String Field Theory and Brane Superpotentials

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$\phi \in \Omega^1(F^* \otimes F)$: gauge field

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$Q(u \bullet v) = (Qu) \bullet v + (-1)^{|u|} u \bullet (Qv)$ derivation

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easily verified

$[u, v] = -(-1)^{|u|\cdot|v|}[v, u]$

$[u, [v, w]] = [[u, v], w] + (-1)^{|u|\cdot|v|}[v, [u, w]]$ Jacobi identity

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$$\int : \mathscr{H} \to \mathbb{C} \quad \text{such that} \quad \langle u, v\rangle = \int u \bullet v \tag{5}$$

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Gauge symmetry $\mathscr{G}$

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Moduli space

$$\mathscr{M} = \left\{ \phi \in \mathscr{H}^1 \,\middle|\, Q\phi + \frac{1}{2}[\phi, \phi] = 0 \right\} \Big/ \mathscr{G} \tag{9}$$

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Orthogonal projector $\pi : \mathscr{H} \to \mathbb{H}$

$$\pi = 1 - [Q, GQ^{\dagger}]. \tag{14}$$

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$P \sim b_0 / L_0$

Potential

$$W[\varphi] = \sum_{n\geq 3} \frac{1}{n}(-1)^{n(n-1)/2} \langle\!\langle \varphi, \varphi, \dots, \varphi \rangle\!\rangle^{(n)}_{\mathrm{tree}}, \quad \varphi \in \mathbb{H}^1. \tag{16}$$

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such that $\langle\!\langle u_1, u_2, \ldots, u_n \rangle\!\rangle^{(n)}_{\text{tree}} = \langle u_1, r_{n-1}(u_2, \ldots, u_n) \rangle.$

Step 1 $\lambda_n : \mathscr{H}^{\otimes n} \to \mathscr{H}$

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$$\lambda_2(u_1, u_2) = u_1 \bullet u_2, \tag{18}$$

$$\begin{aligned}\lambda_n(u_1, \dots, u_n) &= -(-1)^{n|u_1|} u_1 \bullet P\lambda_{n-1}(u_2, \dots, u_n) \\ &- \sum_{k=2}^{n-2} (-1)^s P\lambda_k(u_1, \dots, u_k) \bullet P\lambda_{n-k}(u_{k+1}, \dots, u_n) \\ &+ (-1)^{n-1} P\lambda_{n-1}(u_1, \dots, u_{n-1}) \bullet u_n.\end{aligned} \tag{19}$$

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Example

$\lambda_3(u_1, u_2, u_3) = P(u_1 \bullet u_2) \bullet u_3 - (-1)^{|u_1|} u_1 \bullet P(u_2 \bullet u_3).$

Step 2 $r_n : \mathbb{H}^{\otimes n} \to \mathbb{H}$

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cyclicity (under the assumption $c^2 = 1$)

$$\langle u_1, r_n(u_2, \ldots, u_n, u_{n+1}) \rangle = (-1)^{n(|u_2|+1)} \langle u_2, r_n(u_3, \ldots, u_{n+1}, u_1) \rangle. \tag{22}$$

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$$\delta\varphi = [\eta, \varphi], \qquad \eta \in \mathbb{H}^0. \tag{24}$$

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Moduli space $\mathscr{M}_W$

$$\mathscr{M}_W = \left\{ \varphi \in \mathbb{H}^1 \,\middle|\, \sum_{n\geq 2} (-1)^{n(n+1)/2} r_n(\varphi^{\otimes n}) = 0 \right\} \Big/ \mathscr{G}_W \tag{25}$$

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A_∞ algebra structure on $(\mathbb{H}, \{r_n\}_{n\geq 2})$

$$\sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^s r_k(u_1, \dots, u_j, r_l(u_{j+1}, \dots, u_{j+l}), u_{j+l+1}, \dots, u_n) = 0. \tag{27}$$

$s = l(|u_1| + \cdots + |u_j|) + j(l-1) + (k-1)l$

A_∞ algebra as generalization of DGAA

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$$
\begin{aligned}
&(r_1)^2(u_1) = 0,\\
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A_∞ algebra as generalization of DGAA

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DGAA is A_∞ algebra

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A_∞ relations in $\{r_n\}$ $\iff$ $\delta^2 = 0$.

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$$F_1 = j : \mathbb{H} \hookrightarrow \mathscr{H}, \tag{31}$$

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$\{F_n\} : \mathbb{H} \to \mathscr{H}$ is quasi-isomorphism

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$(\mathbb{H},\{m_n\})$ is L_∞ algebra

$$\sum_{\substack{k+l=n+1\\ \sigma\in\mathrm{Sh}(k,n)}}(-1)^{k(l-1)}\operatorname{sgn}(\sigma)\epsilon(\sigma;\vec{u})\,m_l(m_k(u_{\sigma(1)},\ldots,u_{\sigma(k)}),u_{\sigma(k+1)},\ldots,u_{\sigma(n)})=0. \tag{35}$$

$$G_n(u_1, \ldots, u_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \epsilon(\sigma; \vec{u}) F_n(u_{\sigma(1)}, \ldots, u_{\sigma(n)}) \tag{36}$$

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$$\mathcal{M}_W \xrightarrow{\sim} \mathcal{M}. \tag{38}$$

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Potential

$$W[\varphi] = \sum_{n\geq 2} \frac{(-1)^{n(n+1)/2}}{n+1} \int_X \Omega \wedge \mathrm{Tr}\left(\varphi \bullet r_n(\varphi^{\otimes n})\right). \tag{47}$$