k} \right\| \geq 0$$

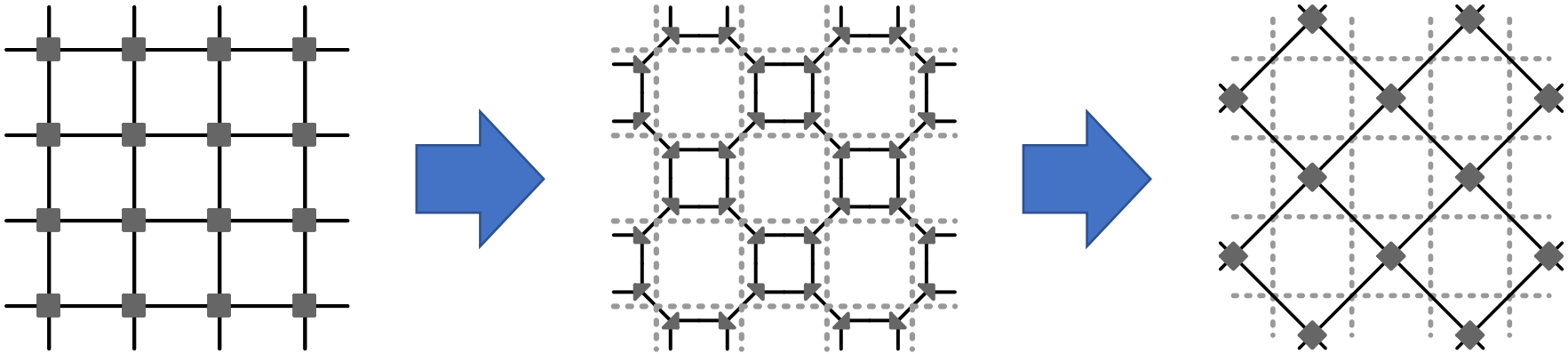
TRG VS HOTRG

"Tensor Renormalization Group"

M. Levin and C. P. Nave, Phys. Rev. Lett. 99 120601 (2007)

In TRG, SVD for $T_{xx'yy'}$ itself is a key in compressing the network.

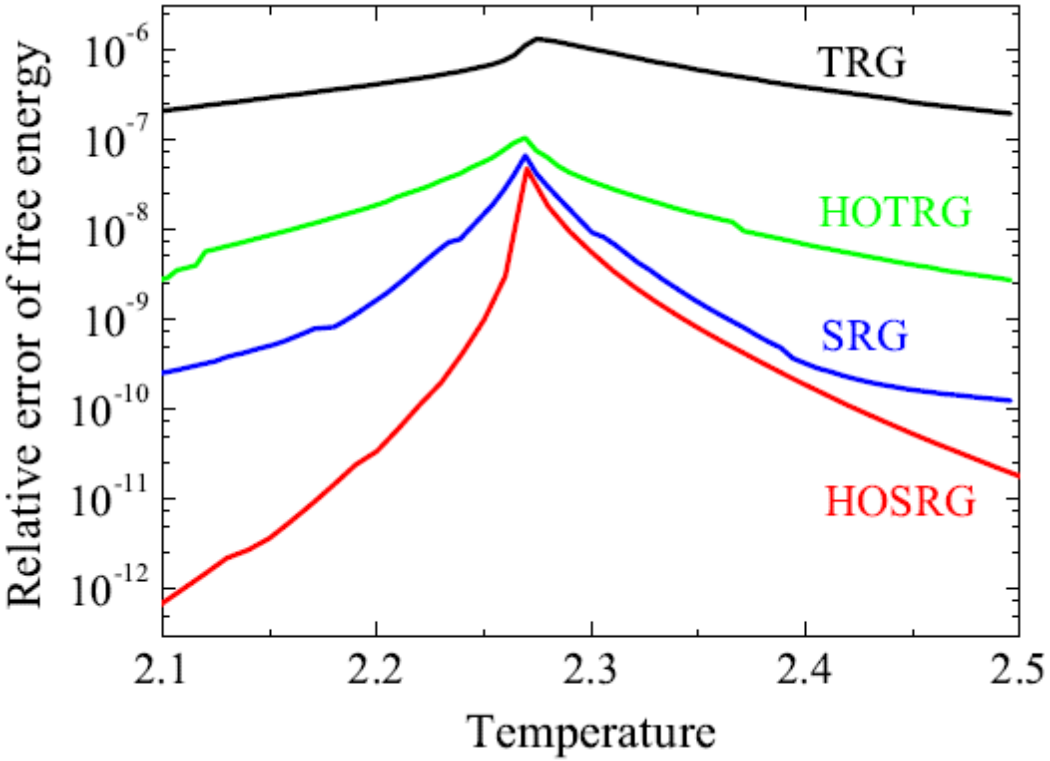
$$T_{xx'yy'} \approx \sum_{l=1}^{D_{\mathrm{cut}}} \sqrt{\lambda_l} U_{x'y,l} \sqrt{\lambda_l} V_{y'x,l} \quad \text{or} \quad T_{xx'yy'} \approx \sum_{l=1}^{D_{\mathrm{cut}}} \sqrt{\lambda_l} U_{xy,l} \sqrt{\lambda_l} V_{x'y',l}$$



TRG VS HOTRG

On the other hand, HOTRG exploits HOSVD of $M = \sum TT$, so T itself is not decomposed.

Ex) 2D Ising model



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Density Matrix Renormalization Group (1/9)

DMRG is a kind of variational method in which **one optimizes the variational wave function expressed by the matrix product of a part of a unitary matrix, which diagonalizes a density matrix**.

The ground state and its energy of the 1D quantum systems can be obtained with high accuracy!

Fundamental Fact

For any projection operator P, such that

$$P^2 = P \text{ and } \mathrm{Tr}P = \chi$$

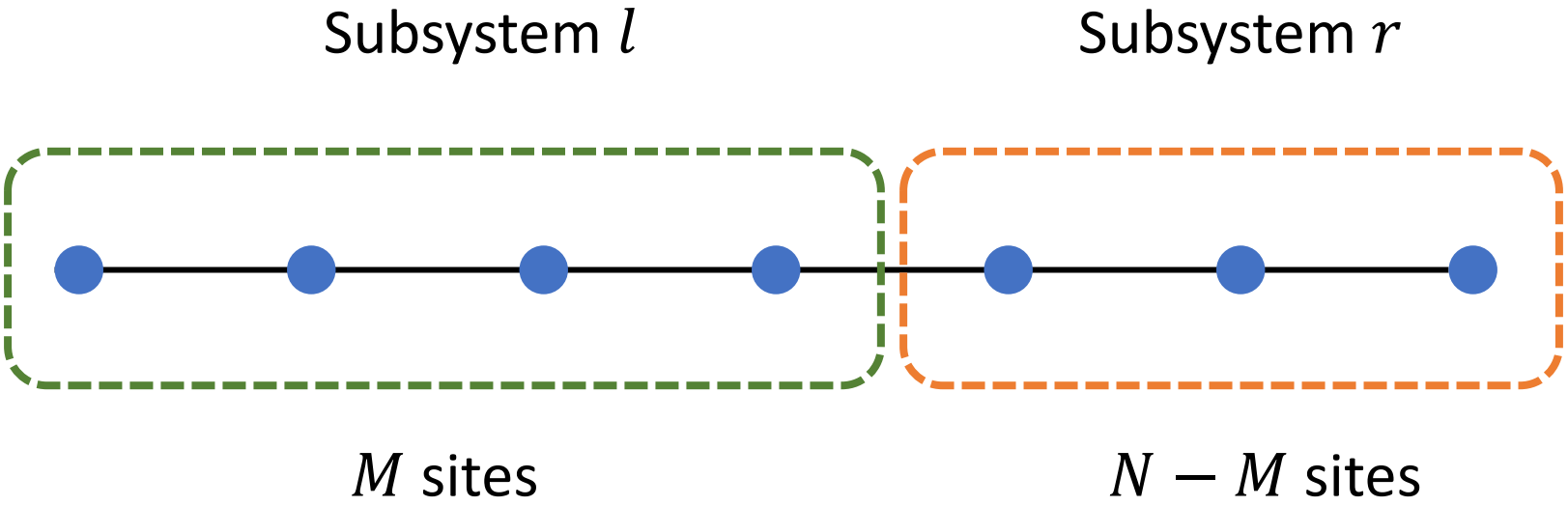
the inequality,

$$\mathrm{Tr}\rho \geq \mathrm{Tr}P\rho$$

holds.

Density Matrix Renormalization Group (2/9)

Ex) 1D N-site $S = 1/2$ Heisenberg Model



$$\hat{H} = \sum_{i=1}^{N-1} \vec{S}_i \cdot \vec{S}_{i+1}$$

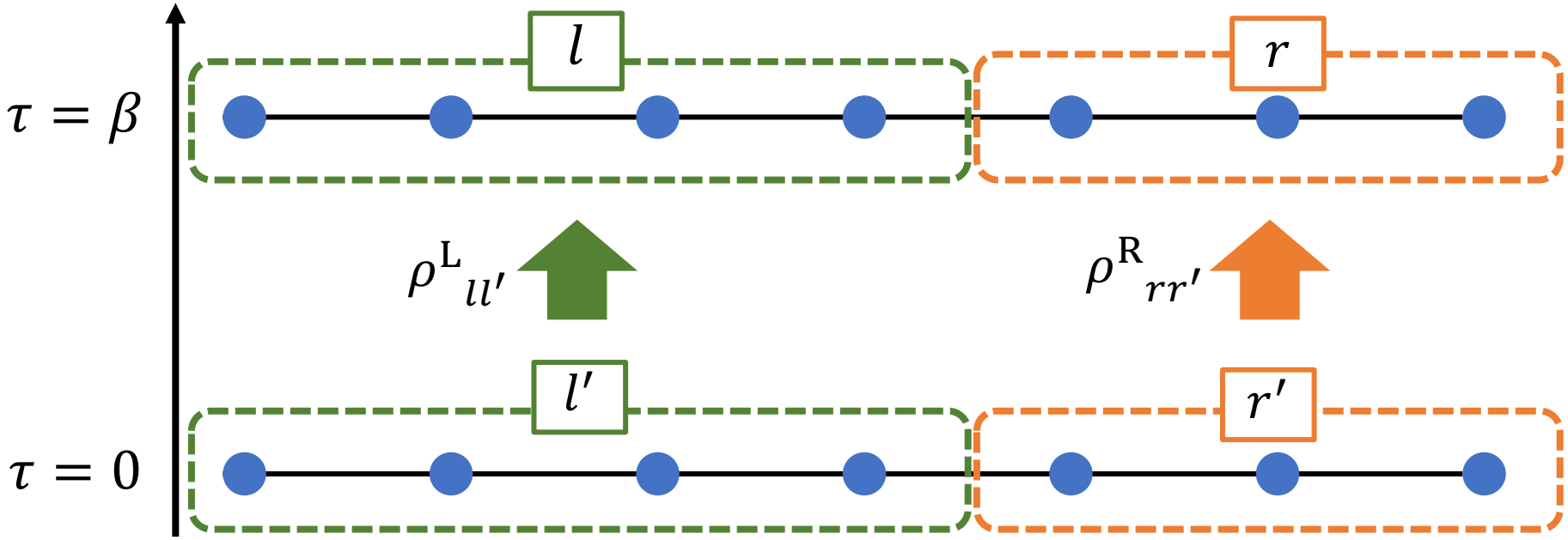
The matrix representation of $\hat{H}$ is given by $2^N \times 2^N$ symmetric matrix.

Density Matrix Renormalization Group (3/9)

$\{|lr\rangle\}$: the basis of the Hilbert space for the total system

$$H_{lrl'r'} := \langle lr|\hat{H}|l'r'\rangle$$

For the density matrix $\hat{\rho} = \mathrm{e}^{-\beta\hat{H}}$, the element $\rho_{lrl'r'}$ describes the following development:



Density Matrix Renormalization Group (4/9)

$$Z = \mathrm{Tr}\hat{\rho} = \mathrm{Tr}_{\mathrm{L(R)}}\hat{\rho}^{\mathrm{L(R)}}$$

Let us compress $2^M \times 2^M$ matrix $\hat{\rho}^{\mathrm{L}}$ into $m \times m$ matrix $\tilde{\rho}$.
Since $\hat{\rho}^{\mathrm{L}}$ is symmetric,

$$\exists \hat{Q} = \left[\vec{q}_1, \vec{q}_2, \cdots, \vec{q}_{2^M}\right] \quad \text{s.t.} \quad \hat{Q}^{\mathrm{T}}\hat{\rho}^{\mathrm{L}}\hat{Q} = \mathrm{diag}\left(\lambda_1, \lambda_2, \cdots, \lambda_{2^M}\right)$$

Assuming the diagonal elements are arranged in the descending order,

$$\tilde{Z} \coloneqq \sum_{i=1}^{m} \lambda_i$$

gives the best approximation of Z. So, we choose $m \times m$ matrix $\tilde{\rho}$ as

$$\tilde{\rho} = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m)$$

Density Matrix Renormalization Group (5/9)

This selection is equally expressed by the transformation

$$\tilde{\rho} \coloneqq \tilde{Q}^{\mathrm{T}}\, \hat{\rho}^{\mathrm{L}} \tilde{Q}$$

$$\tilde{Q} = [\vec{q}_1, \vec{q}_2, \cdots, \vec{q}_m]$$

The matrix $\tilde{Q}$ truncates the size of $\hat{\rho}^{\mathrm{L}}$.

Defining $P \coloneqq \tilde{Q}\tilde{Q}^{\mathrm{T}}$, this transformation is expressed by

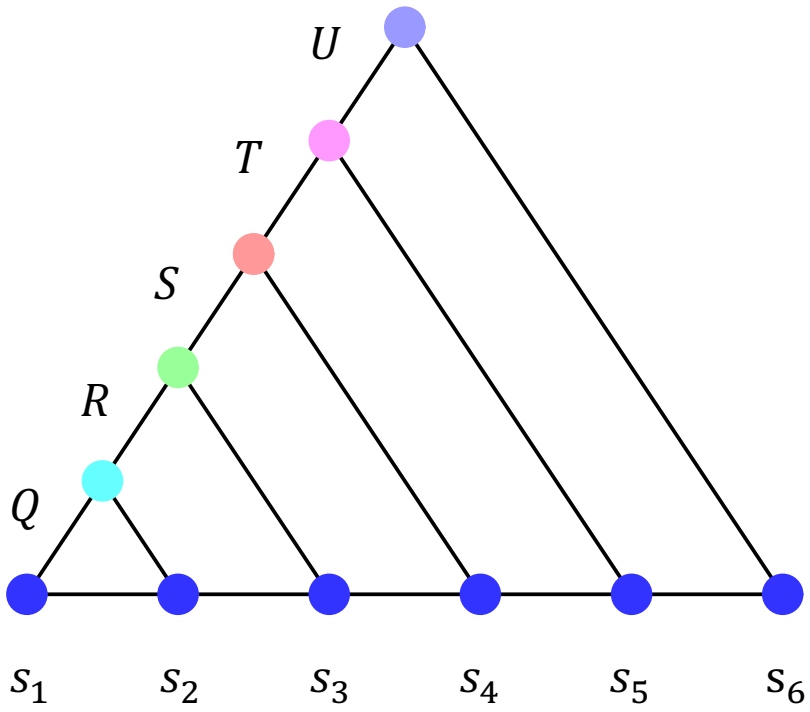
$$\tilde{Z} = \mathrm{Tr}_{\mathrm{L}} P \hat{\rho}^{\mathrm{L}}$$

The projection P can be seen as a block spin transformation from the original system to the coarse-grained system.

Density Matrix Renormalization Group (6/9)

Let us consider the following example

Ex) $N = 6$



Starting the case $M = 2$ and $\hat{\rho}^{\mathrm{L}}$ is diagonalized iteratively.

Density Matrix Renormalization Group (7/9)

Finally, we approach the total $\hat{\rho}$ and

$$(QRSTU)^{\mathrm{T}}\hat{\rho}(QRSTU) = \mathrm{diag}(\mathrm{e}^{-\beta E_0}, \mathrm{e}^{-\beta E_1}, \cdots)$$

This indicates that the energy eigen state is given by

$$\Psi^{\nu}_{s_1 s_2 s_3 s_4 s_5 s_6} = \sum_{\xi\zeta\eta\mu} Q_{s_1 s_2,\xi} R_{\xi s_3,\zeta} S_{\zeta s_4,\eta} T_{\eta s_5,\mu} U_{\mu s_6,\nu}$$

which satisfies

$$\hat{H}\Psi^{\nu} = E_{\nu}\Psi^{\nu}$$

Density Matrix Renormalization Group (8/9)

Now the eigen state is expressed by the unitary matrix product

$$\Psi^{\nu}_{s_1 s_2 s_3 s_4 s_5 s_6} = \sum_{\xi\zeta\eta\mu} Q_{s_1 s_2,\xi} R_{\xi s_3,\zeta} S_{\zeta s_4,\eta} T_{\eta s_5,\mu} U_{\mu s_6,\nu}$$

In the zero-temperature limit, one expects

$$\tilde{\Psi}^{0}_{s_1 s_2 s_3 s_4 s_5 s_6} = \sum_{\xi\zeta\eta\mu} \tilde{Q}_{s_1 s_2,\xi} \tilde{R}_{\xi s_3,\zeta} \tilde{S}_{\zeta s_4,\eta} \tilde{T}_{\eta s_5,\mu} \tilde{U}_{\mu s_6,0}$$

gives a good approximation of Ψ^0!

Density Matrix Renormalization Group (9/9)

The real DMRG algorithm starts with the ansatz: the ground state of the 1D quantum many-body system is given by

$$\widetilde{\Psi}^{0}_{s_1 s_2 s_3 s_4 s_5 s_6 \cdots} = \sum_{\xi\zeta\eta\mu\cdots} \tilde{Q}_{s_1 s_2,\xi} \tilde{R}_{\xi s_3,\zeta} \tilde{S}_{\zeta s_4,\eta} \tilde{T}_{\eta s_5,\mu} \widetilde{U}_{\mu s_6,\nu} \cdots$$

This is called “Matrix Product State” ansatz.

Here, the all elements of the tilde matrices play the role of variational parameters. The goal of DMRG is to minimize

$$\langle E \rangle = \frac{\langle \widetilde{\Psi}^{0} | \hat{H} | \widetilde{\Psi}^{0} \rangle}{\langle \widetilde{\Psi}^{0} | \widetilde{\Psi}^{0} \rangle}$$

Finite$-\chi$ scaling (semi-infinite 1D quantum chain)

In the numerical analysis of critical phenomena, we have to care the correction originating from finite size and finite χ (# of states). Ground state energy is modified by these corrections as

$$E = E_0 + \frac{A}{\xi^2} + \frac{B}{\xi}\epsilon(\chi)$$

The second term is the finite-size correction and the third term is the finite-χ correction:

$$\epsilon(\chi) = \sum_{i=\chi+1}^{\infty} \lambda_i$$

Finite$-\chi$ scaling (semi-infinite 1D quantum chain)

Focusing on the finite-χ correction, it is known that χ-dependence of the entanglement entropy is given by

$$S \sim \frac{1}{\sqrt{\frac{12}{c}+1}} \log\chi =: \frac{c\kappa}{6}\log\chi$$

For the semi-infinite 1D chain, the Calabrese-Cardy formula says

$$S \sim \frac{c}{6}\log\xi$$

From these,

$$\xi \sim \chi^{\kappa}$$