Critical exponents of the 3d Ising and related models from Conformal Bootstrap

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Journal Club Oct. 31 2014, N. Ishibashi

October 30, 2014

Conformal symmetry

- At the critical point

$$\begin{aligned} x^\mu &\rightarrow e^{-\epsilon} x^\mu \\ x^\mu &\rightarrow \frac{x^\mu - \epsilon^\mu x^2}{1 - 2\epsilon \cdot x + \epsilon^2 x^2} \end{aligned}$$

- Recently the critical exponents of the systems like 3D Ising model are calculated using the consequences of this symmetry.
- Gliozzi and Rago propose an alternative method to evaluate them.

Conformal symmetry

- Symmetry generators $P_\mu, M_{\mu\nu}, D, K_\mu \in SO(D,2)$
- Primary fields

$$
\begin{aligned}
[M_{\mu\nu}, \mathcal{O}(0)] &= \Sigma_{\mu\nu}\mathcal{O}(0) \\
[D, \mathcal{O}(0)] &= i\Delta\mathcal{O}(0) \\
[K_\mu, \mathcal{O}(0)] &= 0
\end{aligned}
$$

  classified by their scaling dimension Δ and spin l. Unitarity implies

$$
\Delta \geq \begin{cases} \frac{D}{2} - 1 & (l = 0) \\ l + D - 2 & (l \geq 1) \end{cases}
$$

- Other fields can be expressed as drivatives of primary fields.

Conformal symmetry

- Two and three point functions are fixed by the symmetry

$$\langle \phi(x)\,\phi(y) \rangle = \frac{1}{|x-y|^{2\Delta_\phi}}$$

$$\langle \phi_1(x_1)\,\phi_2(x_2)\,\phi_3(x_3) \rangle = \frac{\lambda_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}\,|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}\,|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}}$$

$$x_{ij} = x_i - x_j$$

ϕ : primary scalar

λ_{123} : constant

OPE

- For scalar primary $\phi_{1,2}$

$$\begin{aligned}\phi_1(x)\,\phi_2(0) &= \sum_{\mathcal{O}} \lambda_{12\mathcal{O}} \left[\frac{1}{|x|^{\Delta_1+\Delta_2-\Delta_{\mathcal{O}}}} \mathcal{O}(0) + \cdots \right] \\ &= \sum_{\mathcal{O}\text{:primary}} \lambda_{12\mathcal{O}} C_{12\mathcal{O}}(x,\partial)\,\mathcal{O}(0)\end{aligned}$$

- Four-point function

$$\begin{aligned}&\langle \phi_1(x_1)\,\phi_2(x_2)\,\phi_3(x_3)\,\phi_4(x_4)\rangle \\ &= \sum_{\mathcal{O}} \lambda_{12\mathcal{O}} \lambda_{34\mathcal{O}} C_{12\mathcal{O}}(x_1-x_2,\partial_2)\, C_{34\mathcal{O}}(x_3-x_4,\partial_4) \langle \mathcal{O}(x_2)\,\mathcal{O}(x_3)\rangle\end{aligned}$$

We can calculate everything once we know all the λ.

OPE

- We get the form of the four point function

$$
\begin{aligned}
&\langle \phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\rangle \\
&\quad = \sum_{\mathcal{O}} \lambda^2_{\phi\phi\mathcal{O}} C_{\phi\phi\mathcal{O}}(x_1 - x_2, \partial_2)\, C_{\phi\phi\mathcal{O}}(x_3 - x_4, \partial_4)\, \langle \mathcal{O}(x_2)\,\mathcal{O}(x_3)\rangle \\
&\quad = |x_{12}|^{-2\Delta_\phi}\, |x_{34}|^{-2\Delta_\phi}\, g(u, v)
\end{aligned}
$$

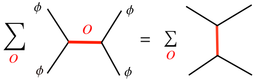
$$
g(u, v) = 1 + \sum_{\Delta, l} \lambda^2_{\phi\phi\mathcal{O}} G_{\Delta, l}(u, v)
$$

$$
u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}
$$

Conformal bootstrap

- There are two ways to get the four point function

$$\langle \phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\rangle = |x_{12}|^{-2\Delta_\phi}\,|x_{34}|^{-2\Delta_\phi}\,g(u,v)$$



- Thus we get the identity

$$v^{\Delta_\phi} g(u,v) = u^{\Delta_\phi} g(v,u)$$

$$g(u,v) = 1 + \sum_{\Delta,l} \lambda^2_{\phi\phi\mathcal{O}} G_{\Delta,l}(u,v)$$

- We should be able to get informations on λ by studying the constraints.

Recent developments

- The identity can be rewritten as

$$\sum_{\Delta,l} \lambda^2_{\phi\phi\mathcal{O}} F_{\Delta_\phi,\Delta,l}(u,v) = 1$$

$$F_{\Delta_\phi,\Delta,l}(u,v) = \frac{v^{\Delta_\phi} G_{\Delta,l}(u,v) - u^{\Delta_\phi} G_{\Delta,l}(u,v)}{u^{\Delta_\phi} - v^{\Delta_\phi}}$$

- from which one gets

$$\sum_{\Delta,l} \lambda^2_{\phi\phi\mathcal{O}} F_{\Delta_\phi,\Delta,l}\left(\frac{1}{4},\frac{1}{4}\right) = 1$$

$$\sum_{\Delta,l} \lambda^2_{\phi\phi\mathcal{O}} \partial_a^{2m} \partial_b^n F_{\Delta_\phi,\Delta,l}\left(\frac{1}{4},\frac{1}{4}\right) = 0$$

$$u = z\bar{z},\ v = (1-z)(1-\bar{z}),\ z = \frac{a+\sqrt{b}}{2}$$

Recent developments

$$
\begin{aligned}
\sum_{\Delta,l} \lambda^2_{\phi\phi\mathcal{O}} F_{\Delta_\phi,\Delta,l}\left(\frac{1}{4},\frac{1}{4}\right) &= 1 \\
\sum_{\Delta,l} \lambda^2_{\phi\phi\mathcal{O}} \partial_a^{2m}\partial_b^{n} F_{\Delta_\phi,\Delta,l}\left(\frac{1}{4},\frac{1}{4}\right) &= 0
\end{aligned}
$$

- Unitarity implies $\lambda^2_{\phi\phi\mathcal{O}} > 0$.
- For 2D and 4D, one can get constraints on Δ and λ.

  (Rattazzi et.al., El-Showk and Paulos)

3D Ising

- There are primary fields $\sigma, \varepsilon, \cdots$ such that

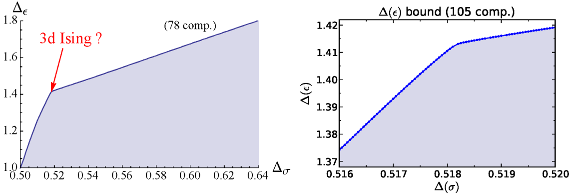
$$\sigma\sigma \sim \varepsilon + \cdots$$

- Bootstrap equation for $\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle$:

$$\begin{aligned}\sum_{\Delta,l} \lambda^2_{\sigma\sigma\mathcal{O}} F_{\Delta_\sigma,\Delta,l}\left(\frac{1}{4},\frac{1}{4}\right) &= 1 \\ \sum_{\Delta,l} \lambda^2_{\sigma\sigma\mathcal{O}} \partial_a^{2m}\partial_b^{n} F_{\Delta_\sigma,\Delta,l}\left(\frac{1}{4},\frac{1}{4}\right) &= 0\end{aligned}$$

- For $\Delta \gg 1$, $\partial_a^{2m}\partial_b^{n} F_{\Delta_\sigma,\Delta,l}\left(\frac{1}{4},\frac{1}{4}\right) \propto r^{\Delta} P(\Delta)$ $(r<1)$ with P a polynomial.
- One can approximate these equations by finite set of (Δ, l) (~ 200) and get an upper bound for Δ_ε.

3D Ising



- From the values of $\Delta_\sigma, \Delta_\varepsilon$ the 3D Ising model is at the cusp of the graph.
- Assuming it is exactly at the cusp, one can get various quantities very accurately. (El-Showk et.al.)

$$\Delta_\sigma = 0.518154(15),\ \Delta_\varepsilon = 1.41267(13),\ \text{etc.}$$

Proposal of Gliozzi and Rago

- Start from the bootstrap equations

$$\begin{aligned}\sum_{\Delta,l}\lambda^2_{\phi\phi\mathcal{O}}F_{\Delta_\phi,\Delta,l}\left(\frac{1}{4},\frac{1}{4}\right) &= 1\\ \sum_{\Delta,l}\lambda^2_{\phi\phi\mathcal{O}}\partial_a^{2m}\partial_b^n F_{\Delta_\phi,\Delta,l}\left(\frac{1}{4},\frac{1}{4}\right) &= 0\end{aligned}$$

  and the fusion rule (first few terms)

$$[\Delta_\phi]\times[\Delta_\phi]=\sum_i[\Delta_i,l_i]$$

- Prepare N set of (Δ,l) and consider $M\,(>N)$ such equations. From the condition that the linear equations have a solution, we get Δ.

Results 1

Lee-Yang edge singularity

$$[\Delta_\phi] \times [\Delta_\phi] = 1 + [\Delta_\phi] + [D, 2] + [\Delta_4, 4] + \left[\Delta'\right] + \cdots$$

D dimensional Yang-Lee model–the edge exponent σ					
D	bootstrap	Ising in H	Fluids	Animals	$\epsilon-$expansion
2	-0.1664(5)	-0.1645(20)	-0.161(8)	-0.165(6)	(exact -1/6)
3	0.085(1)	0.077(2)	0.0877(25)	0.080(7)	0.079-0.091
4	0.2685(1)	0.258(5)	0.2648(15)	0.261(12)	0.262-0.266
5	0.4105(5)	0.401(9)	0.402(5)	0.40(2)	0.399-0.400
6	0.4999(1)	0.460(50)	0.465(35)	—	1/2

- One does not need unitarity to apply this method.

Results 2

3D Ising

- fusion rule

$$[\Delta_\sigma] \times [\Delta_\sigma] = 1 + [\Delta_\varepsilon] + [\Delta_{\varepsilon'}] + [\Delta_{\varepsilon''}] + [\Delta_{\varepsilon'''}]$$
$$+ [3,2] + [\Delta_4, 4] + [\Delta_6, 6] + \cdots$$

- $\Delta_4 = 5.0208(12)$

scalar operators	σ	σ'	ε	ε'	ε''	ε'''
Δ, best estimates	0.51813(5)	$\gtrsim 4.5$	1.41275(25)	3.84(4)	4.67(11)	–
Δ, bootstrap	0.5171(1)	4.05(5)	1.413(2)	3.79(3)	4.60(7)	5.80(4)

spinning operators	$[\Delta_2, 2]$	$[\Delta'_2, 2]$	$[\Delta_4, 4]$	$[\Delta'_4, 4]$	$[\Delta_6, 6]$
Δ, best estimates	–	–	5.0208(12)	–	7.028(8)
Δ, bootstrap	5.117(1)	6.20(1)	input	6.70(1)	7.04(2)