Timelike pion form factor in lattice QCD

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Phys. Rev. D 91 (2015) 054504 [arXiv:1412.6319]

1. Introduction

Space like form factor :

$$\langle \pi(p_f) | V_\mu | \pi(p_i) \rangle = (p_f + p_i)_\mu \, F(q^2) \qquad (\; q = p_f - p_i \; , \; q^2 < 0 \;)$$

Time like form factor :

$$\langle \pi(p_1) \pi(p_2); \mathrm{in} | V_\mu | 0 \rangle = i(p_1 - p_2)_\mu \, F(q^2) \qquad (\; q = p_1 + p_2 \; , \; q^2 = s > 0 \;)$$

$$\left(\; V_\mu = J_\mu^{\mathrm{EM}} = \frac{2}{3} \bar{u} \gamma_\mu u - \frac{1}{3} \bar{d} \gamma_\mu d - \frac{1}{3} \bar{s} \gamma_\mu s \; \right)$$

Time like form factor :

$$\langle \pi(p_1)\pi(p_2);\mathrm{in}|V_\mu|0\rangle = i(p_1-p_2)_\mu F(q^2) \qquad (\ q = p_1+p_2\ ,\ q^2 = s > 0\ )$$

格子上で time like form factor を求めるのは簡単ではない。
格子計算 (有限体積) では、漸近場が作れない。

無限体積 : $|\pi(\mathbf{p})\pi(-\mathbf{p});\mathrm{in}\rangle\ ,\ |\pi(\mathbf{p})\pi(-\mathbf{p});\mathrm{out}\rangle \quad , \quad \langle\pi\pi;\mathrm{out}|\pi\pi;\mathrm{in}\rangle = \mathrm{e}^{i2\delta}$

格子計算で取り出せる状態 :

$|\pi\pi;E\rangle_V$: 相互作用のある場合の energy eigenstate
それぞれの粒子の運動量は quantum number ではない。

有限体積の matrix element は、無限体積の線形結合で書ける。
(up to complex phase)

$$\langle 0|V_\mu|\pi\pi;E\rangle_V = \sum_\omega C(\Omega)\langle 0|V_\mu|\pi(\mathbf{p})\pi(-\mathbf{p});in\rangle$$

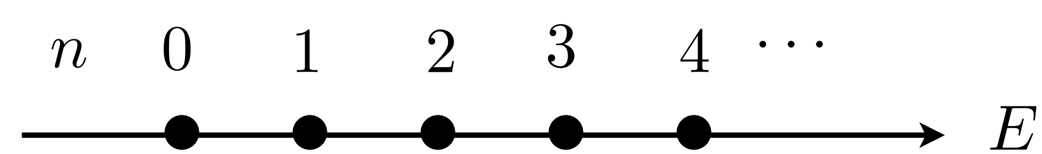
$$G_V(t) = \int_V \mathrm{d}x^3 \, \langle 0 | \, V_\mu(\mathbf{x}, t) \, V_\mu^\dagger(\mathbf{0}, 0) \, | 0 \rangle$$

$$\overset{V \to \infty}{\longrightarrow} \quad \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3 2E_1} \int \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2E_2} (2\pi)^3 \delta(\mathbf{p_1} + \mathbf{p_2}) \, |\langle 0 | V_\mu | \pi\pi \rangle|^2 \mathrm{e}^{-(E_1+E_2)t}$$

$$= \frac{1}{(2\pi)^2} \frac{2}{3} \int \mathrm{d}E \, \frac{k^3}{E} |F(s)|^2 \mathrm{e}^{-Et} \qquad (\ E = E_1 + E_2 = 2\sqrt{m_\pi^2 + k^2} \ , s = E^2 \ )$$

$$\sum_E \overset{V \to \infty}{\longrightarrow} \int \mathrm{d}E \rho(E) \qquad (\ \rho(E) \ : \ \ \text{density of state} \ )$$

$$\overset{V \to \infty}{\longrightarrow} \quad \int \mathrm{d}E \, \rho(E) |\langle 0 | V_\mu | \pi\pi; E \rangle_V|^2 \mathrm{e}^{-Et}$$



$n\pi - \delta(k) = \phi(q)$: Lüscher’s formula

$$\cot \phi(q) = -\frac{1}{\pi^{3/2} q} \frac{1}{\sqrt{4\pi}} \sum_{\mathbf{n} \in \mathbb{Z}^3} \frac{1}{n^2 - q^2}$$

$$\rho(E) = \frac{\mathrm{d}n}{\mathrm{d}E} = \frac{E}{4\pi k^2} \left(k \frac{\partial \delta(k)}{\partial k} + q \frac{\partial \phi(q)}{\partial q} \right)$$

$$(\ E = 2\sqrt{m_\pi^2 + k^2} \ , \ q = kL/(2\pi) \ )$$

$$|F(s)|^2 = \left(k \frac{\partial \delta(k)}{\partial k} + q \frac{\partial \phi(q)}{\partial q} \right) \cdot \frac{3\pi s}{2k^5} \left| \langle 0 | V_\mu | \pi\pi; E \rangle_V \right|^2$$

H.B. Meyer PRL 107 (2011) 072002 [arXiv:1105.1892]

2. 物理

$$\langle \pi(p_1)\pi(p_2); \mathrm{in} | V_\mu | 0 \rangle = +i(p_1 - p_2) F(s - i\epsilon)$$

$$\langle \pi(p_1)\pi(p_2); \mathrm{out} | V_\mu | 0 \rangle = -i(p_1 - p_2) F(s + i\epsilon)$$

(time reversal より)

$$V_\mu = J_\mu^{\mathrm{EM}} \Big|_{I=1} = (\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d)/2$$

$$2m < \sqrt{s} < 3m_\pi$$

$$\langle \pi\pi; \mathrm{out} | V_\mu | 0 \rangle - \langle \pi\pi; \mathrm{in} | V_\mu | 0 \rangle \quad = [1 - \langle \pi\pi; \mathrm{in} | \pi\pi; \mathrm{out} \rangle] \ \langle \pi\pi; \mathrm{out} | V_\mu | 0 \rangle$$

よって

$$\mathrm{Im} F(s) = \sin\delta(k) \mathrm{e}^{-i\delta(k)} F(s)$$

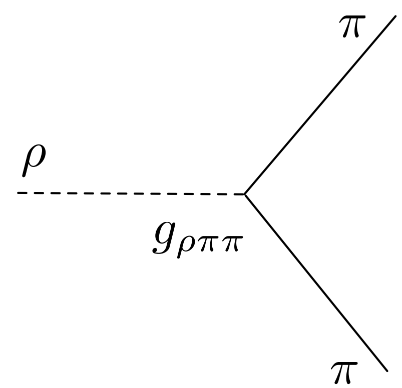
parametarization of from factor :

Gounaris and Sakurai
PRL 21(1968)244.

$$F^{\mathrm{GS}}(s) = \frac{A}{s - m_\rho^2 - \Pi(s)} \qquad (\; A = m_\rho^2 - \Pi(0) \;)$$

effective Lagragian :

$$L = g_{\rho\pi\pi} \cdot \rho_\mu \pi (\partial_\mu \pi)$$



decay width :

$$\Gamma_\rho = \frac{g_{\rho\pi\pi}^2}{6\pi} \frac{k_\rho^3}{m_\rho}$$

$$(\; m_\rho = 2\sqrt{m_\pi^2 + k_\rho^2} \;)$$

optical theory より、

$$\mathrm{Im}\Pi_\rho(s) = -\frac{g_{\rho\pi\pi}^2}{6\pi} \frac{k^3}{\sqrt{s}}$$

(= total cross section)

dispersion relation より、

$$\mathrm{Re}\Pi(s) = c_0 + c_1 s + \frac{s^2}{\pi} P \int_{4m_\pi^2}^{\infty} \mathrm{d}s' \frac{\mathrm{Im}\Pi(s')}{s'^2(s' - s)}$$

$$= \mathrm{Re}\Pi(0)$$

$$= \frac{g_{\rho\pi\pi}^2}{6\pi} \left[k^2 \left(h(\sqrt{s}) - h(m_\rho) \right) - \frac{2k_\rho^2}{m_\rho} h'(m_\rho)(k^2 - k_\rho^2) \right]$$

(c_0 , c_1 は、$\Pi(s)$ の繰り込み条件から決まる)

$$\mathrm{Re}\Pi(m_\rho) = 0 \quad , \quad \mathrm{Re}\Pi'(m_\rho) = 0$$

よって、

$$F^{\mathrm{GS}}(s) = \frac{f_0}{k^2 h(\sqrt{s}) - k_\rho^2 h(m_\rho) + b(k^2 - k_\rho^2) - ik^3/\sqrt{s}}$$

Gounaris and Sakurai
PRL 21(1968)244.

$$h(\sqrt{s}) = \frac{2}{\pi}\frac{k}{\sqrt{s}} \ln\left(\frac{\sqrt{s}+2k}{2m_\pi}\right)$$

$$b = -h(m_\rho) - 24\pi/g_{\rho\pi\pi}^2 - 2k_\rho^2 h'(m_\rho)/m_\rho$$

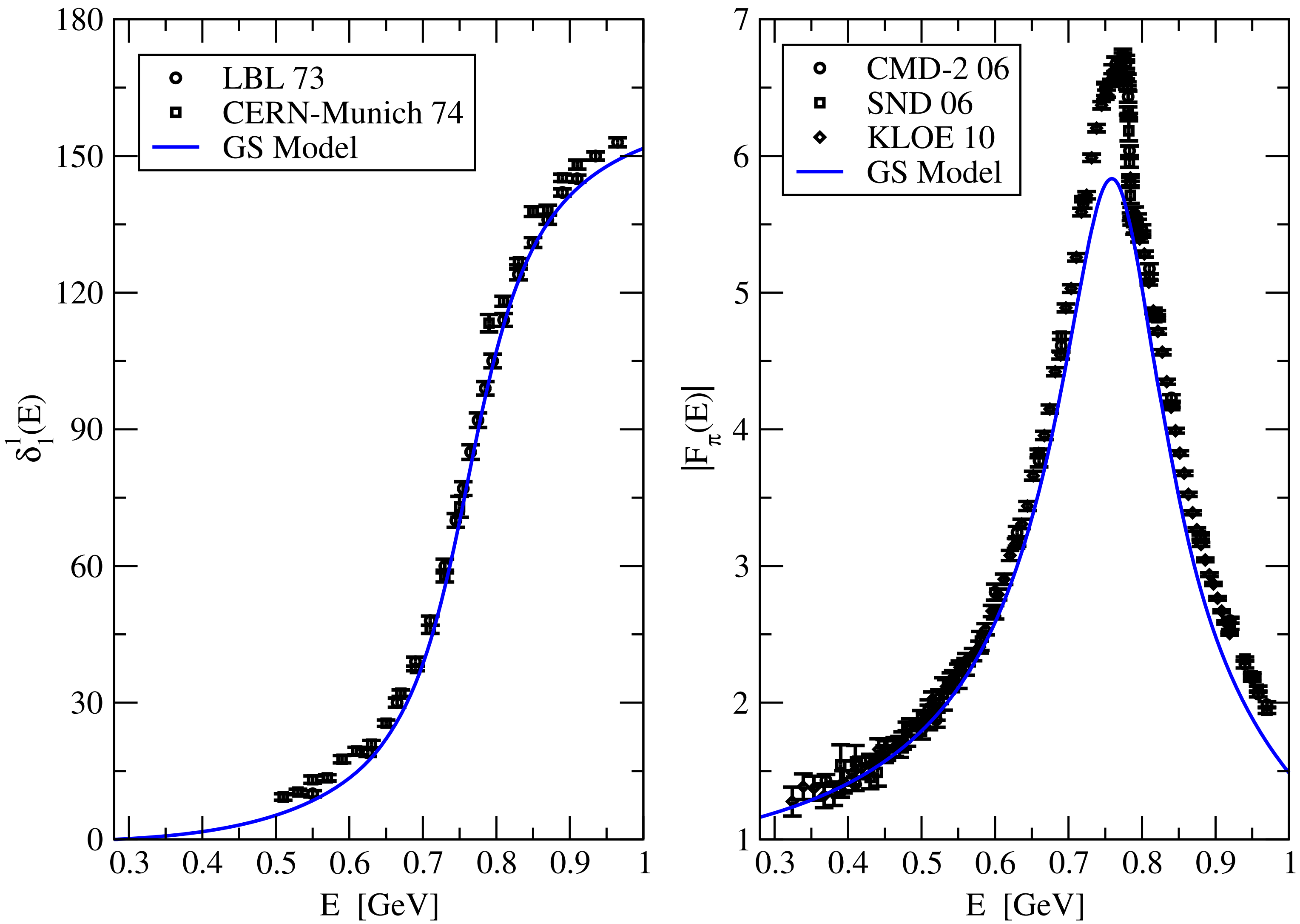
$$f_0 = -m_\pi^2/\pi - k_\rho^2 h(m_\rho) - bm_\rho^2/4$$

$\mathrm{Im}F(s) = \sin\delta(k)\mathrm{e}^{-i\delta(k)}F(s)$ から、

$$\frac{k^3}{\sqrt{s}}\cot\delta(k) = k^2 h(\sqrt{s}) - k_\rho^2 h(m_\rho) + b(k^2 - k_\rho^2)$$

un-known parameter : m_ρ , $g_{\rho\pi\pi}$

実験



$F^{\mathrm{GS}}(s)$ が合わない

$$\longrightarrow \quad F(s) = \frac{A}{s - m_\rho^2 - \Pi(s)} \qquad A = A(s) = \sum_n c_n (s - m_\rho)^n$$

$F(0) = 1$ から $\displaystyle\sum_n c_n (-m_\rho)^n = 1$

3. 格子計算

action： 2+1 overlap fermion

$a = 0.112(1)\,\mathrm{fm}$ $V = 24^3 \times 48$

$m_\pi = 380\,,\, 290\,\mathrm{MeV}$

op： $\rho_\mu = V_\mu$, $\pi(\mathbf{p_a})\pi(\mathbf{p_2})$

No.	$\mathbf{P}$	G	Γ	$j_{\mathbf{b}}^{(\pi\pi,n)}$: $[\mathbf{p}_1,\ \mathbf{p}_2]$	$j_{\mathbf{b}}^{\bar{\psi}\psi}$: $\mathbf{b}$
①	(0, 0, 0)	O_h	T_1^-	[(1, 0, 0), (−1, 0, 0)] [(0, 1, 0), (0, −1, 0)] [(0, 0, 1), (0, 0, −1)]	(1, 0, 0) (0, 1,0) (0, 0, 1)
②	(0, 0, 1)	D_{4h}	A_2^-	[(0, 0, 1), (0, 0, 0)]	(0, 0, 1)
③	(1, 1, 0)	D_{2h}	B_1^-	[(1, 1, 0), (0, 0, 0)]	$\frac{1}{\sqrt{2}}(1, 1, 0)$
④	(1, 1, 1)	D_{3d}	A_2^-	[(1, 1, 1), (0, 0, 0)]	$\frac{1}{\sqrt{3}}(1, 1, 1)$
⑤	(1, 1, 0)	D_{2h}	B_2^-	[(1, 0, 0), (0, 1, 0)]	$\frac{1}{\sqrt{2}}(1, -1, 0)$

$$G_{ij} = \langle 0|\, O_i^\dagger(t)\, O_j(0)\, |0\rangle = \sum_{\alpha=1}^{2} \langle 0|\, O_i^\dagger |\alpha\rangle \mathrm{e}^{-E_\alpha t} \langle \alpha|\, O_j\, |0\rangle$$

$$: \quad G(t) = A^\dagger D(t) A$$

$$D_{\alpha\beta}(t) = \delta_{\alpha\beta}\mathrm{e}^{-E_\alpha t} \;,\quad A_{\alpha j} = \langle \alpha|\, O_j\, |0\rangle$$

$$R(t, t_0) = G(t_0)^{-1/2}\, G(t)\, G(t_0)^{-1/2}$$

$$R(t, t_0)B = B\,(D(t)/D(t_0)) \qquad D(t)/D(t_0) \longrightarrow E \longrightarrow \delta$$

$$D_\alpha(t_0)|A_{\alpha j}|^2 = \left| \left[B^\dagger G(t_0)^{-1/2} G(t) \right]_{\alpha j} \right|^2 \cdot (D(t)/D(t_0))_\alpha^{-2}$$

$j = V_\mu$ で、$A_\alpha = \langle \alpha|\, V_\mu\, |0\rangle$ が求まる

3. 計算結果

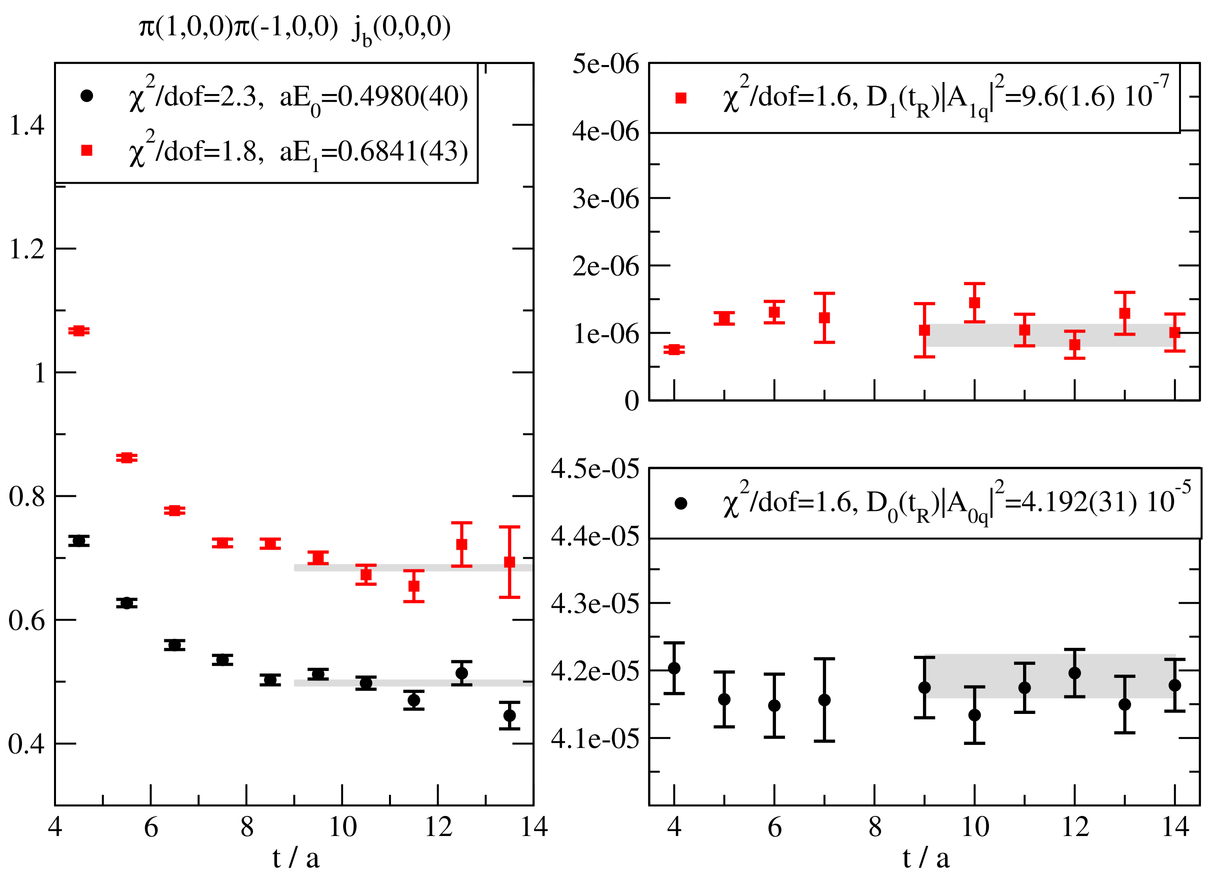


FIG. 3 (color online). Effective energies and amplitudes for the operator set ① and $m_\pi = 380$ MeV.

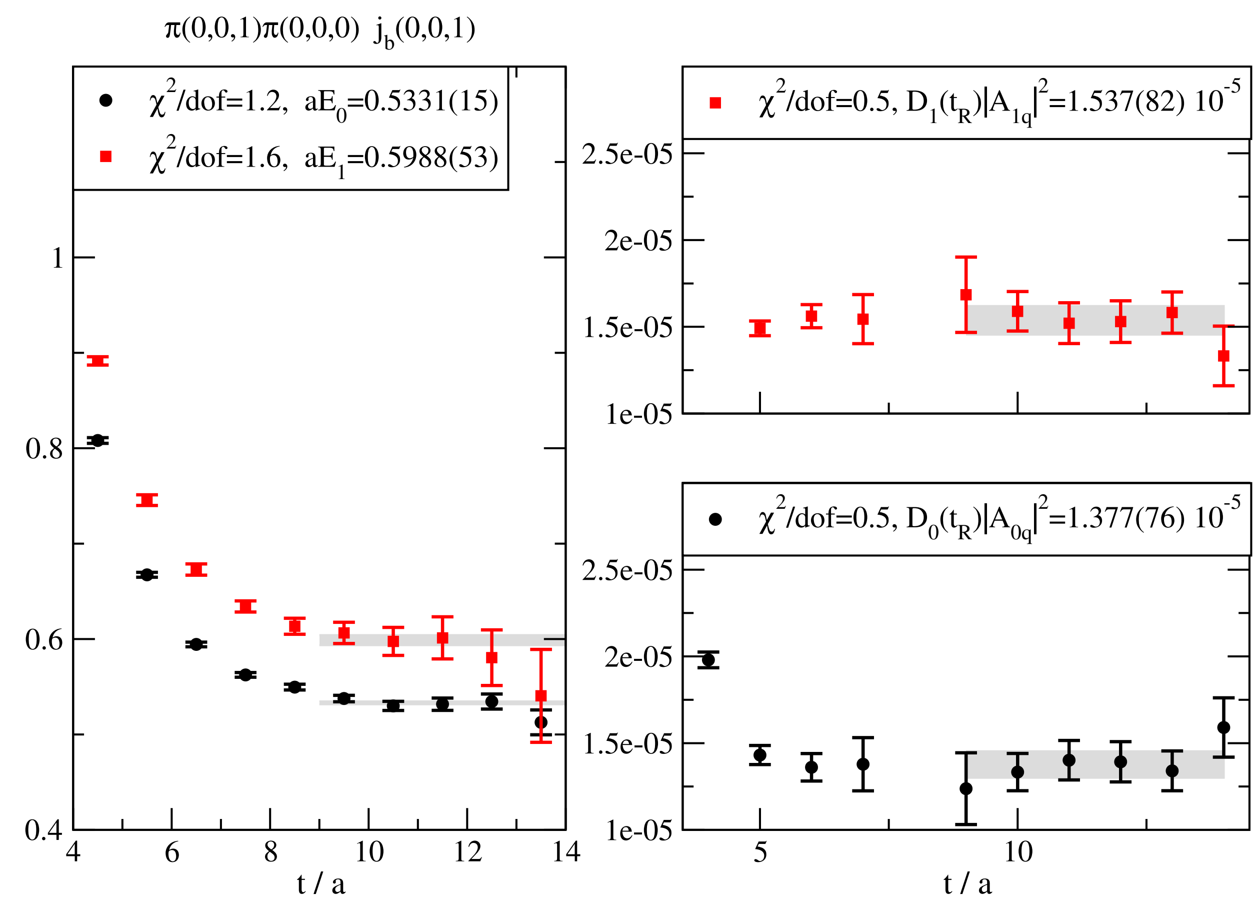
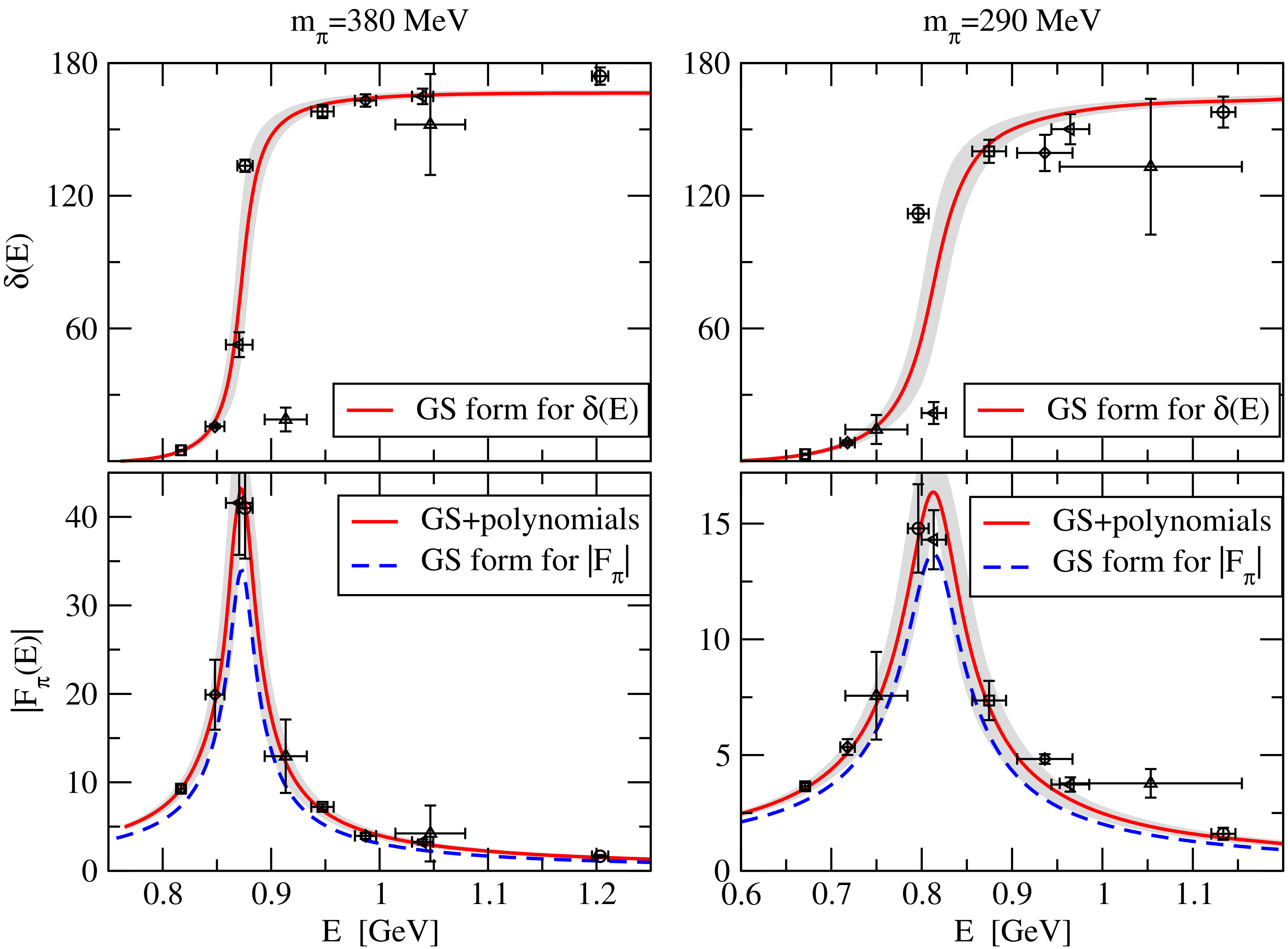


FIG. 4 (color online). Same as Fig. 3, but for the operator set ② and $m_\pi = 380$ MeV.



$$\frac{|F_\pi(s)|}{|F_\pi^{\rm GS}(s)|} - 1 = \sum_{n=0}^{N} c_n((s-m_\rho^2)^n - (-m_\rho^2)^n)$$
$$= s(c_1 + c_2(s-2m_\rho^2) + \cdots).$$

fitted well

charge radius :

$$\langle r_\pi^2 \rangle = 6 \frac{\partial |F(S)|}{\partial s}$$
$$= 6 \left(-\frac{1}{f_0} \left(\frac{b}{4} + \frac{1}{3\pi} \right) + c_1 + c_2(-2m_\rho^2) \right)$$

Lattice	m_π = "380 MeV"	m_π = "290 MeV"
m_π (MeV)	378.6(7)	291.8(1.1)
m_ρ (MeV)	875(7)	819(14)
$g_{\rho\pi\pi}$	5.85(19)	5.78(23)
(time-like) $\langle r_\pi^2 \rangle$ (fm^2)	0.377(38)	0.392(41)
(space-like) $\langle r_\pi^2 \rangle$ (fm^2)	$0.334(10)(^{+00}_{-32})$	$0.366(19)(^{+00}_{-42})$

4. まとめ

ρ decay をやったとき、余分な計算なしで求められる。

他の 系への応用。