Sachdev-Ye-Kitaev (SYK) model

KItaev, a talk at KITP

Polchinski and Rosenhaus arXiv:1601.06768[hep-th]

Maldacena and Stanford arXiv:1604.07818[hep-th]

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SYK model

A statistical system with Hamiltonian

$$H = \frac{1}{4!} \sum_{i,j,k,l} J_{ijkl} \psi_i \psi_j \psi_k \psi_l$$

$$\langle \mathcal{O} \rangle = \int \prod \left[dJ_{ijkl} P\left(J_{ijkl}\right) \right] \mathrm{Tr}\left(e^{-\beta H} \mathcal{O} \right)$$

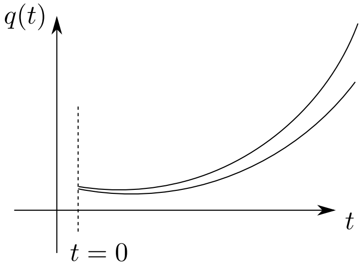
- $\psi_i \ (i = 1, \cdots N)$ are Grassmann odd real variables satisfying
  $\{\psi_i, \psi_j\} = \delta_{i,j}$
- J_{ijkl} has the distribution $P\left(J_{ijkl}\right) \propto \exp\left(-\frac{N^3 J_{ijkl}^2}{12 J^2} \right)$

This model describes a quantum gravity in the limit $N \to \infty$.

SYK model

1. How did Kitaev come up with this kind of model?
2. How can one calculate things in this model?

Chaos in classical systems



$$q\,(0) = q_0 \quad \to \quad q\,(t)$$

$$q'\,(0) = q_0 + \delta q \quad \to \quad q'\,(t)$$

$$\delta q(t) \sim \delta q \exp\left(\lambda_{\mathrm{L}} t\right)$$

$\lambda_{\mathrm{L}}\ (\geq 0)$: Lyapunov exponent

- $\delta q(t)$ grows until it becomes $\mathcal{O}(1)$
- This behavior can be rewritten as $\frac{dq(t)}{dq(0)} \propto \exp\left(\lambda_{\mathrm{L}} t\right)$

Chaos in quantum systems

$$[q(t), p(0)] = i\hbar \frac{dq\,(t)}{dq\,(0)} \propto \exp\left(\lambda_{\mathrm{L}} t\right)$$

- The chaotic behavior will be found in the correlation function $-\left\langle \left([q(t), p(0)]\right)^2 \right\rangle$
- Here we discuss finite temperature systems

$$-\left\langle \left([q(t), p(0)]\right)^2 \right\rangle = \frac{-\mathrm{Tr} e^{-\beta H} \left([q(t), p(0)]\right)^2}{\mathrm{Tr} e^{-\beta H}} \sim \hbar^2 \exp\left(2\lambda_{\mathrm{L}} t\right)$$

for $\frac{1}{\lambda_{\mathrm{L}}} \ll t \ll t_*$

Out of time order correlation functions

$$
\begin{aligned}
&-\frac{1}{Z}\mathrm{Tr} e^{-\beta H}\left([q(t),p(0)]\right)^2 \\
&\quad = \frac{1}{Z}\mathrm{Tr}\left[q(t)e^{-\beta H}q(t)p(0)p(0) + q(t)q(t)p(0)e^{-\beta H}p(0)\right] \\
&\qquad - \frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}q(t)p(0)q(t)p(0) + e^{-\beta H}p(0)q(t)p(0)q(t)\right]
\end{aligned}
$$

- The chaotic behavior seems to come from the second term.
- In general, we expect that out of time order correlation functions behave as

$$\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}V(0)W(t)V(0)W(t)\right] \sim f_0 - f_1 e^{2\lambda_{\mathrm{L}} t}$$

for $\frac{1}{\lambda_{\mathrm{L}}} \ll t \ll t_*$

A bound on chaos

- AdS/CFT: Some strongly coupled statistical systems can be described by gravity theory with a black hole.
- Systems corresponding to black holes have (Shenker-Stanford)

$$\mathrm{Tr}\left[yV(0)yW(t)yV(0)yW(t)\right] \sim f_0 - f_1 e^{\frac{2\pi}{\beta}t}$$

with $y = (\frac{1}{Z}e^{-\beta H})^{\frac{1}{4}}$.
- For $\beta \ll t$, this implies

$$\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}V(0)W(t)V(0)W(t)\right] \sim f_0 - f_1 e^{2\lambda_{\mathrm{L}} t}$$

with $\lambda_{\mathrm{L}} = \frac{\pi}{\beta}$

A bound on chaos

- One can show that for any systems (satisfying plausible conditions)

$$\frac{1}{Z}\mathrm{Tr}\left[yV(0)yW(t)yV(0)yW(t)\right] \sim f_0 - f_1 e^{2\lambda_{\mathrm{L}} t}$$

with $\lambda_{\mathrm{L}} \leq \frac{\pi}{\beta} = \frac{\pi k_{\mathrm{B}} T}{\hbar}$. Therefore B.H. are maximally chaotic.

(Maldacena-Shenker-Stanford)

- Generalizing Sachdev-Ye model

$$H = \sum_{i>j} J_{ij} \hat{S}_i \cdot \hat{S}_j$$

Kitaev tried to construct a maximally chaotic but tractable model.

SYK model

One can solve the model for $N, \beta J \gg 1$

$$H = \frac{1}{4!} \sum_{i,j,k,l} J_{ijkl} \psi_i \psi_j \psi_k \psi_l$$

$$\begin{aligned}
\langle \mathcal{O} \rangle &= \int \prod \left[dJ_{ijkl} P\left(J_{ijkl}\right)\right] \mathrm{Tr}\left(e^{-\beta H} \mathcal{O}\right) \\
&= \int \prod \left[dJ_{ijkl} P\left(J_{ijkl}\right)\right] \int \left[d\psi_i\right] \exp\left[-\frac{1}{2}\int_0^\beta d\tau \psi_i \partial_\tau \psi_i - \int d\tau H\right] \mathcal{O} \\
&= \int \left[d\psi_i\right] \exp\left[-\frac{1}{2}\int d\tau \psi_i \partial_\tau \psi_i - \frac{NJ^2}{8} \int d\tau d\tau' \left(\frac{1}{N}\sum_i \psi_i\left(\tau\right)\psi_i\left(\tau'\right)\right)^4\right] \mathcal{O}
\end{aligned}$$

SYK model

$$\int [d\psi] \exp \left[-\frac{1}{2} \int d\tau \psi_i \partial_\tau \psi_i - \frac{NJ^2}{8} \int d\tau d\tau' \left(\frac{1}{N} \sum_i \psi_i(\tau) \psi_i(\tau') \right)^4 \right]$$

$$= \int [d\psi d\Sigma dG] \exp \left[-\frac{1}{2} \int d\tau \psi_i \partial_\tau \psi_i - \frac{NJ^2}{8} \int d\tau d\tau' \left(G(\tau, \tau') \right)^4 \right.$$
$$\left. - \frac{N}{2} \int d\tau d\tau' \Sigma(\tau, \tau') \left\{ G(\tau, \tau') - \frac{1}{N} \sum_i \psi_i(\tau) \psi_i(\tau') \right\} \right]$$

$$= \int [d\Sigma dG] \exp \left[\frac{N}{2} \mathrm{Tr} \ln (\partial_\tau - \Sigma) \right.$$
$$\left. - \frac{N}{2} \int d\tau d\tau' \left\{ \Sigma(\tau, \tau') G(\tau, \tau') - \frac{J^2}{4} \left(G(\tau, \tau') \right)^4 \right\} \right]$$

SYK model

$$\int [d\Sigma dG] \exp\left[\frac{N}{2}\mathrm{Tr}\ln\left(\partial_\tau - \Sigma\right) - \frac{N}{2}\int d\tau d\tau' \left\{\Sigma\left(\tau,\tau'\right)G\left(\tau,\tau'\right) - \frac{J^2}{4}\left(G\left(\tau,\tau'\right)\right)^4\right\}\right]$$

- For $N \gg 1$, this integral can be evaluated by the saddle point method.
- The saddle point eq.

$$\begin{aligned} \left(\frac{1}{\partial_\tau - \Sigma}\right)_{\tau,\tau'} + G\left(\tau,\tau'\right) &= 0 \\ \Sigma\left(\tau,\tau'\right) - J^2\left(G\left(\tau,\tau'\right)\right)^3 &= 0 \end{aligned}$$

can be solved exactly in the strong coupling limit $\beta J \gg 1$.

SYK model

- The four point function can be evaluated (next to leading order in $\frac{1}{N}$) and we find

$$\frac{1}{Z}\mathrm{Tr}\left[y\psi_i(t)y\psi_j(0)y\psi_i(t)y\psi_j(t)\right] \sim f_0 - f_1 e^{\frac{2\pi}{\beta}t}$$

- From the four point function, one can derive the effective action of some modes, which coincides with that from the two dimensional gravity

$$I = -\frac{1}{16\pi G}\left[\int d^2x\phi\sqrt{g}\left(R+2\right) + 2\int_{\mathrm{bdy}}\phi_{\mathrm{b}}K\right]$$