Hadronic vacuum polarization and muon $g-2$ from magnetic susceptibilities on the lattice

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PRD92(2015)5:054506; arXiv:1506.086381 [hep-lat]

Muon $g-2$

muon **磁気モーメント** $\vec{M} = g_\mu \frac{e}{2m_\mu}\vec{S}$

古典的には $g_\mu = 2$**、量子効果により僅かに**2**からずれる**: $a_\mu = (g_\mu - 2)/2$

実験(BNL**など**)**で非常に高い精度で測定されている**

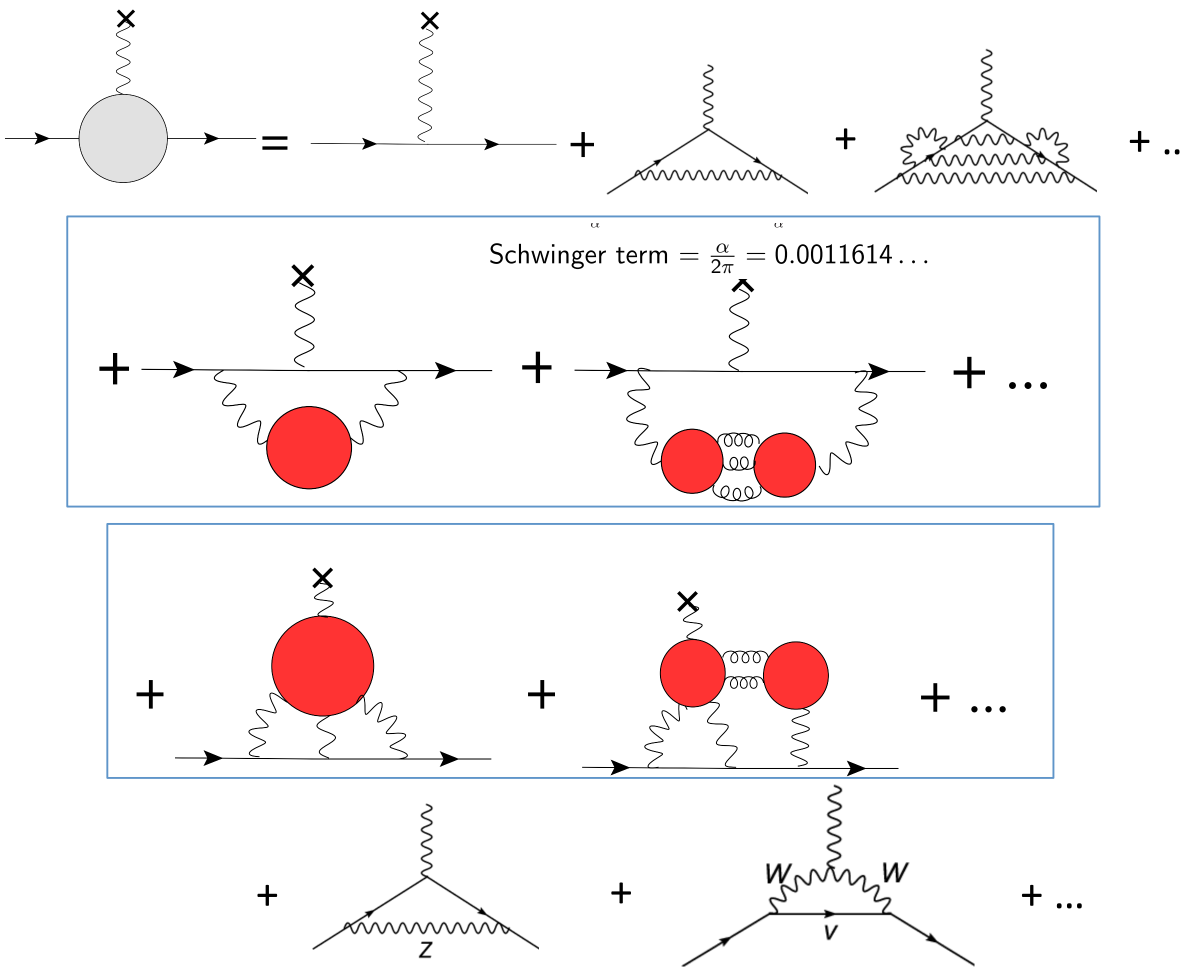
$$a_\mu[\text{exp}] = 116592091(63) \times 10^{-11}$$

将来 Fermi Lab **や** J-PARC **の実験で誤差は** 10×10^{-11} **程度になると期待**

Muon $g-2$ 理論

Slide from Taku Izubuchi @ Lat15, plenary

- QED, hadronic, EW contributions



QED (5-loop)
Aoyama et al.
PRL109,111808 (2012)

Hadronic vacuum polarization (HVP)

Hadronic light-by-light (Hlbl)

Electroweak (EW)
Knecht et al 02
Czarnecki et al. 02

Muon $g-2$ 比較

Slide from Taku Izubuchi @ Lat15, plenary

PDG2014

SM Theory prediction

- QED, EW, Hadronic contributions

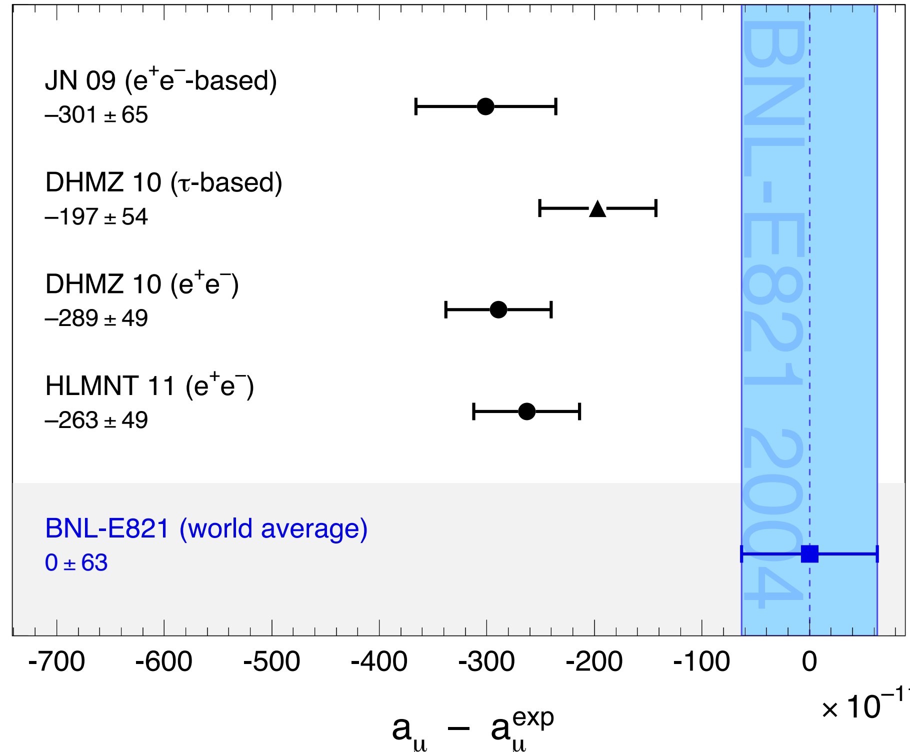
K. Hagiwara et al. , J. Phys. G: Nucl. Part. Phys. 38 (2011) 085003

$$a_\mu^{\rm SM} = (11\ 659\ 182.8\ \pm 4.9\ ) \times 10^{-10}$$

$$a_\mu^{\rm QED} = (11\ 658\ 471.808\ \pm 0.015\ ) \times 10^{-10}$$
$$a_\mu^{\rm EW} = (\ 15.4\ \pm 0.2\ ) \times 10^{-10}$$
$$\boxed{a_\mu^{\rm had,LOVP} = (\ 694.91\ \pm 4.27\ ) \times 10^{-10}}$$
$$a_\mu^{\rm had,HOVP} = (\ -9.84\ \pm 0.07\ ) \times 10^{-10}$$
$$\boxed{a_\mu^{\rm had,lbl} = (\ 10.5\ \pm 2.6\ ) \times 10^{-10}}$$

$$a_\mu^{\rm EXP} - a_\mu^{\rm SM} = (26.1 \pm 8.0) \times 10^{-10}$$

3+ σ !



e^+e^- 3.6σ, $\tau^+\tau^-$ 2.4σ の違い

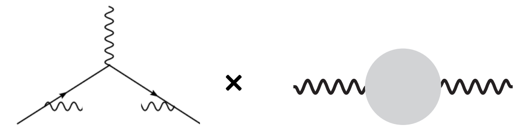
標準模型を超える物理の効果?

２番目に大きな寄与をする

$a_\mu^{\rm had,LOVP}$ は実験値 $R(e^+e^- \to {\rm hadrons})$ か $R(\tau^+\tau^- \to {\rm hadrons})$ で計算

$a_\mu^{\rm had,LO}$ を純粋な理論計算から見積もりたい

$a_\mu^{\text{had,LOVP}}$ Lattice計算

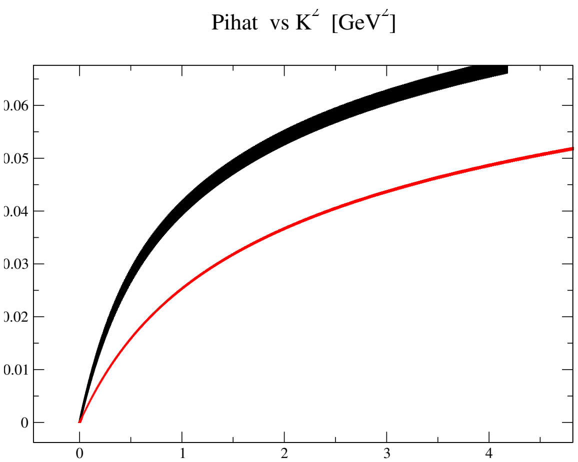
$a_\mu^{\text{had}} = \int ds$ 

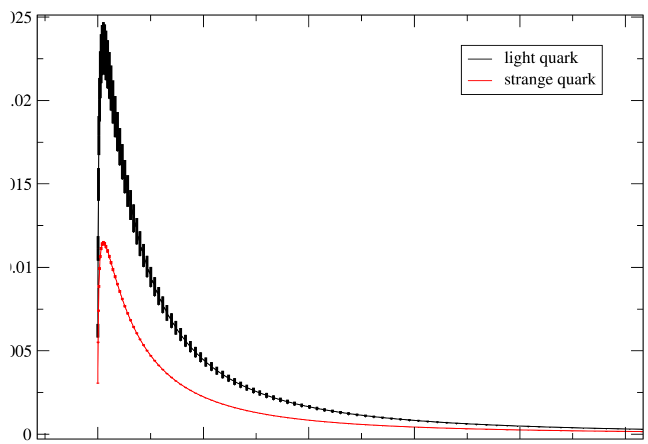
$$a_\mu^{\text{had,LOVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dp^2 f(p^2)\hat{\Pi}(p^2)$$ （既知の関数 $f(p^2) \sim 1/p^2$）

$$\int d^4x e^{ipx}\langle J_\mu(x) J_\nu(0)\rangle = (p_\mu p_\nu - \delta_{\mu\nu}p^2)\Pi(p^2), \quad \hat{\Pi}(p^2) = \Pi(p^2) - \Pi(0)$$

EM current $J_\mu(x) = \sum_f^{u,d,s} q_f \overline{\psi}_f(x)\gamma_\mu\psi_f(x), \quad q_u = 2/3, q_d = q_s = -1/3$

いろいろなpでvector current correlatorのフーリエ変換を計算





Taku Izubuchi @ Lat15, plenary

$\hat{\Pi}(p^2)$

$f(p^2)\hat{\Pi}(p^2)$ ピーク $\sim (m_\mu/2)^2 = 0.028$ GeV^2

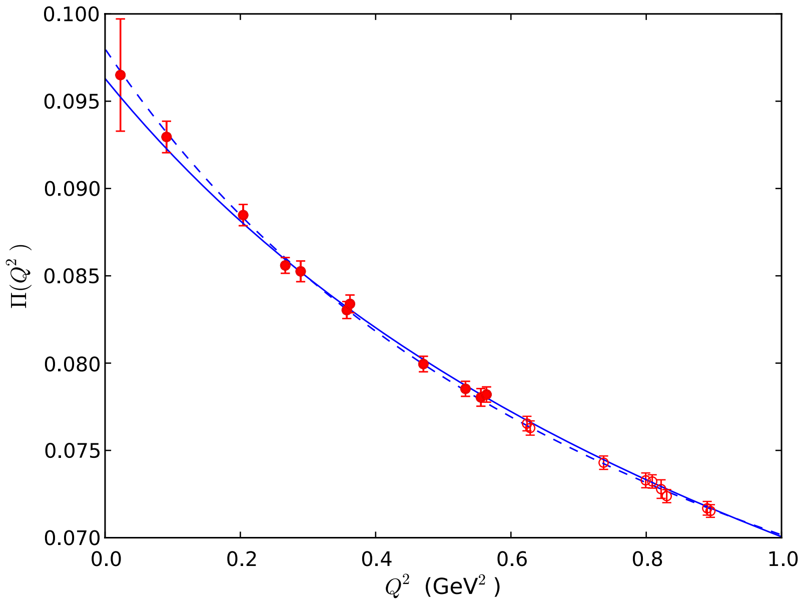
$p^2 \lesssim 0.03$ GeV^2 の $\hat{\Pi}(p^2)$ $\left(\Pi(p^2)\right)$ の計算が重要 $\to$ **新しい計算方法を提案する**

これまでのLattice計算

$$\int d^4x e^{ipx}\langle J_\mu(x)J_\nu(0)\rangle = (p_\mu p_\nu - \delta_{\mu\nu}p^2)\Pi(p^2), \quad \hat{\Pi}(p^2) = \Pi(p^2) - \Pi(0)$$

特に最小の$p(=2\pi/T)$の計算が重要

四次元体積$V_4 = L^3 \cdot T$(多くの計算で$T > L$)



Aubin *et al.*, PRD86:054509

これまでの計算方法

1. $\Pi(p^2)$**を**$\langle J_\mu(x)J_\nu(0)\rangle$**から計算する**
2. $\Pi(p^2)$**を適当な関数でフィットする**
3. $\hat{\Pi}(p^2)$**を見積もる**

問題点1. **最小のpでの誤差が大きいため、$p\sim 0$での$\hat{\Pi}(p^2)$の不定性が大きい**

問題点2. **周期的境界条件では最小のpが**$2\pi/T$ ($p^2 \sim 0.31\text{GeV}^2$@$T\sim 7$fm)

$p\sim 0$で誤差を抑えられる計算方法が必要

問題点3. disconnected **ダイアグラムの計算が難しいので無視している**

磁化率を用いた$\Pi(p^2)$

$$\int d^4x e^{ipx} \langle J_\mu(x) J_\nu(0) \rangle = (p_\mu p_\nu - \delta_{\mu\nu} p^2) \Pi(p^2), \quad \widehat{\Pi}(p^2) = \Pi(p^2) - \Pi(0)$$

$p = (p_1, 0, 0, 0), p_1 \neq 0$で　　　　四次元体積$V_4 = L^3 \cdot T$

$-\frac{1}{V_4 p_1^2} \langle \tilde{J}_2(p) \tilde{J}_2(-p) \rangle = \Pi(p^2)$計算を考える $(\tilde{J}_\mu(p) = \int d^4x e^{ipx} J_\mu(x))$

連続理論

磁化率を用いた$\Pi(p^2)$

$$\int d^4x e^{ipx}\langle J_\mu(x)J_\nu(0)\rangle = (p_\mu p_\nu - \delta_{\mu\nu}p^2)\mathbf{\Pi}(p^2), \quad \hat{\mathbf{\Pi}}(p^2) = \mathbf{\Pi}(p^2) - \mathbf{\Pi}(0)$$

$p = (p_1, 0, 0, 0), p_1 \neq 0$ で　　**四次元体積** $V_4 = L^3 \cdot T$

$$-\frac{1}{V_4 p_1^2}\langle \tilde{J}_2(p)\tilde{J}_2(-p)\rangle = \mathbf{\Pi}(p^2)$$ **計算を考える** $(\tilde{J}_\mu(p) = \int d^4x e^{ipx} J_\mu(x))$

連続理論 Background EM (BEM) interaction $\mathcal{L}_{\mathrm{QCD}} + ieJ_\mu(x)A_\mu^{p,\cos}(x)$

$$A_2^{p,\cos}(x) = B\sin(p_1x_1)/p_1, \; A_1^{p,\cos}(x) = A_2^{p,\cos}(x) = A_3^{p,\cos}(x) = 0$$

$$\vec{B} = \nabla \times \vec{A}^{p,\cos} = B\cos(p_1x_1)\vec{e}_3$$

作用 $S = S_{\mathrm{QCD}} + S_{\mathrm{BEM}}(A^{p,\cos})$, $S_{\mathrm{BEM}}(A^{p,\cos}) = eB\left(\tilde{J}_2(p) - \tilde{J}_2(-p)\right)/(2p_1)$

磁化率を用いた$\Pi(p^2)$

$$\int d^4x e^{ipx}\langle J_\mu(x)J_\nu(0)\rangle = (p_\mu p_\nu - \delta_{\mu\nu}p^2)\Pi(p^2), \quad \hat{\Pi}(p^2) = \Pi(p^2) - \Pi(0)$$

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$$\chi^{p,\cos} = \frac{1}{V_4}\frac{\partial^2 \log \mathcal{Z}(A^{p,\cos})}{\partial(eB)\partial(eB)}\bigg|_{B=0} = \frac{1}{4V_4p_1^2}\left\langle\left(\tilde{J}_2(p) - \tilde{J}_2(-p)\right)^2\right\rangle$$

磁化率を用いた$\Pi(p^2)$

$\int d^4x e^{ipx}\langle J_\mu(x)J_\nu(0)\rangle = (p_\mu p_\nu - \delta_{\mu\nu}p^2)\Pi(p^2),\quad \hat{\Pi}(p^2) = \Pi(p^2) - \Pi(0)$

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連続理論 Background EM (BEM) interaction $\mathcal{L}_{\rm QCD} + ieJ_\mu(x)A_\mu^{p,\cos}(x)$

$$A_2^{p,\cos}(x) = B\sin(p_1x_1)/p_1,\quad A_1^{p,\cos}(x) = A_2^{p,\cos}(x) = A_3^{p,\cos}(x) = 0$$

$$\vec{B} = \nabla \times \vec{A}^{p,\cos} = B\cos(p_1x_1)\vec{e}_3$$

作用 $S = S_{\rm QCD} + S_{\rm BEM}(A^{p,\cos}),\ S_{\rm BEM}(A^{p,\cos}) = eB\left(\tilde{J}_2(p) - \tilde{J}_2(-p)\right)/(2p_1)$

$$\chi^{p,\cos} = \frac{1}{V_4}\frac{\partial^2 \log \mathcal{Z}(A^{p,\cos})}{\partial(eB)\partial(eB)}\bigg|_{B=0} = \frac{1}{4V_4p_1^2}\left\langle\left(\tilde{J}_2(p) - \tilde{J}_2(-p)\right)^2\right\rangle$$

一方、 $$\chi^{p,\sin} = \frac{1}{V_4}\frac{\partial^2 \log \mathcal{Z}(A^{p,\sin})}{\partial(eB)\partial(eB)}\bigg|_{B=0} = -\frac{1}{4V_4p_1^2}\left\langle\left(\tilde{J}_2(p) + \tilde{J}_2(-p)\right)^2\right\rangle$$

$$\chi^{p,\cos} + \chi^{p,\sin} = \Pi(p^2)$$

$\chi^{p,\cos}$ と $\chi^{p,\sin}$ は定位相$\pi/2$違うだけなので$\chi^{p,\cos} = \chi^{p,\sin} = \chi^p$

磁化率を用いた$\Pi(p^2)$(続)

$$2\chi^p = \mathbf{\Pi}(p^2),\; p = (p_1, 0, 0, 0)$$

格子理論 BEM $U_2(x_1) \to U_2(x_1)\exp(ieq_f B \sin(p_1 x_1)/p_1)$

$$\overline{\psi}_f M_f \psi_f \to \overline{\psi}_f M_f^{p,\cos} \psi_f$$

磁化率を用いた$\Pi(p^2)$(続)

$$2\chi^p = \mathbf{\Pi}(p^2),\ p = (p_1, 0, 0, 0)$$

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$$\overline{\psi}_f M_f \psi_f \to \overline{\psi}_f M_f^{p,\cos} \psi_f$$

$$\chi^p = \frac{1}{V_4} \frac{\partial^2 \log \mathcal{Z}(A^{p,\cos})}{\partial(eB)\partial(eB)}\bigg|_{B=0} = \frac{1}{V_4}\left\langle \mathcal{C}_p^2 + \frac{\partial \mathcal{C}_p}{\partial(eB)} \right\rangle_{B=0}$$

$$\mathcal{C}_p = \frac{1}{4}\sum_f q_f \mathrm{Tr}\left[\left(M_f^{p,\cos}\right)^{-1} \dot{M}_f^{p,\cos}\right], \quad \dot{M}_f^{p,\cos} = \frac{\partial M_f^{p,\cos}}{\partial(eB)},$$ 1/4: starggered quark

磁化率を用いた $\Pi(p^2)$(続)

$$2\chi^p = \Pi(p^2),\ p = (p_1, 0, 0, 0)$$

格子理論 BEM $U_2(x_1) \to U_2(x_1)\exp(ieq_f B\sin(p_1 x_1)/p_1)$

$$\overline{\psi}_f M_f \psi_f \to \overline{\psi}_f M_f^{p,\cos}\psi_f$$

$$\chi^p = \frac{1}{V_4}\frac{\partial^2 \log \mathcal{Z}(A^{p,\cos})}{\partial(eB)\partial(eB)}\bigg|_{B=0} = \frac{1}{V_4}\left\langle \mathcal{C}_p^2 + \frac{\partial \mathcal{C}_p}{\partial(eB)}\right\rangle_{B=0}$$

$$\mathcal{C}_p = \frac{1}{4}\sum_f q_f \mathrm{Tr}\left[\left(M_f^{p,\cos}\right)^{-1}\dot{M}_f^{p,\cos}\right],\quad \dot{M}_f^{p,\cos} = \frac{\partial M_f^{p,\cos}}{\partial(eB)},\ \text{1/4: starggered quark}$$

$$\chi^p = \frac{1}{4V_4}\left\langle \frac{5}{9}\mathrm{Tr}(M_l^{-1}\ddot{M}_l - M_l^{-1}\dot{M}_l M_l^{-1}\dot{M}_l) + \frac{1}{9}\mathrm{Tr}(M_s^{-1}\ddot{M}_s - M_s^{-1}\dot{M}_s M_s^{-1}\dot{M}_s)\right\rangle$$
$$+\frac{1}{16V_4}\left\langle \frac{1}{9}\mathrm{Tr}(M_l^{-1}\dot{M}_l)\mathrm{Tr}(M_l^{-1}\dot{M}_l) + \frac{1}{9}\mathrm{Tr}(M_s^{-1}\dot{M}_s)\mathrm{Tr}(M_s^{-1}\dot{M}_s) - \frac{2}{9}\mathrm{Tr}(M_l^{-1}\dot{M}_l)\mathrm{Tr}(M_s^{-1}\dot{M}_s)\right\rangle$$
$$M_l = M_u = M_d (m_u = m_d)$$

第1項 connected **ダイアグラム、第2項** disconnected **ダイアグラムに対応**

結局 $B = 0$ **のゲージ場で** χ^p **を計算する**(?)(p **依存性は** $\dot{M}_l, \ddot{M}_l$)

磁化率を用いた$\Pi(p^2)$(続)

$$2\chi^p = \Pi(p^2),\ p = (p_1, 0, 0, 0)$$

$$\chi^p = \frac{1}{4V_4}\left\langle \frac{5}{9}\mathrm{Tr}(M_l^{-1}\ddot{M}_l - M_l^{-1}\dot{M}_l M_l^{-1}\dot{M}_l) + \frac{1}{9}\mathrm{Tr}(M_s^{-1}\ddot{M}_s - M_s^{-1}\dot{M}_s M_s^{-1}\dot{M}_s)\right\rangle$$
$$+\frac{1}{16V_4}\left\langle \frac{1}{9}\mathrm{Tr}(M_l^{-1}\dot{M}_l)\mathrm{Tr}(M_l^{-1}\dot{M}_l) + \frac{1}{9}\mathrm{Tr}(M_s^{-1}\dot{M}_s)\mathrm{Tr}(M_s^{-1}\dot{M}_s) - \frac{2}{9}\mathrm{Tr}(M_l^{-1}\dot{M}_l)\mathrm{Tr}(M_s^{-1}\dot{M}_s)\right\rangle$$

$$M_l = M_u = M_d(m_u = m_d)$$

第１項 connected ダイアグラム、第２項 disconnected ダイアグラムに対応

利点 1. 運動量に制限がない(ツイスト境界条件に対応?) $\to p \sim 0$の計算が可能

利点 2. 精度の良い disconnected ダイアグラムの計算が可能 →後のページ
これまでの多くの計算では無視(計算が難しい: 寄与が小さく、誤差が大きい)

inversion の回数 $N_{\mathrm{inv}} = 2N_\xi(1 + N_p)$ (2: l, s, N_ξ: 乱数, N_p: 運動量)

l, sでそれぞれN_ξ個の乱数を使うと、各ゲージ配位で
connected はN_ξ個で乱数平均
disconnected の第1,2項は$N_\xi(N_\xi - 1)/2$個, 第３項はN_ξ^2個で乱数平均

計算結果

シミュレーションパラメータ

stout improved staggered quark action

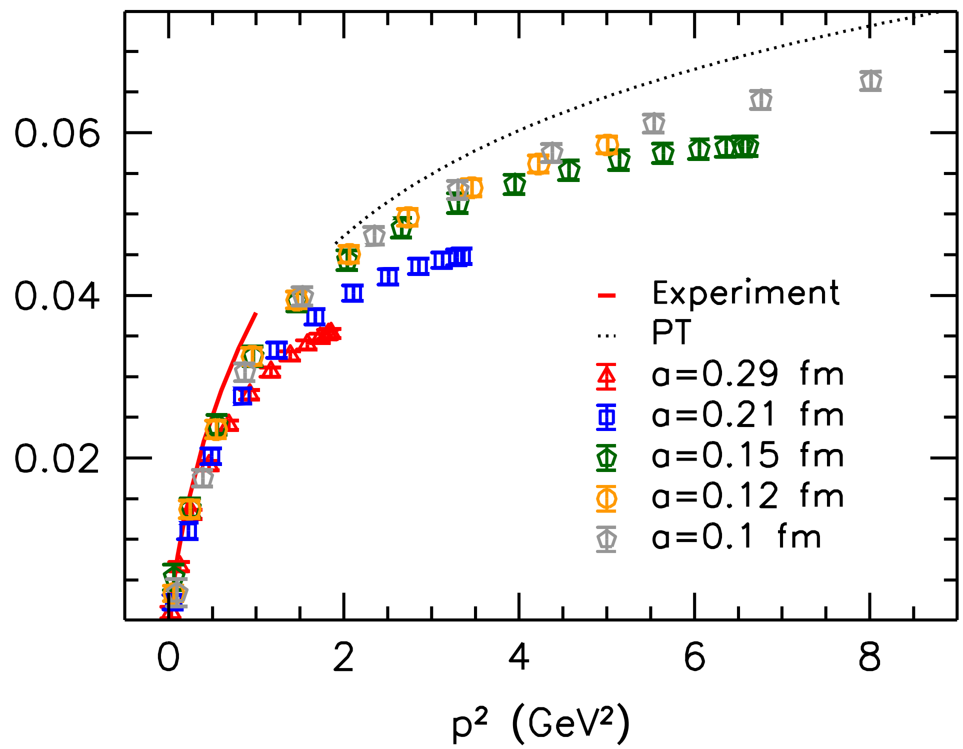
tree-level improved Symanzik gauge action

physical pion and kaon masses

TABLE I. Lattice ensembles investigated; the largest lattice spacing reads $a_0 = 0.29$ fm.

N_s	N_t	β	a [fm]	$\log(a/a_0)$
24	32	3.45	0.290	0
24	32	3.55	0.216	−0.295
32	48	3.67	0.153	−0.636
40	48	3.75	0.125	−0.843
40	48	3.85	0.099	−1.078

$\hat{\Pi}(p^2)$



格子間隔依存性が見える

aを小さくすると$\hat{\Pi}(p^2)$は実験値に近づく

計算結果(続)

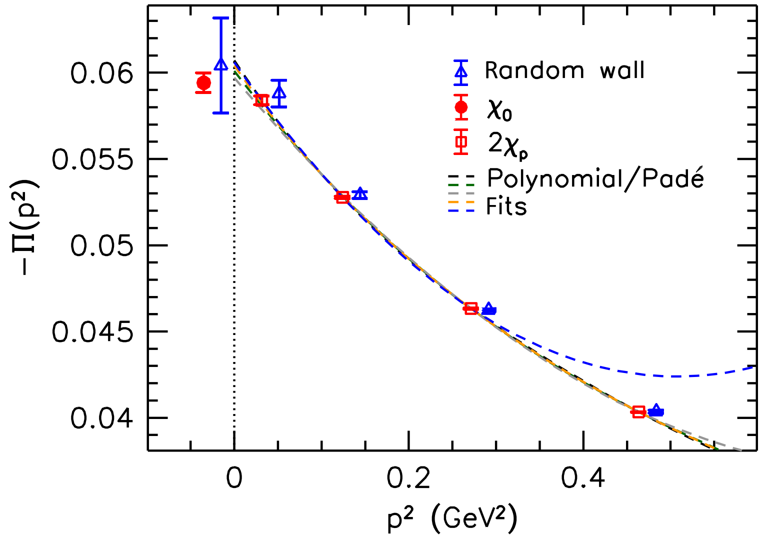


FIG. 2 (color online). The low-momentum region of the oscillatory susceptibilities as measured on the $24^3 \times 32$ configurations at $\beta = 3.45$. The curves correspond to polynomial- and Padé-type extrapolations of $2\chi_p$ to $p = 0$. The direct determination χ_0 is shifted horizontally to the left for better visibility. Also included are results obtained using random wall sources, displaced horizontally to the right.

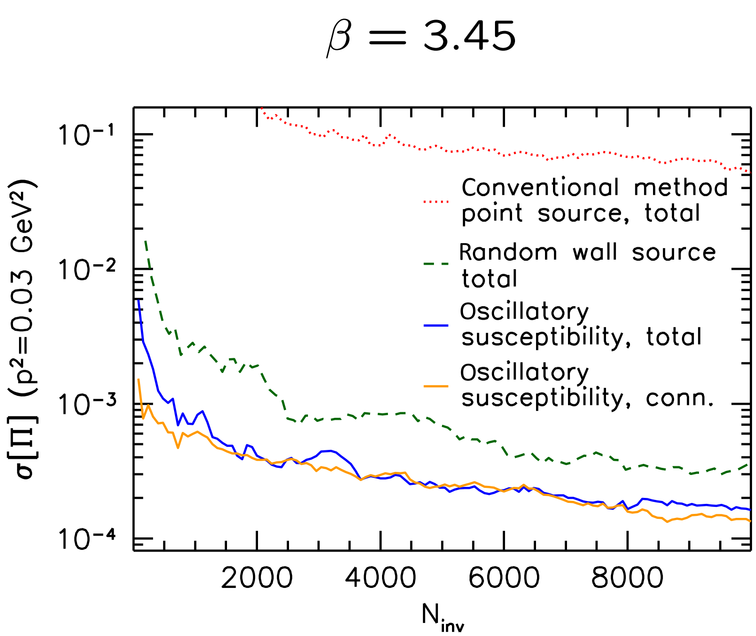


FIG. 6 (color online). Statistical error of the total (connected plus disconnected) $\Pi(p^2 = 0.03\ \mathrm{GeV}^2)$ as a function of the number of inversions. Compared are the results obtained from oscillatory susceptibilities, using point sources and random wall sources. In addition, the error of the connected oscillatory susceptibility alone is shown. Note the logarithmic scale.

- これまでの方法を改良した相関関数(Random wall)を使った結果と一致
- 最小の$p \neq 0$ではRandom wallと同じ計算量で誤差が小さい
- $p \to 0$外挿値が直接計算(説明省略)した結果と一致
    - disconnected を入れた point sourceと比べて誤差はかなり小さい
    - disconnectedを入れても誤差が悪化しない

計算結果(続)

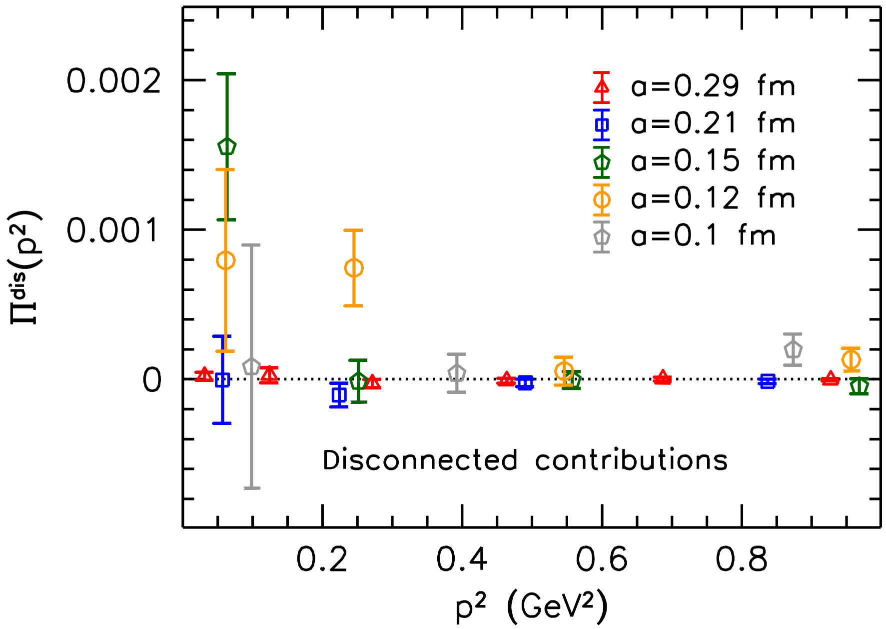


FIG. 7 (color online). Disconnected contribution to $\Pi(p^2)$ as a function of p^2 for our five lattice spacings.

Most accurate determination in this paper

$p^2 = 0.03\ \mathrm{GeV}^2$ at $a = 0.29$ fm

$N_{\rm inv} = 20000$

$\Pi(p^2) = -0.058362(117)$

$\Pi^{\rm dis}(p^2) = 0.0000021(026)$

$\hat{\Pi}(p^2) = 0.002355(198)$

$\Pi(0)$ from random wall

最小の p **以外は** $N_{\rm inv} = 800$

- disconnected **は統計的にゼロと無矛盾、誤差が小さい(大きい)のもある**

$p \sim 0$ **では格子間隔が小さくなる(格子サイズが大きくなる)と誤差が大きくなる**(?)

一番良い計算結果では

- disconnected **の誤差は** connected **よりも小さい**
- $\hat{\Pi}(p^2)$ **の誤差は** 8%← **主に** $\Pi(0)$ **の誤差**

計算結果(続)

$p^2 \sim 0.03\ \mathrm{GeV}^2$での誤差比較(異なる$m_\pi, a$, 相対誤差ではない)

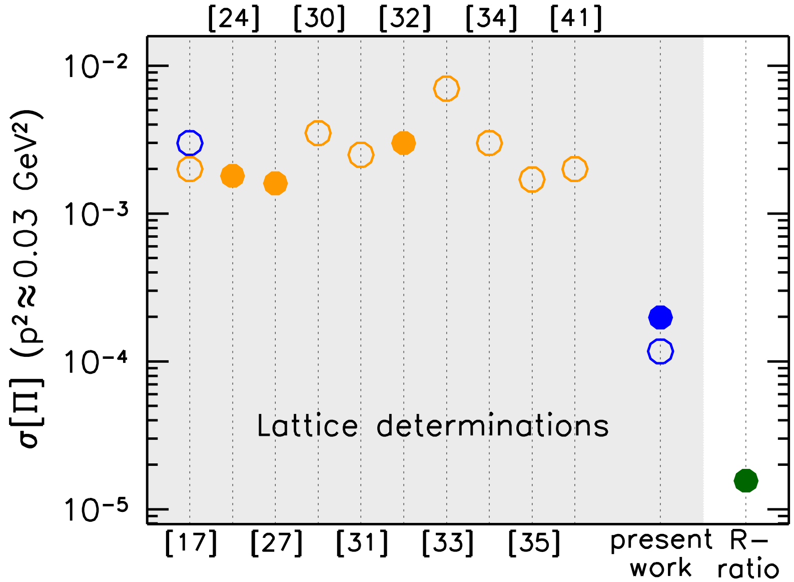


FIG. 8 (color online). The statistical error of the vacuum polarization at low momenta around $p^2 = 0.03\ \mathrm{GeV}^2$ for several lattice studies in the literature and for the present work (shaded area). Open points denote the error of the unsubtracted $\Pi(p^2)$, while full symbols indicate that of the renormalized $\Pi_R(p^2)$. Studies involving only the connected contribution are indicated in yellow, while those also taking into account the disconnected terms are indicated in blue. The determination using the experimental R ratio is also included for comparison (solid green point).

$\Pi(p^2)$: 白抜き

$\hat{\Pi}(p^2)$: 塗りつぶし

connected のみ: 黄色

connected + disconnected: 青

結論

これまでよりも$p^2 \sim 0.03\ \mathrm{GeV}^2$で誤差を抑えられた

$a_\mu^{\mathrm{had,LOVP}}$に使われている実験値よりはまだ誤差が大きい

Izubuchiさん談 @ Lat15

最近の結果と比べれば同程度の誤差

$\hat{\Pi}(p^2 \sim 0.044\mathrm{GeV}^2) \sim 0.0035(4)$, BMW@Lat15

多分 $m_\pi = 130$ MeV, connected のみ

でもdisconnectedが入ればこの方法の方が誤差は小さいかも