New symmetries of massless QED

T. He, P. Mitra, A. P. Porfyriadis and A. Strominger, JHEP 10 (2014) 112

Weinberg's soft photon theorem (1965).

$$\Updownarrow$$

Ward identity for a symmetry (2014),

soft photon = Goldstone mode of the spontaneously symmetry breaking.

Let us recall the case of Ginsparg-Wilson relation on the lattice,

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D, \quad \mathcal{L}_a = \bar{\psi} D \psi,$$

$$\Downarrow$$

$$\gamma_5 D + D\gamma_5(1 - aD) = 0,$$

$$\Downarrow$$

$$\delta(\bar{\psi} D \psi) = 0, \quad \delta\bar{\psi} = \bar{\psi}\gamma_5, \quad \delta\psi = \gamma_5(1 - aD)\psi.$$

Ginsparg-Wilson relation (1982) $\Leftrightarrow$ Chiral symmetry on the lattice (1998).

1. Soft photon theorem and conclusion

$$\lim_{\omega\to 0}\omega\mathcal{M}^{(+)}_{A\to B}(q)=\omega\left[\sum_{n\in B}e_n\frac{\epsilon_+\cdot p_n}{q\cdot p_n}-\sum_{n\in A}e_n\frac{\epsilon_+\cdot p_n}{q\cdot p_n}\right]\mathcal{M}_{A\to B}$$

- $\mathcal{M}^{(+)}_{A\to B}(q)$: amplitude $A\to B$ for emitting a soft photon with four momentum $q=(\omega,\vec{q})$ and polarization ϵ_+.
- $\mathcal{M}_{A\to B}$: amplitude $A\to B$.
- p_n : four momentum of nth particle in A/B.
- e_n : charge of nth particle in A/B.

- They showed the following correspondence:

$$\lim_{\omega\to 0}\omega\mathcal{M}^{(+)}_{A\to B}(q)-\omega\left[\sum_{n\in B}e_n\frac{\epsilon_+\cdot p_n}{q\cdot p_n}-\sum_{n\in A}e_n\frac{\epsilon_+\cdot p_n}{q\cdot p_n}\right]\mathcal{M}_{A\to B}=0$$

$$\Updownarrow$$

$$\langle B|\left(Q^+\mathcal{S}-\mathcal{S}Q^-\right)|A\rangle=0,$$

where soft photons are Goldstone modes of broken generators $Q^\pm$.

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- Model : massless QED in $d = 4$ Minkowski spacetime.
- Fields : abelian gauge field $\mathcal{A}_\mu$, massless matter fields Ψ_i with charges e_i.
- Maxwell equation :

$$\nabla^\mu \mathcal{F}_{\mu\nu} = J_\nu.$$

- Gauge symmetry :

$$\mathcal{A}_\mu \to \mathcal{A}_\mu + \partial_\mu \lambda, \qquad \Psi_i \to e^{ie_i\lambda}\Psi_i.$$

2. Future/past null infinities $\mathcal{I}^+/\mathcal{I}^-$

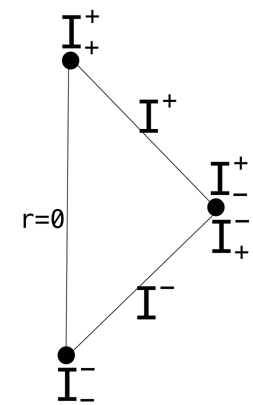
- Future null infinity $\mathcal{I}^+$ = null hypersurface $(r=\infty, u, z, \bar{z})$ in retarded coordinates $(u, r, z, \bar{z})$ $= \mathbb{R}\times S^2$.
- Past null infinity $\mathcal{I}^-$ = null hypersurface $(r=\infty, v, z, \bar{z})$ in advanced coordinates $(v, r, z, \bar{z})$ $= \mathbb{R}\times S^2$.

$$ds^2 = -dt^2 + (dx^i)^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}, \quad (1)$$

$$= -dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}, \quad (2)$$

where $u = t - r$, $r = \sqrt{x_i x^i}$, unit S^2 coordinates $(z, \bar{z})$ with $\gamma_{z\bar{z}} = 2/(1+z\bar{z})^2$ and advanced coordinates are given by $u \to v - 2r$, $r \to r$ and $z \to -1/\bar{z}$.

- Points on S^2 with the same value of z in retarded and advanced coordinate are antipodal.
- Future $u=\infty$ (past $u=-\infty$) boundary of $\mathcal{I}^+ = \mathcal{I}^+_+$ $(\mathcal{I}^+_-)$.
- Future $v=\infty$ (past $v=-\infty$) boundary of $\mathcal{I}^- = \mathcal{I}^-_+$ $(\mathcal{I}^-_-)$.



3. Phase space at $\mathcal{I}^{\pm} = \Gamma^{\pm}$

- $\Gamma^{+} = \{A_z(u,z,\bar{z}), A_{\bar{z}}(u,z,\bar{z}), \phi_{+}(u=+\infty,z,\bar{z}), \phi_{-}(u=-\infty,z,\bar{z})\}$.

In retarded coordinates $(u, r, z, \bar{z})$,

- Retarded radial gauge : $\mathcal{A}_r = 0, \qquad \mathcal{A}_u|_{\mathcal{I}^{+}} = 0$.

- Near $\mathcal{I}^{+}$, $\mathcal{A}_u = r^{-1} A_u(u,z,\bar{z}) + O(r^{-2})$, $\mathcal{A}_z = A_z(u,z,\bar{z}) + O(r^{-1})$,

$$\mathcal{F}_{ur} = r^{-2} A_u + O(r^{-3}),\ \mathcal{F}_{uz} = \partial_u A_z + O(r^{-1}),\ \mathcal{F}_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + O(r^{-1}),$$

- Boundary condition at $\mathcal{I}^{+}_{+}$: $\quad A_u|_{\mathcal{I}^{+}_{+}} = 0, \quad \partial_u A_z|_{\mathcal{I}^{+}_{+}} = 0.$

- Maxwell eq. at $\mathcal{I}^{+}$: $\gamma_{z\bar{z}} \partial_u A_u = \partial_u(\partial_z A_{\bar{z}} + \partial_{\bar{z}} A_z) + \gamma_{z\bar{z}} j_u, \qquad j_u = \lim_{r\to\infty}[r^2 J_u]$.
  $\rightarrow$ free data $= A_z, A_{\bar{z}}$.

- Boundary condition at $\mathcal{I}^{+}_{\pm}$: $(\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z)|\mathcal{I}^{+}_{\pm} = 0 \rightarrow A_z|_{\mathcal{I}^{+}_{\pm}} = \partial_z \phi_{\pm}(z,\bar{z})$="soft photon".
  (zero frequency mode $F^{\omega=0}_{uz} = \int du F_{uz} = A_z|_{\mathcal{I}^{+}_{+}} - A_z|_{\mathcal{I}^{+}_{-}} = \partial_z(\phi_{+} - \phi_{-})$)

- Similarly, $\Gamma^{-} = \{B_z(v,z,\bar{z}), B_{\bar{z}}(v,z,\bar{z}), \psi_{+}(v=+\infty,z,\bar{z}), \psi_{-}(v=-\infty,z,\bar{z})\}$.

4. New symmetries on $\Gamma^{\pm}$

- Residual gauge transformation of Γ^+ : $\delta_\lambda A_z(u,z,\bar{z}) = \partial_z \lambda^+(z,\bar{z})$.
  (large gauge transformation)

- The associated charge $Q^+_\lambda = \underbrace{2\int_{S^2} d^2z\, \lambda^+\, \partial_z \partial_{\bar{z}}(\phi_+ - \phi_-)}_{Q^{+\text{soft}}_\lambda} + \underbrace{\int_{\mathcal{I}^+} du d^2 z\, \lambda^+\, \gamma_{z\bar{z}} j_u}_{Q^{+\text{hard}}_\lambda}.$

- Canonical formulation on Γ^+ :

$$[Q^+_\lambda, \Psi_i(u,z,\bar{z})] = -e_i \lambda^+(z,\bar{z}) \Psi_i(u,z,\bar{z}), \tag{3}$$

$$[Q^+_\lambda, A_z(u,z,\bar{z})] = i\partial_z \lambda^+(z,\bar{z}), \tag{4}$$

$$\Downarrow$$

$$[A_z(u,z,\bar{z}), A_{\bar{w}}(u',w,\bar{w})] = -\frac{i}{4}\Theta(u-u')\delta^2(z-w), \tag{5}$$

$$[\phi_\pm(z,\bar{z}), A_{\bar{z}}(u',w,\bar{w})] = \mp \frac{i}{8\pi}\frac{1}{z-w}, \tag{6}$$

$$[\phi_+(z,\bar{z}), \phi_-(w,\bar{w})] = \mp \frac{i}{4\pi} \log|z-w|^2. \tag{7}$$

$$\Downarrow$$

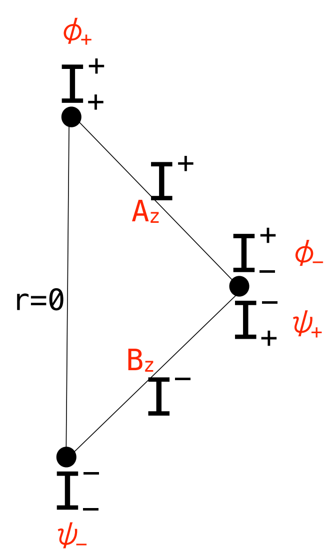
$$[Q^+_\lambda, \phi_\pm(z,\bar{z})] = i\lambda^+(z,\bar{z}), \quad Q^+_\lambda|0\rangle \neq 0 \; for \; \phi_-|0\rangle = 0. \tag{8}$$

5. Matching $\mathcal{I}^+_-$ to $\mathcal{I}^-_+$

- $\Gamma^+ = \{A_z(u,z,\bar{z}), A_{\bar{z}}(u,z,\bar{z}), \phi_+(u=+\infty,z,\bar{z}), \phi_-(u=-\infty,z,\bar{z})\}$.
- $\Gamma^- = \{B_z(v,z,\bar{z}), B_{\bar{z}}(v,z,\bar{z}), \psi_+(v=+\infty,z,\bar{z}), \psi_-(v=-\infty,z,\bar{z})\}$.
- Matching $\mathcal{I}^+_-$ to $\mathcal{I}^-_+$: $\phi_-(z,\bar{z}) = \psi_+(z,\bar{z}) \quad \to \quad \lambda^+(z,\bar{z}) = \lambda^-(z,\bar{z})$.
- Diagonal subgroup of large gauge transformations at $\mathcal{I}^+$ and $\mathcal{I}^-$ = symmetry of the S-matrix.

$$\langle \mathrm{out}|(Q^+_\lambda \mathcal{S} - \mathcal{S} Q^-_\lambda)|\mathrm{in}\rangle = 0, \tag{9}$$

$$\to \quad \langle \mathrm{out}|(Q^{+\mathrm{soft}}_\lambda \mathcal{S} - \mathcal{S} Q^{-\mathrm{soft}}_\lambda)|\mathrm{in}\rangle = -\langle \mathrm{out}|(Q^{+\mathrm{hard}}_\lambda \mathcal{S} - \mathcal{S} Q^{-\mathrm{hard}}_\lambda)|\mathrm{in}\rangle. \tag{10}$$



6. Soft theorem $\Leftrightarrow$ Ward identity

$$\text{Ward identity} \;:\; \langle \text{out}|(Q_\lambda^{+\text{soft}}\mathcal{S} - \mathcal{S}Q_\lambda^{-\text{soft}})|\text{in}\rangle = -\langle \text{out}|(Q_\lambda^{+\text{hard}}\mathcal{S} - \mathcal{S}Q_\lambda^{-\text{hard}})|\text{in}\rangle, \quad (11)$$

where $\langle \text{out}| = \langle z_1^{\text{out}}, \cdots, z_m^{\text{out}}|$, $|\text{in}\rangle = |z_1^{\text{in}}, \cdots, z_n^{\text{in}}\rangle$ and $\lambda(\zeta, \bar\zeta) = \frac{1}{\zeta - z}$,

$$Q^{+\text{soft}}_{\lambda=\frac{1}{\zeta-z}} = -\frac{1}{2}\frac{\sqrt{2}}{1+z\bar z}\lim_{\omega\to 0+}\left[\omega a_+^{\text{out}}(\omega, z, \bar z) + \omega a_-^{\text{out}}(\omega, z, \bar z)^\dagger\right], \quad (12)$$

$$Q^{-\text{soft}}_{\lambda=\frac{1}{\zeta-z}} = \frac{1}{2}\frac{\sqrt{2}}{1+z\bar z}\lim_{\omega\to 0+}\left[\omega a_+^{\text{in}}(\omega, z, \bar z) + \omega a_-^{\text{in}}(\omega, z, \bar z)^\dagger\right], \quad (13)$$

$$\to \lim_{\omega\to 0}\left[\omega \langle \text{out}|\, a_+^{\text{out}}(\omega, z, \bar z)\, \mathcal{S}\, |\text{in}\rangle\right]$$

$$= \frac{1+z\bar z}{\sqrt{2}}\left[\sum_{i=1}^{m}\frac{e_i}{z - z_i^{\text{out}}} - \sum_{i=1}^{n}\frac{e_i}{z - z_i^{\text{in}}}\right]\langle \text{out}|\,\mathcal{S}\,|\text{in}\rangle, \quad (14)$$

$$= \omega\left[\sum_{i=1}^{m} e_i\frac{\epsilon_+\cdot p_i^{\text{out}}}{q\cdot p_i^{\text{out}}} - \sum_{i=1}^{n} e_i\frac{\epsilon_+\cdot p_i^{\text{in}}}{q\cdot p_i^{\text{in}}}\right]\langle \text{out}|\,\mathcal{S}\,|\text{in}\rangle \;:\; \text{soft theorem.} \quad (15)$$