# The Cremmer-Scherk Mechanism in F-Theory Compactifications on K3 Manifolds

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arXiv : 1403.1595

## 1 type IIB string の 7-brane 解

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^{10}x\sqrt{-g}\left(R - g^{MN}\frac{\partial_M\tau\partial_N\bar{\tau}}{2\tau_2^2} - M_{IJ}F^I_{(3)}\cdot F^J_{(3)}\right) + \frac{\epsilon_{IJ}}{8\kappa^2}\int C^+_{(4)}\wedge F^I_{(3)}\wedge F^J_{(3)} + \cdots\ , \quad (1)$$

ここで

$$\tau = C_{(0)} + \mathrm{i}\exp(-\phi), \quad \mathcal{M}_{IJ} = \frac{1}{\tau_2}\begin{pmatrix} |\tau|^2 & -\tau_1 \\ -\tau_1 & 1 \end{pmatrix}. \quad (2)$$

$SL(2;\mathbb{Z})$ 不変性. $\Lambda = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \in SL(2;\mathbb{Z})$,

$$\tau \to \frac{a\tau+b}{c\tau+d}, \quad \left(\tau_2 \to \frac{\tau_2}{|c\tau+d|^2}, \quad \partial_M\tau \to \frac{\partial_M\tau}{(c\tau+d)^2}\right), \quad (3)$$

$$\boldsymbol{B}_{(2)} \to \Lambda\boldsymbol{B}_{(2)}, \quad (4)$$

$$\mathcal{M} \to {}^t(\Lambda^{-1})\mathcal{M}\Lambda^{-1}. \quad (5)$$

SUSY を半分保つ (**32** $\to$ **16**) 7-brane 解の構成 on $\mathbb{R}^{1,7}\times\mathbb{C}$.

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^7)^2 + e^{\varphi(z,\bar{z})}\mathrm{d}z\mathrm{d}\bar{z}, \quad (6)$$

$$\tau = \tau(z,\bar{z}), \quad (7)$$

$$B_{(2)} = C_{(2)} = C^+_{(4)} = 0. \quad (8)$$

$\tau$ の運動方程式

$$\partial_z\partial_{\bar{z}}\tau + \frac{2\partial_z\tau\partial_{\bar{z}}\tau}{\bar{\tau}-\tau} = 0, \quad (9)$$

Instanton 解

$$\partial_{\bar{z}}\tau = 0. \tag{10}$$

$\tau$ は必然的に $z$ に関して多価正則関数になる.

$j$-不変量の導入

$$j(\tau) := \frac{1}{8}\frac{\left(\theta_2(\tau)^8 + \theta_3(\tau)^8 + \theta_4(\tau)^8\right)^3}{\eta(\tau)^{24}} = q^{-1} + 744 + 196884q + O(q^2), \tag{11}$$

$$j(\tau) = j\left(\frac{a\tau + b}{c\tau + d}\right), \tag{12}$$

$$j(\tau) \sim \exp(-2\pi\,\mathrm{i}\,\tau), \qquad \tau_2 \gg 1, \tag{13}$$

7-brane 1 個が原点 $z = 0$ にある解

$$j(\tau(z)) = \frac{1}{z}. \tag{14}$$

$|z| \to +0$ で

$$\tau(z) \sim -\frac{\mathrm{i}}{2\pi}\log(z). \tag{15}$$

$$C_{(0)} \to C_{(0)} + 1 \qquad \left(z \to e^{2\pi\,\mathrm{i}}z\right). \tag{16}$$

Einstein 方程式

$$R_{00} - \frac{1}{2}g_{00}R = T_{00} = -g_{00}e^{-\varphi}\frac{\partial_z\tau\partial_{\bar{z}}\bar{\tau}}{2\tau_2^2} \tag{17}$$

$$\Rightarrow \partial_z\partial_{\bar{z}}\varphi = \partial_z\partial_{\bar{z}}\log\tau_2. \tag{18}$$

$\varphi(z)$ の一価条件から

$$\varphi = \log\left(\tau_2|\eta(\tau)|^4\right) + f(z) + \overline{f(z)} \;\Rightarrow\; \exp(\varphi) = \tau_2|\eta(\tau)|^4|g(z)|^2. \tag{19}$$

ここで $j(z) = \dfrac{P(z)}{(z-z_1)(z-z_2)\cdots(z-z_n)}, \quad \deg P \le n,$ とせよ.

$$e^{2\pi\,\mathrm{i}\,\tau(z)} \sim z - z_i \;\Rightarrow\; |\eta(\tau(z))|^4 \sim |z - z_i|^{1/6}, \qquad z - z_i \to 0, \tag{20}$$

そこで (19) において, $g(z) = ((z-z_1)\cdots(z-z_n))^{-1/12}$ とすると, $\exp(\varphi)$ は 0 にならない. 最終的に

$$\exp(\varphi) = \tau_2|\eta(\tau)|^4 \left|\prod_{i=1}^{n}(z-z_i)^{-1/12}\right|^2. \tag{21}$$

$|z| \to \infty$ で

$$\mathrm{d}s^2 \sim z^{-n/12}\bar{z}^{-n/12}\mathrm{d}z\mathrm{d}\bar{z} = \mathrm{d}\tilde{z}\mathrm{d}\bar{\tilde{z}}, \tag{22}$$

ここで $\tilde{z} = z^{1-n/12}$, その欠損角は

$$\tilde{z} \to e^{2\pi\,\mathrm{i} - \frac{n\pi\,\mathrm{i}}{6}}\,\tilde{z}, \quad (z \to e^{2\pi\,\mathrm{i}}z). \tag{23}$$

複素構造 $\tau$ を持つトーラス

$$E_\tau = \mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau), \quad w = y^{10} + \tau y^{11}. \tag{24}$$

単位面積を持つ計量

$$G_{ab} = \frac{1}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix} \;\Rightarrow\; G_{ab}\mathrm{d}y^a\mathrm{d}y^b = \frac{1}{\tau_2}\mathrm{d}w\mathrm{d}\bar{w}. \tag{25}$$

仮想的な楕円曲線をファイバーに持つ曲面

$$S = \coprod_{z\in\mathbb{C}} E_{\tau(z)} \xrightarrow{\pi} \mathbb{C} \tag{26}$$

の Kähler 計量

$$\mathrm{d}s^2 = \exp(\varphi)\mathrm{d}z\mathrm{d}\bar{z} + \frac{1}{\tau_2}\left(\mathrm{d}w - \frac{w_2\partial_z\tau}{\tau_2}\mathrm{d}z\right)\left(\mathrm{d}\bar{w} - \frac{w_2\partial_{\bar{z}}\bar{\tau}}{\tau_2}\mathrm{d}\bar{z}\right) \tag{27}$$

の Ricci-flat 条件

$$R_{\alpha\bar{\beta}} = -\partial_\alpha\partial_{\bar{\beta}}\log\det g = 0 \tag{28}$$

が (18) に一致する.

Semi-flat Kähler 計量とは

$$\mathrm{d}s^2 = \frac{2^{1/3}(2\pi)^2}{t}\exp(\varphi)\mathrm{d}z\mathrm{d}\bar{z} + \frac{t}{\tau_2}\left(\mathrm{d}w - \frac{w_2\partial_z\tau}{\tau_2}\mathrm{d}z\right)\left(\mathrm{d}\bar{w} - \frac{w_2\partial_{\bar{z}}\bar{\tau}}{\tau_2}\mathrm{d}\bar{z}\right) \tag{29}$$

で, $t \to +0$ としたもの.

平面代数曲線

$$y^2 = x^3 + fx + g = (x - e_1)(x - e_2)(x - e_3), \tag{30}$$

の discriminant

$$\Delta = -(e_1 - e_2)^2(e_2 - e_3)^2(e_3 - e_1)^2 = 4f^3 + 27g^2, \tag{31}$$

(30) の絶対不変量は

$$j(\tau) = 1728\,\frac{4f^3}{4f^3 + 27g^2} = 1728\frac{4f^3}{\Delta}. \tag{32}$$

逆に, 複素構造 $\tau$ を持つ平面代数曲線は

$$y^2 = x^3 - \frac{1}{3}\left(\frac{\pi}{\omega_1}\right)^4 E_4(\tau)x - \frac{2}{27}\left(\frac{\pi}{\omega_1}\right)^6 E_6(\tau), \tag{33}$$

ここで $\lambda = \dfrac{\mathrm{d}x}{2y}$ としておくと

$$\omega_1 = \int_\alpha \lambda, \quad \omega_2 = \int_\beta \lambda, \quad \tau = \frac{\omega_2}{\omega_1}. \tag{34}$$

$$j(\tau) = \frac{E_4(\tau)^3}{\eta(\tau)^{24}} = 1728\frac{4f^3}{\Delta}, \qquad f = -\frac{1}{3}\left(\frac{\pi}{\omega_1}\right)^4 E_4(\tau), \tag{35}$$

を用いて $E_4$ を消去すると,

$$\frac{|\eta(\tau)|^4}{|\Delta|^{1/6}} = 16^{-1/3}\pi^{-2}|\omega_1|^2. \tag{36}$$

一方

$$\int_E \bar{\lambda}\wedge\lambda = \int_\alpha \bar{\lambda}\int_\beta \lambda - \int_\beta \bar{\lambda}\int_\alpha \lambda, \tag{37}$$

を使って

$$\tau_2 = \frac{1}{2\,\mathrm{i}}|\omega_1|^{-2}\int_E \bar{\lambda}\wedge\lambda, \tag{38}$$

$$\exp(\varphi) = \frac{\tau_2|\eta(\tau)|^4}{|\Delta|^{1/6}} = \frac{2^{-1/3}}{4\,\mathrm{i}\,\pi^2}\int_E \bar{\lambda}\wedge\lambda. \tag{39}$$

また

$$\exp(-\varphi)|\omega_1|^2 = \frac{2^{1/3}(2\pi)^2}{\tau_2}. \tag{40}$$

楕円曲線上の座標

$$\zeta = \omega_1 x_1 + \omega_2 x_2 = \omega_1 w, \tag{41}$$

Douglas の 1-form の計算

$$\theta^w = \mathrm{d}\zeta - \frac{(\partial_z\lambda,\bar{\lambda})\zeta - (\partial_z\lambda,\lambda)}{(\lambda,\bar{\lambda})}\mathrm{d}z, \tag{42}$$

$$(\lambda,\bar{\lambda}) = \int_\alpha \lambda \int_\beta \bar{\lambda} - \int_\beta \lambda \int_\alpha \bar{\lambda} = \omega_1\bar{\omega}_2 - \omega_2\bar{\omega}_1 = -2\,\mathrm{i}\,|\omega_1|^2\tau_2, \tag{43}$$

$$(\partial_z\lambda,\bar{\lambda}) = \int_\alpha \partial_z\lambda \int_\beta \bar{\lambda} - \int_\beta \partial_z\lambda \int_\alpha \bar{\lambda} = \partial_z\omega_1\bar{\omega}_2 - \partial_z\omega_2\bar{\omega}_1, \tag{44}$$

$$(\partial_z\lambda,\lambda) = \int_\alpha \partial_z\lambda \int_\beta \lambda - \int_\beta \partial_z\lambda \int_\alpha \lambda = \partial_z\omega_1\omega_2 - \partial_z\omega_2\omega_1, \tag{45}$$

更に

$$\omega_2 = \omega_1\tau, \quad \mathrm{d}\omega_2 = \partial_z\omega_1\tau\mathrm{d}z + \omega_1\partial_z\tau\mathrm{d}z, \tag{46}$$

を使って変形すると

$$\theta^w = \omega_1\left(\mathrm{d}w - \frac{w_2\partial_z\tau}{\tau_2}\mathrm{d}z\right) = \omega_1(\mathrm{d}x_1 + \tau\mathrm{d}x_2), \tag{47}$$

(29) の　Kähler form は $t_0 = 2^{1/3}(2\pi)^2$ として,

$$-\,\mathrm{i}\,J = \frac{t_0}{t}\exp(\varphi)\mathrm{d}z\wedge\mathrm{d}\bar{z} + \frac{t}{\tau_2}\left(\mathrm{d}w - \frac{w_2}{\tau_2}\partial_z\tau\mathrm{d}z\right)\wedge\left(\mathrm{d}\bar{w} - \frac{w_2}{\tau_2}\partial_{\bar{z}}\bar{\tau}\mathrm{d}\bar{z}\right). \tag{48}$$

正規化した 1-forms

$$\rho^z = \sqrt{\frac{t_0}{t}} e^{\varphi/2} \mathrm{d}z, \tag{49}$$

$$\rho^w = \sqrt{\frac{t}{\tau_2}} \left( \mathrm{d}w - \frac{w_2}{\tau_2} \partial_z \tau \mathrm{d}z \right) \tag{50}$$

を使って

$$J = \mathrm{i}\, \rho^z \wedge \rho^{\bar{z}} + \mathrm{i}\, \rho^w \wedge \rho^{\bar{w}}. \tag{51}$$

正則 2-form

$$\Omega = \lambda \wedge \mathrm{d}z. \tag{52}$$

体積要素

$$*1 = \mathrm{i}\, \rho^z \wedge \rho^{\bar{z}} \wedge \mathrm{i}\, \rho^w \wedge \rho^{\bar{w}} = -\frac{t_0}{\tau_2} e^{\varphi} \mathrm{d}z \wedge \mathrm{d}\bar{z} \wedge \mathrm{d}w \wedge \mathrm{d}\bar{w} = \Omega \wedge \bar{\Omega}. \tag{53}$$

# 2 8d String Duality

Moduli 空間 (の主要な連結成分)

$$\mathcal{M} = (SO(2) \times SO(18)) \backslash SO(2,18) / SO(2,18;\mathbb{Z}) \times \mathbb{R}. \tag{54}$$

Hetero side: Narain 格子の moduli 空間と heterotic dilaton $\phi_{\mathrm{Het}}$,

F side: 楕円 K3 の複素構造の moduli と size$(S^2) = \exp(\phi_{\mathrm{Het}})$.

hetero side で Wilson lines の自由度を凍結すると

$$\begin{aligned} \mathcal{M}_{\mathrm{res}} &= (SO(2) \times SO(2)) \backslash SO(2,2) / SO(2,2;\mathbb{Z}) \\ &= \mathbb{Z}_2 \backslash \left( \mathbb{H}_\tau / SL(2;\mathbb{Z}) \times \mathbb{H}_\sigma / SL(2;\mathbb{Z}) \right), \end{aligned} \tag{55}$$

$$\begin{pmatrix} G_{88} & G_{89} \\ G_{98} & G_{99} \end{pmatrix} = \frac{\sqrt{\det G}}{\tau_2} \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}, \tag{56}$$

$$\sigma = B_{89} + \mathrm{i}\sqrt{\det G}. \tag{57}$$

F side で対応する楕円 K3 曲面の 2-parameters 族は

$$y^2 = x^3 + \alpha z^4 x + (z^5 + \beta z^6 + z^7), \tag{58}$$

$z = 0, \infty$ に $\mathrm{II}^*$ タイプの特異点を持つ $\Rightarrow E_8 \times E_8$ gauge symmetry.

Toric 構成

| charge | $Z_0$ | $Z_1$ | $X$ | $Y$ | $W$ | tot |
|---|---|---|---|---|---|---|
| $\lambda$ | 1 | 1 | 0 | 0 | $-2$ | 0 |
| $\mu$ | 0 | 0 | 2 | 3 | 1 | 6 |

(58) を斉次座標 (Cox 座標) で書くと

$$Y^2 = X^3 + \alpha Z_0^4 Z_1^4 X W^4 + (Z_0^7 Z_1^5 + \beta Z_0^6 Z_1^6 + Z_0^5 Z_1^7) W^6. \tag{59}$$

両者の対応

$$j(\tau) j(\sigma) = -1728^2\, \frac{\alpha^3}{8}, \tag{60}$$

$$(j(\tau) - 1728)(j(\sigma) - 1728) = 1728^2\, \frac{\beta^2}{4}. \tag{61}$$

# 3 Cremmer-Scherk 機構

$(p, q)$ 7-brane world sheet の結合

$$-\frac{\mu_8}{4} \int \mathrm{d}^8 x \sqrt{-g} (F_{\mu\nu} + p B_{\mu\nu} + q C_{\mu\nu})^2 \tag{62}$$

F の K3 compact 化の場合

$$-\sum_{i=1}^{24}\frac{\mu_8}{4}\int_{W_i}\mathrm{d}^8x\sqrt{-g}(F^{(i)}_{\mu\nu}+p_iB_{\mu\nu}+q_iC_{\mu\nu})^2. \tag{63}$$

7-brane world volume の埋め込み写像

$$g_i : W_i = \mathbb{R}^{1,7}\times\{z_i\}\hookrightarrow\mathbb{R}^{1,7}\times S^2. \tag{64}$$

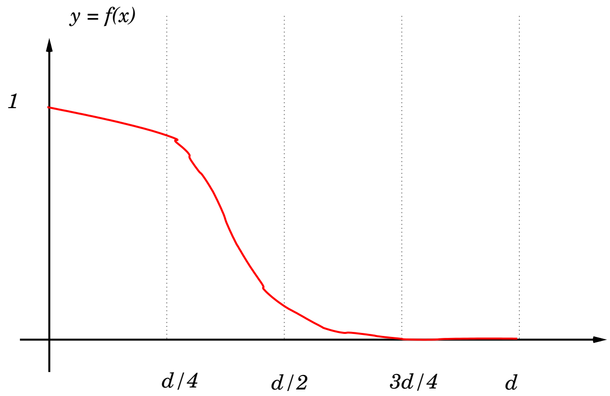
Cremmer-Scherk gauge 変換

$$B^I\to B^I+\mathrm{d}\Gamma^I, \tag{65}$$

$$A^{(i)}\to A^{(i)}-g_i^*(p_i\Gamma^1+q_i\Gamma^2), \tag{66}$$

で (63) は gauge 不変.

Bump 関数 $f(x)$ の導入




$$\begin{pmatrix}\Gamma^1\\ \Gamma^2\end{pmatrix}=\frac{f(|z-z_i|)}{p_i^2+q_i^2}\begin{pmatrix}p_i\\ q_i\end{pmatrix}\Lambda(x^\mu), \tag{67}$$

$$A^{(i)}\to A^{(i)}-\Lambda, \tag{68}$$

$$A^{(j)}\to A^{(j)},\qquad j\neq i, \tag{69}$$

Unitary gauge での結合 (63) は

$$-\sum_{i=1}^{24} \frac{\mu_8}{4} \int_{W_i} \mathrm{d}^8 x \sqrt{-g} (p_i B_{\mu\nu} + q_i C_{\mu\nu})^2. \tag{70}$$

Picard-Lefshetz monodromy 公式.

$\boldsymbol{\gamma} \in H_1(E)$, 消滅サイクル $\boldsymbol{\delta} \in H_1(E)$ に対して

$$\boldsymbol{\gamma} \to \boldsymbol{\gamma}' = \boldsymbol{\gamma} - (\boldsymbol{\gamma} \cdot \boldsymbol{\delta}) \boldsymbol{\delta}. \tag{71}$$

$H_1(E)$ の基底の交叉数は

$$\boldsymbol{\alpha} \cdot \boldsymbol{\alpha} = 0, \quad \boldsymbol{\beta} \cdot \boldsymbol{\beta} = 0, \quad \boldsymbol{\alpha} \cdot \boldsymbol{\beta} = 1, \quad \boldsymbol{\beta} \cdot \boldsymbol{\alpha} = -1, \tag{72}$$

であるから, $\boldsymbol{\delta} = p\boldsymbol{\alpha} + q\boldsymbol{\beta}$ に対して

$$\boldsymbol{\alpha} \to (1 - pq)\boldsymbol{\alpha} - q^2 \boldsymbol{\beta}, \tag{73}$$

$$\boldsymbol{\beta} \to p^2 \boldsymbol{\alpha} + (1 + pq)\boldsymbol{\beta}, \tag{74}$$

$\boldsymbol{\gamma} = m\boldsymbol{\alpha} + n\boldsymbol{\beta}$ の係数の変化で見ると

$$\begin{pmatrix} m' \\ n' \end{pmatrix} = \begin{pmatrix} 1 - pq & p^2 \\ -q^2 & 1 + pq \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}. \tag{75}$$

このとき,

$$\begin{pmatrix} \mathrm{d}B \\ \mathrm{d}C \end{pmatrix} \to \begin{pmatrix} 1 - pq & -q^2 \\ p^2 & 1 + pq \end{pmatrix} \begin{pmatrix} \mathrm{d}B \\ \mathrm{d}C \end{pmatrix}. \tag{76}$$

### 3.1 $\boldsymbol{SL(2, \mathbb{Z})}$ doublet 調和形式

以下の形の KK reduction を考える:

$$B^I = \xi^I(z) \wedge a(x), \tag{77}$$

ここで, $S^2$ 上の調和 1-form $\xi^I$ の $(p,q)$ 7-brane まわりの monodoromy, 不変内積は

$$\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} \to \begin{pmatrix} 1-pq & -q^2 \\ p^2 & 1+pq \end{pmatrix} \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}. \tag{78}$$

$$\langle \xi^{(k)}, \xi^{(l)} \rangle = \int_{S^2} \mathcal{M}_{IJ} \xi^{I(k)} \wedge * \xi^{J(l)}, \tag{79}$$

$a(x)$ が massless ⇔ 調和条件

$$\mathrm{d}\xi^I = 0, \tag{80}$$

$$\mathrm{d} * \mathcal{M}_{IJ} \xi^J = 0. \tag{81}$$

# 未完成