Cyclic Leibniz rule

- M. Kato, M. Sakamoto, H. So, JHEP 05 (2013) 089,
  A criterion for lattice supersymmetry: cyclic Leibniz rule.

- M. Kato, M. Sakamoto, H. So, arXiv:1311.4962 [hep-lat],
  Cyclic Leibniz rule: a formulation of supersymmetry on lattice.

- M. Kato, M. Sakamoto, H. So, JHEP 05 (2008) 057,
  Taming the Leibniz rule on the lattice.

Contents

1. From continuum SUSY to lattice SUSY : Leibniz rule vs locality

- continuum SUSY algebra : $\{Q, \bar{Q}\} = i\partial$
- SUSY invariant action : $\delta S = \int dx \delta\mathcal{L} = \int dx \partial(*) = 0$
- Leibniz rule is used for $\delta\mathcal{L} \ni \partial A \cdot B \cdot C + A \cdot \partial B \cdot C + A \cdot B \cdot \partial C = \partial(ABC)$.

- continuum $\rightarrow$ lattice : derivative op. ∂ $\rightarrow$ difference op. Δ
- lattice SUSY algebra : $\{Q, \bar{Q}\} = i\Delta$
- violation of Leibniz rule for Δ : $\Delta\phi \cdot \psi + \phi \cdot \Delta\psi \neq \Delta(\phi\psi)$
- it is possible to make a lattice SUSY invariant free action due to $\sum_n (\Delta\phi)_n \psi_n + \phi_n (\Delta\psi)_n = 0$.
- it is difficult to realize lattice SUSY with interaction $\phi\phi F, \phi\bar{\psi}\psi$.

- reconsider a general "local" difference op.

$$\Delta_{mn} = \Delta(m-n), \ \ |\Delta(m-n)| \leq C_0 e^{-K_0|m-n|}, \ \ |m-n| >> 1. \tag{1}$$

and a general "local" product rule

$$\{\phi, \psi\}_l^M = \sum_{mn} M_{lmn} \phi_m \psi_n = \sum_{mn} M_{lmn} \phi_n \psi_m. \tag{2}$$

$$M_{lmn} = M(m-l, n-l), \ \ |M(m-l.n-l) \leq C_1 e^{-K_1|m-l|-K_2|n-l|}|. \tag{3}$$

- and a inner product on a lattice space

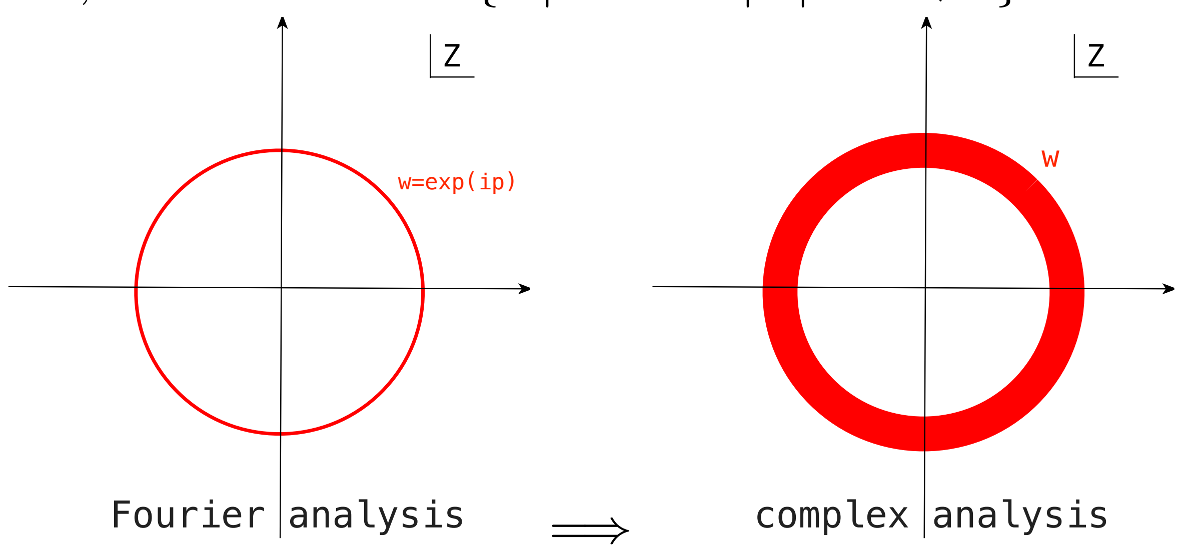
$$(\phi, \psi) \quad = \quad \sum_n \phi_n \psi_n. \tag{4}$$

- $\sum_n F_n \phi_n \phi_n \to \sum_{lmn} M_{lmn} F_l \phi_m \phi_n = (F, \{\phi, \phi\}^M)$.
- Momentum representation (w-expression) :

$$\hat{\Delta}(w) \quad = \quad \sum_{n=-\infty}^{\infty} w^n \Delta(n), \tag{5}$$

$$\hat{M}(w, z) \quad = \quad \sum_{m,n=-\infty}^{\infty} w^m z^n M(m, n) = \hat{M}(z, w). \tag{6}$$

where $w = e^{ipa}$, $z = e^{iqa} \Rightarrow \mathcal{D} = \{w | 1 - \epsilon < |w| < 1 + \epsilon\}$.



- $\Delta(n)$ is exponentially local. $=$ $\hat{\Delta}(w)$ is holomorphic on $\mathcal{D}$.
- $M(m,n)$ is exponentially local. $=$ $\hat{M}(w,z)$ is holomorphic on $\mathcal{D}\otimes\mathcal{D}$.

- Leibniz rule :

$$(\chi, \Delta\{\phi,\psi\}^M) = (\chi, \{\Delta\phi,\psi\}^M) + (\chi, \{\phi, \Delta\psi\}^M) \tag{7}$$

in w-expressin

$$(\underbrace{\hat{\Delta}(wz) - \hat{\Delta}(w) - \hat{\Delta}(z)}_{=0})\hat{M}(w,z) = 0. \tag{8}$$

$$\hat{\Delta}(w) = \beta \log w. \tag{9}$$

$\log w$ is not holomorphic on $\mathcal{D}$, so that $\Delta(n)$ is not exponentially local.
Leibniz rule and locality are incompatible with each other.

2. Cyclic Leibniz rule and locality

- Introduce a new rule "cyclic Leibniz rule"

$$(\Delta\chi, \{\phi, \psi\}^M) + (\Delta\phi, \{\psi, \chi\}^M) + (\Delta\psi, \{\chi, \phi\}^M) = 0. \tag{10}$$

- In w-expressin

$$\hat{M}(w, z)\hat{\Delta}(\frac{1}{wz}) + \hat{M}(z, \frac{1}{wz})\hat{\Delta}(w) + \hat{M}(\frac{1}{wz}, w)\hat{\Delta}(z) = 0 \tag{11}$$

- A holomorphic (= local) solution is

$$\hat{\Delta}(w) = \frac{w - w^{-1}}{2}, \quad \hat{M}(wz) = \frac{1}{6}(2wz + wz^{-1} + w^{-1}z + 2(wz)^{-1}). \tag{12}$$

3. D=1, N=2 supersymmetric quantum mechanics

- D=1, N=2 super quantum mechanics with (ϕ_n, ψ_n, F_n)
- N=2 SUSY transformation

$$
\begin{aligned}
\delta\phi_n &= \epsilon\bar{\psi}_n - \bar{\epsilon}\psi_n, & (13)\\
\delta\psi_n &= \epsilon((i\Delta\phi)_n + F_n), & (14)\\
\delta\bar{\psi}_n &= \bar{\epsilon}((-i\Delta\phi)_n + F_n), & (15)\\
\delta F_n &= -\epsilon(i\Delta\bar{\psi}) - \bar{\epsilon}(i\Delta\psi). & (16)
\end{aligned}
$$

- SUSY invariant action under only ϵ transformation.

$$
\begin{aligned}
S &= \frac{1}{2}(\Delta\phi, \Delta\phi) + i(\bar{\psi}, \Delta\psi) + \frac{1}{2}(F, F) \\
&\quad +(F, (im + iW)\phi) + (\bar{\psi}, (im + iW)\psi) \\
&\quad +ig(F, \{\phi, \phi\}) - 2ig(\psi, \{\phi, \bar{\psi}\}). & (17)\\
\delta_\epsilon S &= -i\epsilon((\Delta\bar{\psi}, \{\phi, \phi\}) + (\Delta\phi, \{\phi, \bar{\psi}\}) + (\Delta\phi, \{\bar{\psi}\,\phi\})) = 0. & (18)\\
\delta_{\bar{\epsilon}} S &\neq 0. & (19)
\end{aligned}
$$