Calabi's Diastasis and Interface Entropy

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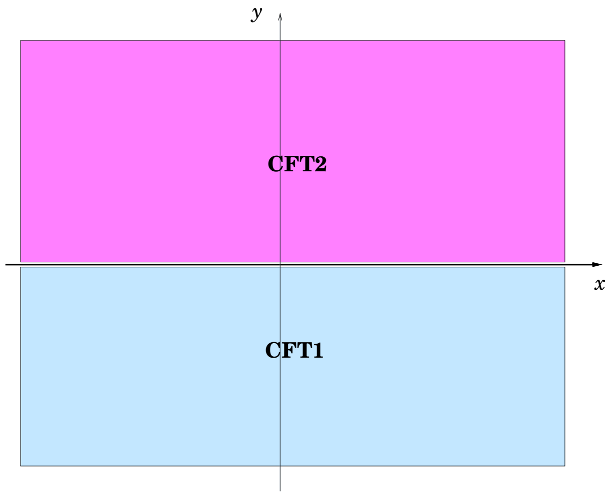
1 Conformal Interface

1.1 Interface と Folding trick

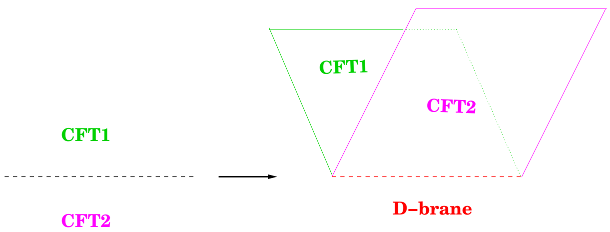


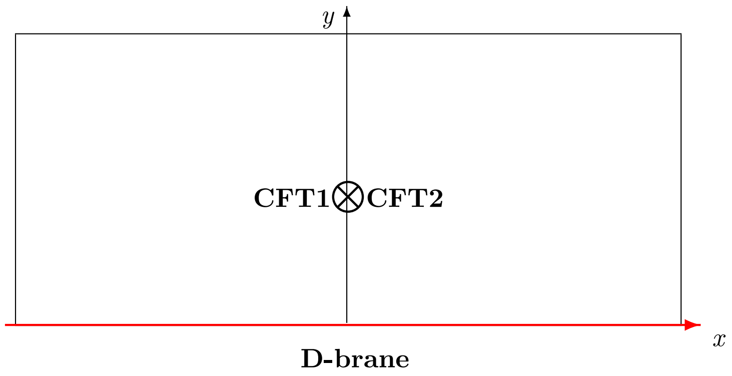
整合性条件

$$T^{(1)}_{tx} = T^{(2)}_{tx} \qquad \text{at the boundary.} \tag{1}$$

light cone 座標 $x^{\pm} = t \pm x$ の成分で $T_{tx} = T_{++}(x^+) - T_{--}(x^-)$.

Folding により, Interface $\Rightarrow$ CFT1 $\bigotimes$ CFT2.





$$(L_n^{(1)}+L_n^{(2)}-\bar{L}_{-n}^{(1)}-\bar{L}_{-n}^{(2)})|\mathcal{B}\rangle\rangle=0. \tag{2}$$

例 完全反射型の場合 $(L_n^{(1)}-\bar{L}_{-n}^{(1)})|\mathcal{B}\rangle\rangle=0,\ (L_n^{(2)}-\bar{L}_{-n}^{(2)})|\mathcal{B}\rangle\rangle=0,$

$\Rightarrow |\mathcal{B}\rangle\rangle=|\mathcal{B}_1\rangle\rangle\otimes|\mathcal{B}_2\rangle\rangle.$

境界状態 $|\mathcal{B}\rangle\rangle=\sum_{\lambda_1,\bar{\lambda}_1,\lambda_2,\bar{\lambda}_2}B_{\lambda_1,\bar{\lambda}_1;\lambda_2,\bar{\lambda}_2}|\lambda_1,\bar{\lambda}_1\rangle\otimes|\lambda_2,\bar{\lambda}_2\rangle$ に対して

$$\mathcal{I}=\sum_{\lambda_1,\bar{\lambda}_1,\lambda_2,\bar{\lambda}_2}B_{\lambda_1,\bar{\lambda}_1;\lambda_2,\bar{\lambda}_2}|\lambda_1,\bar{\lambda}_1\rangle\otimes\langle\bar{\lambda}_2,\lambda_2|:\mathcal{H}_2\to\mathcal{H}_1, \tag{3}$$

を定義すると, これは $T(z)-\bar{T}(\bar{z})$ の作用と可換, つまり以下は可換図になる.

$$\begin{array}{ccc} \mathcal{H}_2 & \xrightarrow{\mathcal{I}} & \mathcal{H}_1 \\ {\scriptstyle L_n-\bar{L}_{-n}}\downarrow & & \downarrow{\scriptstyle L_n-\bar{L}_{-n}} \\ \mathcal{H}_2 & \xrightarrow[\mathcal{I}]{} & \mathcal{H}_1 \end{array} \tag{4}$$

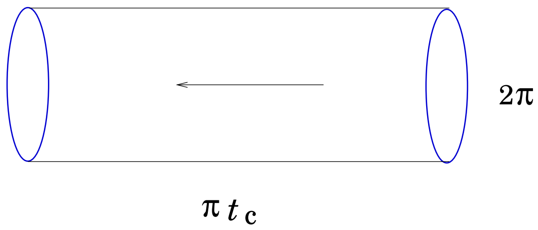
例 完全透過型の場合 $\mathcal{I}$ は $T(z),\bar{T}(\bar{z})$ とそれぞれ可換 (topological interface).

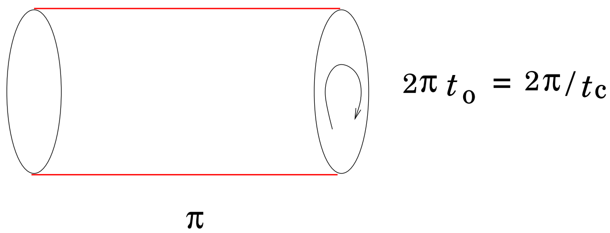
例　特に自明な interface では, $\mathcal{I}=\mathrm{id}:\mathcal{H}_1\to\mathcal{H}_1$.

1.2　D-brane の Cardy 条件

$$Z_{\beta|\alpha}=\langle\langle\beta|\exp\left[-\pi t_c(L_0^{\text{closed}}+\bar{L}_0^{\text{closed}}-c/12)\right]|\alpha\rangle\rangle, \qquad \text{closed string channel}, \tag{5}$$

$$=\mathrm{Tr}_{\mathcal{H}_{\beta|\alpha}}\exp\left[-2\pi t_o(L_0^{\text{open}}-c/24)\right], \qquad \text{open string channel}. \tag{6}$$





以下 RCFT における Cardy 条件の解の構成.

$$\text{Ishibashi state} \qquad |i\rangle\rangle_{\mathrm{I}} = \sum_n |i;n\rangle \otimes \overline{|i;n\rangle}, \tag{7}$$

$$\text{Cardy state} \qquad |\alpha\rangle\rangle = \sum_l \frac{\psi_\alpha^l}{\sqrt{S_1{}^l}} |l\rangle\rangle_{\mathrm{I}}. \tag{8}$$

$${}_{\mathrm{I}}\langle\langle j| q_{\mathrm{c}}^{1/2(L_0^{\mathrm{closed}}+\bar{L}_0^{\mathrm{closed}}-c/12)} |i\rangle\rangle_{\mathrm{I}} = \delta_{i,j}\chi_i(q_{\mathrm{c}}), \quad q_{\mathrm{c}} := \exp(-2\pi t_{\mathrm{c}}),$$

Open string channel での分配関数

$$Z_{\beta|\alpha} = \sum_i n^i_{\beta^\vee,\alpha}\chi_i(q_{\mathrm{o}}), \qquad q_{\mathrm{o}} = \exp(-2\pi t_{\mathrm{o}}). \tag{9}$$

Closed string channel では

$$Z_{\beta|\alpha} = \langle\langle\beta| q_{\mathrm{c}}^{1/2(L_0^{\mathrm{closed}}+\bar{L}_0^{\mathrm{closed}}-c/12)} |\alpha\rangle\rangle = \sum_l \frac{\psi_\alpha^l \overline{\psi_\beta^l}}{S_1{}^l}\chi_l(q_{\mathrm{c}}) = \sum_{l,i} \frac{\psi_\alpha^l \overline{\psi_\beta^l}}{S_1{}^l} S_l{}^{i^\vee}\chi_i(q_{\mathrm{o}}). \tag{10}$$

ここで

$$\chi_i(q_{\mathrm{o}}) = \sum_j S_i{}^j \chi_j(q_{\mathrm{c}}), \quad \chi_i(q_{\mathrm{c}}) = \sum_j S_i{}^{j^\vee}\chi_j(q_{\mathrm{o}}). \tag{11}$$

Fusion 係数に関する Verlinde の公式

$$N_{\alpha,\beta}^{\gamma} = \sum_l \frac{S_\alpha{}^l S_\beta{}^l \overline{S_\gamma{}^l}}{S_1{}^l} \tag{12}$$

に基づく Cardy の解

$$\psi_\alpha^l = S_\alpha{}^l, \qquad n^i_{\beta^\vee,\alpha} = N^i_{\beta^\vee,\alpha}, \tag{13}$$

1.3 Affleck-Ludwig の境界エントロピー

$t_\mathrm{c} \gg 1$(高温展開) において,

$$\log Z_{\beta|\alpha} \sim \frac{c}{12}\pi t_\mathrm{c} + \log\langle 0|\alpha\rangle\!\rangle + \log\langle\!\langle\beta|0\rangle + \cdots, \tag{14}$$

そこで D-brane α に同伴するエントロピー因子

$$g_\alpha := \langle 0|\alpha\rangle\!\rangle. \tag{15}$$

例 CFT として半径 R の S^1 を target にする σ 模型をとる ($\alpha' = 2$).

$$X_L(\tau,\sigma) = x_L - \mathrm{i}(\tau - \mathrm{i}\,\sigma)p_L + \mathrm{i}\sum_{n\neq 0}\frac{\alpha_n}{n}e^{-n(\tau-\mathrm{i}\,\sigma)}, \tag{16}$$

$$X_R(\tau,\sigma) = x_R - \mathrm{i}(\tau + \mathrm{i}\,\sigma)p_R + \mathrm{i}\sum_{n\neq 0}\frac{\bar\alpha_n}{n}e^{-n(\tau+\mathrm{i}\,\sigma)}, \tag{17}$$

ここで

$$p_L = \frac{\hat N}{R} + \frac{R\hat M}{2}, \qquad x_L = \frac{1}{2}(\hat\phi_0 + \hat{\tilde\phi}_0), \tag{18}$$

$$p_R = \frac{\hat N}{R} - \frac{R\hat M}{2}, \qquad x_R = \frac{1}{2}(\hat\phi_0 - \hat{\tilde\phi}_0), \tag{19}$$

として, 交換関係は $[\hat\phi_0, \hat N] = R\,\mathrm{i}, \qquad [\hat{\tilde\phi}_0, \hat M] = \dfrac{2\,\mathrm{i}}{R}$.

$$|\mathrm{D}\rangle\!\rangle = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}\bar\alpha_{-n}\right)\left(\frac{1}{\sqrt{2R}}\sum_{N=-\infty}^{\infty}e^{-\mathrm{i}\,N\phi_0/R}|N;0\rangle\right), \tag{20}$$

$$|\mathrm{N}\rangle\!\rangle = \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}\bar\alpha_{-n}\right)\left(\sqrt{R}\sum_{M=-\infty}^{\infty}e^{-\mathrm{i}\,RM\tilde\phi_0/2}|0;M\rangle\right), \tag{21}$$

係数 $g_\mathrm{D} = \dfrac{1}{\sqrt{2R}}$, $g_\mathrm{N} = \sqrt{R}$, は open-closed duality から定まる.

Graviton vertex operator[5]: $\overbrace{\epsilon_{\mu\nu}\partial X^{\mu}\bar{\partial}X^{\nu}e^{\mathrm{i}\,kx}}^{\text{flat part}} \bigotimes \overbrace{\text{identity}}^{\text{CFT}}$,

$$(l_{\mathrm{s}}m_{\alpha})^2 \propto |\langle 0|\alpha\rangle\rangle|^2. \tag{22}$$

2 大半径 CYd でのエントロピー

2.1 Torus

$\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$: $x \sim x+1,\, y \sim y+1.$

複素構造 $\rho \in \mathbb{H}, \quad z = x + \rho y,\, z \sim z+1,\, z \sim z+\rho.$

正則 1-form $\Omega = \mathrm{d}z.$

Kähler 構造

$$G_{\mu\nu} = \frac{\tau_2}{\rho_2}\begin{pmatrix} 1 & \rho_1 \\ \rho_1 & |\rho|^2 \end{pmatrix}, \qquad B_{\mu\nu} = \begin{pmatrix} 0 & \tau_1 \\ -\tau_1 & 0 \end{pmatrix}. \tag{23}$$

Kähler 形式 $k = \frac{\mathrm{i}}{2} G_{z\bar{z}}\, \mathrm{d}z \wedge \mathrm{d}\bar{z} = \tau_2\, \mathrm{d}x \wedge \mathrm{d}y, \quad b = B_{xy}\, \mathrm{d}x \wedge \mathrm{d}y = \tau_1\, \mathrm{d}x \wedge \mathrm{d}y,$

複素化した Kähler 形式 $\omega = b + \mathrm{i}\, k = \tau\, \mathrm{d}x \wedge \mathrm{d}y.$

Kähler potentials for moduli spaces

$$\mathcal{K}_{\text{複素}} = -\log\left(\int_M \frac{\mathrm{i}}{2}\Omega \wedge \bar{\Omega}\right) = -\log(\rho_2), \tag{24}$$

$$\mathcal{K}_{\text{Käh}} = -\log\left(\int_M k\right) = -\log(\tau_2). \tag{25}$$

その解析接続を以下で定義出来る:

$$\mathcal{K}_{\text{Käh}}(\tau, \bar{\tau}') = -\log(\tau - \bar{\tau}') \tag{26}$$

$M = M(\tau, \rho),\, M' = M(\tau', \rho')$ の interface entropy

$$g = \frac{\det^{1/2}(G - B + G' + B')}{\sqrt{4\tau_2\tau_2'}} = \left[\frac{(\tau-\bar{\tau}')(\tau'-\bar{\tau})}{(\tau-\bar{\tau})(\tau'-\bar{\tau}')} + \frac{(\rho-\bar{\rho}')(\rho'-\bar{\rho})}{(\rho-\bar{\rho})(\rho'-\bar{\rho}')} - 1\right]^{1/2}. \tag{27}$$

$$\begin{aligned} 2\log g|_{\rho=\rho'} &= -\log(\tau-\bar{\tau}) - \log(\tau'-\bar{\tau}') + \log(\tau-\bar{\tau}') + \log(\tau'-\bar{\tau}), \\ &= \mathcal{K}_{\text{Käh}}(\tau,\bar{\tau}) + \mathcal{K}_{\text{Käh}}(\tau',\bar{\tau}') - \mathcal{K}_{\text{Käh}}(\tau,\bar{\tau}') - \mathcal{K}_{\text{Käh}}(\tau',\bar{\tau}), \end{aligned} \tag{28}$$

右辺は diastasis $D_{\text{Käh}}(\tau, \tau')$ である.

2.2 Kähler 構造の 変形

CFT1: $M = M(u,t)$ を target にする super σ-model,

CFT2: $M' = M(u',t)$ を target にする super σ-model,

Folding of conformal Interface $\Rightarrow$ diagonal D-brane $\Delta(M) \subset M \times M'$.

Dirac-Born-Infeld 作用による近似

$$g_{\mathrm{hol}} \simeq \frac{\int_M \det^{1/2}(G - B + G' + B')}{\sqrt{4^d \operatorname{vol}(M) \operatorname{vol}(M')}} \tag{29}$$

の Kähler forms $\omega = \omega(u)$, $\omega' = \omega(u')$ による書き換え.

$$b = \sum_{ij} B_{i\bar{j}} \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j, \quad k = \frac{\mathrm{i}}{2} \sum_{ij} G_{i\bar{j}} \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j, \quad \omega := b + \mathrm{i}\, k,$$

$B_{i\bar{j}}$ は反エルミート, $\mathrm{d}s^2 = \sum_{ij} G_{i\bar{j}} \mathrm{d}z_i \mathrm{d}\bar{z}_j$,

$$\omega = \sum_{ij} (B_{i\bar{j}} - \frac{1}{2} G_{i\bar{j}}) \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j, \quad \omega' = \sum_{ij} ({B'}_{i\bar{j}} - \frac{1}{2} {G'}_{i\bar{j}}) \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j,$$

$$\bar{\omega} = \sum_{ij} (B_{i\bar{j}} + \frac{1}{2} G_{i\bar{j}}) \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j, \quad \bar{\omega}' = \sum_{ij} ({B'}_{i\bar{j}} + \frac{1}{2} {G'}_{i\bar{j}}) \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j,$$

$$G - B + G' + B' = \left[\begin{array}{c|c} O & \frac{1}{2}(G_{i\bar{j}} + {G'}_{i\bar{j}}) + {B'}_{i\bar{j}} - B_{i\bar{j}} \\ \hline \frac{1}{2}(G_{j\bar{i}} + {G'}_{j\bar{i}}) - {B'}_{j\bar{i}} + B_{j\bar{i}} & O \end{array} \right],$$

$$\int_M \frac{1}{d!} \left(\frac{\omega - \bar{\omega}'}{2\,\mathrm{i}} \right)^d = \int_M \prod_{k=1}^d \left(\frac{\mathrm{i}}{2} \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k \right) \det(\tfrac{1}{2}(G_{i\bar{j}} + {G'}_{i\bar{j}}) + {B'}_{i\bar{j}} - B_{i\bar{j}}), \tag{30}$$

$$\int_M \frac{1}{d!} \left(\frac{\omega' - \bar{\omega}}{2\,\mathrm{i}} \right)^d = \int_M \prod_{k=1}^d \left(\frac{\mathrm{i}}{2} \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k \right) \det(\tfrac{1}{2}(G_{i\bar{j}} + {G'}_{i\bar{j}}) - {B'}_{i\bar{j}} + B_{i\bar{j}}), \tag{31}$$

特に $$\int_M \frac{1}{d!} \left(\frac{\omega - \bar{\omega}}{2\,\mathrm{i}} \right)^d = \int_M \prod_{k=1}^d \left(\frac{\mathrm{i}}{2} \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k \right) \det(G_{i\bar{j}}) = \operatorname{vol}(M), \tag{32}$$

他方 $$\int_M \det^{1/2}(G - B + G' + B') \mathrm{d}^{2d}x = 2^d \int_M \prod_{k=1}^d \left(\frac{\mathrm{i}}{2} \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k \right)$$

$$\det^{1/2}(\tfrac{1}{2}(G_{i\bar{j}} + {G'}_{i\bar{j}}) + {B'}_{i\bar{j}} - B_{i\bar{j}}) \cdot \det^{1/2}(\tfrac{1}{2}(G_{i\bar{j}} + {G'}_{i\bar{j}}) - {B'}_{i\bar{j}} + B_{i\bar{j}}), \tag{33}$$

すると (29) は

$$g_{\rm hol}^2 \simeq \frac{\dfrac{1}{d!}\displaystyle\int_M \left(\frac{\omega-\bar{\omega}'}{2\,\mathrm{i}}\right)^d \dfrac{1}{d!}\displaystyle\int_M \left(\frac{\omega'-\bar{\omega}}{2\,\mathrm{i}}\right)^d}{\dfrac{1}{d!}\displaystyle\int_M \left(\frac{\omega-\bar{\omega}}{2\,\mathrm{i}}\right)^d \dfrac{1}{d!}\displaystyle\int_M \left(\frac{\omega'-\bar{\omega}'}{2\,\mathrm{i}}\right)^d} \tag{34}$$

と近似出来る.

Kähler potential

$$e^{-\mathcal{K}_{\rm K\ddot{a}h}(u,\bar{u})} \simeq \mathrm{vol}(M) = \frac{1}{d!}\int_M k^d = \frac{1}{d!}\int_M \left(\frac{\omega(u)-\overline{\omega(u)}}{2\,\mathrm{i}}\right)^d, \tag{35}$$

を解析接続出来て

$$\mathcal{K}_{\rm K\ddot{a}h}(u,\bar{u}') \simeq -\log\left[\frac{1}{d!}\int_M \left(\frac{\omega(u)-\overline{\omega(u')}}{2\,\mathrm{i}}\right)^d\right], \tag{36}$$

$$2\log g_{\rm hol} \simeq \mathcal{K}_{\rm K\ddot{a}h}(u,\bar{u}) + \mathcal{K}_{\rm K\ddot{a}h}(u',\bar{u}') - \mathcal{K}_{\rm K\ddot{a}h}(u',\bar{u}) - \mathcal{K}_{\rm K\ddot{a}h}(u,\bar{u}') = D_{\rm K\ddot{a}h}(u,u'). \tag{37}$$

2.3 複素構造の変形

CFT1: $M = M(u,t)$ を target にする super σ-model,

CFT2: $M' = M(u,t')$ を target にする super σ-model,

Folding of Interface ⇒ special Lagrangian D-brane $\Delta_f(M) \subset \overline{M} \times M'$(複素共役!).

$s = p_2^* k - p_1^* k$: symplectic form, calibrated by $\varphi := p_1^* \bar{\Omega} \wedge p_2^* \Omega'$:

$\Omega(t)$ 自体が calibrated 正則 d-form on M, つまり $\frac{\mathrm{i}^{d^2}}{2^d} \Omega \wedge \bar{\Omega} = \frac{k^d}{d!}$ として

$$\frac{1}{2^{2d}} \varphi \wedge \bar{\varphi} = \frac{s^{2d}}{(2d)!} = \mathrm{d}\,\mathrm{vol}_{\overline{M} \times M'} . \tag{38}$$

$\Delta_f(M)$ が special Lagrangian になる条件

1. Lagrangian 条件: $s\big|_{\Delta_f(M)} = 0 \Leftrightarrow k = f^* k$, つまり $f : M \to M'$ は symplectomorphism.

2. Calibration 条件: $\mathrm{Im}(e^{\mathrm{i}\theta} \varphi) = \mathrm{Im}(e^{\mathrm{i}\theta} \bar{\Omega} \wedge f^* \Omega') = 0$ for $^\exists\theta$.

Interface entropy 因子は

$$g_{\mathrm{sLag}} = \frac{e^{\mathrm{i}\theta}\, \mathrm{i}^d \int_M \bar{\Omega} \wedge f^* \Omega'}{\sqrt{4^d\, \mathrm{vol}(M) \cdot \mathrm{vol}(M')}}, \tag{39}$$

今 Kähler potential

$$\exp(-\mathcal{K}_{\mathrm{comp}}(t, \bar{t})) = \frac{\mathrm{i}^{d^2}}{2^d} \int_M \Omega(t) \wedge \overline{\Omega(t)}, \tag{40}$$

を解析接続出来て

$$\mathcal{K}_{\mathrm{comp}}(t, \bar{t}') = -\log \left[\frac{\mathrm{i}^{d^2}}{2^d} \int_M \Omega(t) \wedge f^* \overline{\Omega(t')} \right], \tag{41}$$

$$2 \log(g_{\mathrm{sLag}}) = \mathcal{K}_{\mathrm{comp}}(t, \bar{t}) + \mathcal{K}_{\mathrm{comp}}(t', \bar{t}') - \mathcal{K}_{\mathrm{comp}}(t, \bar{t}') - \mathcal{K}_{\mathrm{comp}}(t', \bar{t}) = D_{\mathrm{comp}}(t, t'). \tag{42}$$

3 $N=2$ SCFT によるアプローチ

Intertwiner of $N=2$ SCA $\quad [T_L - T_R, \mathcal{I}] = 0$ に加えて,

$$[G_L^+ - \mathrm{i}\, G_R^-, \mathcal{I}_A] = [G_L^- - \mathrm{i}\, G_R^+, \mathcal{I}_A] = [J_L - J_R, \mathcal{I}_A] = 0, \tag{43}$$

$$[G_L^+ - \mathrm{i}\, G_R^+, \mathcal{I}_B] = [G_L^- - \mathrm{i}\, G_R^-, \mathcal{I}_B] = [J_L + J_R, \mathcal{I}_B] = 0, \tag{44}$$

$$e^{-\mathrm{i}\theta\phi}\mathcal{I}_A e^{\mathrm{i}\theta\phi} = e^{-\mathrm{i}\theta\phi_0}\mathcal{I}_A, \quad e^{-\mathrm{i}\theta\tilde{\phi}}\mathcal{I}_B e^{\mathrm{i}\theta\tilde{\phi}} = e^{-\mathrm{i}\theta\tilde{\phi}_0}\mathcal{I}_B, \tag{45}$$

$|B\rangle\rangle = g|0\rangle \otimes |0'\rangle + \cdots \Rightarrow \mathcal{I} = g|0\rangle \otimes \langle 0'| + \cdots$ (但し NS 真空が規格化されている場合)

$|0\rangle_s$: パラメータ $s \in \mathcal{M}$ に正則に依存する真空の族を扱うとき

$$g = g(s, s') = \frac{\langle 0|\mathcal{I}|0'\rangle}{\sqrt{\langle 0|0\rangle\langle 0'|0'\rangle}}. \tag{46}$$

ここで $\theta = 1/2$ のスペクトル・フローで RR セクターの状態に移行すると,

$\langle 0|\mathcal{I}|0'\rangle = \langle 0|e^{+\mathrm{i}\phi/2} \cdot e^{-\mathrm{i}\phi/2}\mathcal{I}e^{+\mathrm{i}\phi/2} \cdot e^{-\mathrm{i}\phi/2}|0'\rangle = e^{-\mathrm{i}\phi_0/2}\,{}_{\mathrm{RR}}\langle\bar{0}|\mathcal{I}|0'\rangle_{\mathrm{RR}}$, よって

$$g^2 = \frac{{}_{\mathrm{RR}}\langle\bar{0}|\mathcal{I}|0'\rangle_{\mathrm{RR}} \cdot {}_{\mathrm{RR}}\langle\bar{0}'|\mathcal{I}^\dagger|0\rangle_{\mathrm{RR}}}{{}_{\mathrm{RR}}\langle\bar{0}|0\rangle_{\mathrm{RR}} \cdot {}_{\mathrm{RR}}\langle\bar{0}'|0'\rangle_{\mathrm{RR}}}. \tag{47}$$

ここで, ${}_{\mathrm{RR}}\langle\bar{0}|0\rangle_{\mathrm{RR}} = e^{-K(s,\bar{s})}$, $\quad {}_{\mathrm{RR}}\langle\bar{0}'|0'\rangle_{\mathrm{RR}} = e^{-K(s',\bar{s}')}$,

同様に ${}_{\mathrm{RR}}\langle\bar{0}|\mathcal{I}|0'\rangle_{\mathrm{RR}} = e^{-K(s',\bar{s})}$, 従って

$$g^2 = \frac{e^{-K(s',\bar{s})} \cdot e^{-K(s,\bar{s}')}}{e^{-K(s,\bar{s})} \cdot e^{-K(s',\bar{s}')}} \Rightarrow 2\log g(s,s') = D_{\mathcal{M}}(s,s'). \tag{48}$$

4 Geometric Quantization

4.1 1 自由度系 $\mathbb{C}$

$\mathrm{d}s^2 = |\mathrm{d}z|^2,\ \ K = |z|^2, \quad k = \frac{\mathrm{i}}{2}\mathrm{d}z \wedge \mathrm{d}\bar{z}.$

Kähler 偏極: 波動関数 $=z$ の正則関数，その内積は

$$\langle\psi|\psi'\rangle = \int_{\mathbb{C}} \frac{\mathrm{i}}{2\pi}\mathrm{d}z\wedge\mathrm{d}\bar{z}\exp(-|z|^2)\overline{\psi(z)}\psi'(z), \tag{49}$$

正規直交完全系 $\psi_n(z) = \frac{z^n}{\sqrt{n!}}, \quad n \geq 0,$ つまり $a^\dagger \sim z,\ \ a \sim \partial/\partial z,\ \ |n\rangle \sim \psi_n.$

Coherent 状態 (non-normalized)

$$\varphi_\lambda(z) = \sum_{n=0}^{\infty}\frac{\bar{\lambda}^n}{\sqrt{n!}}\psi_n(z) = \exp(\bar{\lambda}z). \tag{50}$$

$$\langle\varphi_\mu|\varphi_\lambda\rangle = \int_{\mathbb{C}} \frac{\mathrm{i}}{2\pi}\mathrm{d}z\wedge\mathrm{d}\bar{z}\ \exp(-|z|^2)\exp(\bar{\lambda}z+\mu\bar{z}) = \exp(\bar{\lambda}\mu) = \exp(K(\mu,\bar{\lambda})). \tag{51}$$

$$\log\left(\frac{\langle\varphi_\mu|\varphi_\lambda\rangle\langle\varphi_\lambda|\varphi_\mu\rangle}{\langle\varphi_\lambda|\varphi_\lambda\rangle\langle\varphi_\mu|\varphi_\mu\rangle}\right) = -|\lambda-\mu|^2. \tag{52}$$

4.2 $\mathbb{PC}^1$

$K = \log(1+|z|^2),\ \ \mathrm{d}s^2 = \frac{\mathrm{d}z\mathrm{d}\bar{z}}{(1+|z|^2)^2},\ \ k = \frac{\mathrm{i}}{2}\frac{\mathrm{d}z\wedge\mathrm{d}\bar{z}}{(1+|z|^2)^2}.$

Kähler 偏極: 波動関数 $=z$ の正則関数，その内積は

$$\langle\psi|\psi'\rangle = \int_{\mathbb{C}} \frac{\mathrm{i}}{2\pi}\frac{\mathrm{d}z\wedge\mathrm{d}\bar{z}}{(1+|z|^2)^3}\overline{\psi(z)}\psi'(z), \tag{53}$$

正規直交完全系 $e_0(z) = \sqrt{2},\ \ e_1(z) = \sqrt{2}z.$

Coherent 状態 $\varphi_\lambda(z) = \overline{e_0(\lambda)}e_0(z) + \overline{e_1(\lambda)}e_1(z) \Rightarrow \langle\varphi_\lambda|\varphi_\mu\rangle = 2(1+\bar{\mu}\lambda) = 2\exp(K(\lambda,\bar{\mu})).$

$$\log\left(\frac{\langle\varphi_\mu|\varphi_\lambda\rangle\langle\varphi_\lambda|\varphi_\mu\rangle}{\langle\varphi_\lambda|\varphi_\lambda\rangle\langle\varphi_\mu|\varphi_\mu\rangle}\right) = -K(\lambda,\bar{\lambda}) - K(\mu,\bar{\mu}) + K(\lambda,\bar{\mu}) + K(\mu,\bar{\lambda}) = -D(\lambda,\mu). \tag{54}$$

4.3 $\mathbb{CP}^1$ higher level

$K_\ell = \ell \log(1+|z|^2)$, $\mathrm{d}s^2 = \dfrac{\ell \mathrm{d}z\mathrm{d}\bar{z}}{(1+|z|^2)^2}$, $k = \dfrac{\mathrm{i}\,\ell}{2}\dfrac{\mathrm{d}z\wedge\mathrm{d}\bar{z}}{(1+|z|^2)^2}$.

Kähler 偏極: 波動関数 $=z$ の正則関数, その内積は

$$\langle\psi|\psi'\rangle = \int_{\mathbb{C}} \frac{\mathrm{i}\,\ell}{2\pi}\frac{\mathrm{d}z\wedge\mathrm{d}\bar{z}}{(1+|z|^2)^{2+\ell}}\overline{\psi(z)}\psi'(z), \tag{55}$$

$\psi_k(z) = z^k$ として内積を考えると, $\langle\psi_k|\psi_{k'}\rangle = 0$ for $k\neq k'$,

$$\begin{aligned}\langle\psi_k|\psi_k\rangle &= \ell\int_{\mathbb{C}}\frac{\mathrm{d}x\mathrm{d}y}{(1+x^2+y^2)^{\ell+2}}(x^2+y^2)^{2k}\\ &= \ell\int_0^\infty\frac{t^k\mathrm{d}t}{(1+t)^{\ell+2}} = \ell B(k+1,\ell-k+1) = \ell\,\frac{k!(\ell-k)!}{(\ell+1)!},\end{aligned} \tag{56}$$

従って, $e_k(z) = \sqrt{\dfrac{(\ell+1)!}{\ell k!(\ell-k)!}}\,z^k$, $0\le k\le\ell$, が正規直交系.

Coherent 状態

$$\varphi_\lambda(z) = \sum_{k=0}^{\ell}\overline{e_k(\lambda)}e_k(z) \tag{57}$$

として,

$$\langle\varphi_\lambda|\varphi_\mu\rangle = \sum_{k=0}^{\ell}e_k(\bar{\mu})e_k(\lambda) = \frac{\ell+1}{\ell}\sum_{k=0}^{\ell}\frac{\ell!}{k!(\ell-k)!}(\bar{\mu}\lambda)^k = \frac{\ell+1}{\ell}(1+\bar{\mu}\lambda)^\ell. \tag{58}$$

$$\log\left(\frac{\langle\varphi_\mu|\varphi_\lambda\rangle\langle\varphi_\lambda|\varphi_\mu\rangle}{\langle\varphi_\lambda|\varphi_\lambda\rangle\langle\varphi_\mu|\varphi_\mu\rangle}\right) = -K_\ell(\lambda,\bar{\lambda}) - K_\ell(\mu,\bar{\mu}) + K_\ell(\lambda,\bar{\mu}) + K_\ell(\mu,\bar{\lambda}) = -\ell D(\lambda,\mu). \tag{59}$$

4.4 $\mathbb{CP}^n$

$$K = \log(1 + \sum_{i=1}^{n} |z_i|^2), \;\; \mathrm{d}s^2 = \frac{\sum_{i,j=1}^{n} \{(1+\sum_{k=1}^{n}|z_k|^2)\delta_{ij} - \bar{z}_i z_j\} \, \mathrm{d}z_i \mathrm{d}\bar{z}_j}{(1+\sum_{k=1}^{n}|z_k|^2)^2},$$

$$k = \frac{\mathrm{i}}{2} \frac{\sum_{i,j=1}^{n} \{(1+\sum_{k=1}^{n}|z_k|^2)\delta_{ij} - \bar{z}_i z_j\} \, \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j}{(1+\sum_{k=1}^{n}|z_k|^2)^2},$$

Kähler 偏極で, 状態は z_i の正則関数,

$$\langle \psi | \psi' \rangle = \frac{1}{\pi^n} \int_{\mathbb{C}^n} \frac{k^n}{n!} \exp(-K) \overline{\psi(z)} \psi'(z). \tag{60}$$

$n=2$ のとき, 正規直交系は $e_0(z) = \sqrt{6}$, $e_1(z) = \sqrt{6} z_1$, $e_2(z) = \sqrt{6} z_2$,

Coherent 状態 $\varphi_\lambda(z) = \overline{e_0(\lambda)} e_0(z) + \overline{e_1(\lambda)} e_1(z) + \overline{e_2(\lambda)} e_2(z)$,

$\langle \varphi_\lambda | \varphi_\mu \rangle = 6(1 + \bar{\mu}_1 \lambda_1 + \bar{\mu}_2 \lambda_2) = 6 \exp(K(\lambda, \bar{\mu}))$.

$$\log \left(\frac{\langle \varphi_\mu | \varphi_\lambda \rangle \langle \varphi_\lambda | \varphi_\mu \rangle}{\langle \varphi_\lambda | \varphi_\lambda \rangle \langle \varphi_\mu | \varphi_\mu \rangle} \right) = -K(\lambda, \bar{\lambda}) - K(\mu, \bar{\mu}) + K(\lambda, \bar{\mu}) + K(\mu, \bar{\lambda}) = -D(\lambda, \mu). \tag{61}$$

$n=3$ のとき, 正規直交系は $e_0(z) = \sqrt{24}$, $e_1(z) = \sqrt{24} z_1$, $e_2(z) = \sqrt{24} z_2$, $e_3(z) = \sqrt{24} z_3$,

Coherent 状態 $\varphi_\lambda(z) = \sum_{k=0}^{3} \overline{e_k(\lambda)} e_k(z)$,

$\langle \varphi_\lambda | \varphi_\mu \rangle = 24(1 + \bar{\mu}_1 \lambda_1 + \bar{\mu}_2 \lambda_2 + \bar{\mu}_3 \lambda_3) = 24 \exp(K(\lambda, \bar{\mu}))$.

一般の n, level ℓ の場合

$$\exp(-\ell K) \frac{(\ell k)^n}{n!} = \ell^n \frac{\mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}y_n}{(1 + x_1^2 + y_1^2 + x_2^2 + \cdots + y_n^2)^{n+\ell+1}}, \tag{62}$$

正規直交系は

$$e_{\boldsymbol{a}}(\boldsymbol{z}) = \sqrt{\frac{(\ell+n)!(\ell-|\boldsymbol{a}|)!}{\ell^n (\boldsymbol{a})!}} \boldsymbol{z}^{\boldsymbol{a}}, \tag{63}$$

ここで, $\boldsymbol{z}^{\boldsymbol{a}} := (z_1)^{a_1} (z_2)^{a_2} \cdots (z_n)^{a_n}$, $\quad 0 \le |\boldsymbol{a}| := a_1 + a_2 + \cdots + a_n \le \ell$ に限る.

Coherent state とその内積は

$$\varphi_{\boldsymbol{\lambda}}(\boldsymbol{z}) = \sum_{\boldsymbol{a}} \overline{e_{\boldsymbol{a}}(\boldsymbol{\lambda})} e_{\boldsymbol{a}}(\boldsymbol{z}) = \frac{(\ell+n)!}{\ell! \cdot \ell^n}(1 + \bar{\boldsymbol{\lambda}} \cdot \boldsymbol{z})^{\ell}, \tag{64}$$

$$\langle \varphi_{\boldsymbol{\lambda}} | \varphi_{\boldsymbol{\mu}} \rangle = \frac{(\ell+n)!}{\ell! \cdot \ell^n} \exp(\ell K(\boldsymbol{\lambda}, \bar{\boldsymbol{\mu}})). \tag{65}$$

測地線長と diastasis の関係

$$\cos^2(d(\boldsymbol{z}, \boldsymbol{w})) = \frac{(1 + \boldsymbol{z} \cdot \bar{\boldsymbol{w}})(1 + \boldsymbol{w} \cdot \bar{\boldsymbol{z}})}{(1 + |\boldsymbol{z}|^2)(1 + |\boldsymbol{w}|^2)} = \exp(-D(\boldsymbol{z}, \boldsymbol{w})). \tag{66}$$

4.5 Unit Disk $\mathbb{B}^1$

$\mathbb{B}^1 = \{z \in \mathbb{C} | |z| < 1\}$ は $f(\tau) = \dfrac{\tau - \mathrm{i}}{\tau + \mathrm{i}}$, $g(z) = \mathrm{i}\,\dfrac{1+z}{1-z}$ により, 上半平面 $\mathbb{H}$ と等価.

Kähler potentails $K_{\mathbb{H}}(\tau, \bar{\tau}) = -\log(\tau_2)$, $K_{\mathbb{B}^1}(z, \bar{z}) = -\log(1 - |z|^2)$,

$f^* K_{\mathbb{B}^1}(\tau, \bar{\tau}) = -\log(\tau_2) + \log(\tau + \mathrm{i}) + \log(\bar{\tau} - \mathrm{i}) + \log(4)$,

$f^* \left(\dfrac{\mathrm{i}\,\mathrm{d}\tau \wedge \mathrm{d}\bar{\tau}}{4\tau_2^2} \right) = \dfrac{\mathrm{i}\,\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1 - |z|^2)^2}$.

level $\ell \geq 2$ 量子化: $K_\ell(z, \bar{z}) = -\ell \log(1 - |z|^2)$, $k_\ell = \dfrac{\ell}{2} \dfrac{\mathrm{i}\,\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1 - |z|^2)^2}$,

内積 $\langle \psi | \psi' \rangle = \dfrac{\ell}{\pi} \displaystyle\int_{\mathbb{B}^1} \mathrm{d}x \mathrm{d}y (1 - x^2 - y^2)^{\ell - 2}\, \overline{\psi(z)} \psi'(z)$,

正規直交系 $e_k(z) = \sqrt{\dfrac{(\ell + k - 1)!}{\ell(\ell - 2)!k!}}\, z^k$, $k = 0, 1, 2, \ldots$,

Coherent state $\varphi_\lambda(z) = \displaystyle\sum_{k=0}^{\infty} \overline{e_k(\lambda)} e_k(z)$,

$$\langle \varphi_\lambda | \varphi_\mu \rangle = \frac{\ell - 1}{\ell} \sum_{k=0}^{\infty} \binom{k + \ell - 1}{\ell - 1} (\bar{\mu}\lambda)^k = \frac{\ell - 1}{\ell} (1 - \bar{\mu}\lambda)^{-\ell} = \frac{\ell - 1}{\ell} \exp(K_\ell(\lambda, \bar{\mu})). \tag{67}$$

測地線距離と diastasis (for $\ell = 1$)

$$d(z, w) = \operatorname{arctanh} \left| \frac{z - w}{1 - \bar{w}z} \right|, \tag{68}$$

$$D(z, w) = \log(1 - \bar{z}w) + \log(1 - \bar{w}z) - \log(1 - |z|^2) - \log(1 - |w|^2), \tag{69}$$

$$\cosh^2(d(z, w)) = \exp(D(z, w)). \tag{70}$$

4.6 Unit Ball $\mathbb{B}^n$

$\mathbb{B}^n = \{(z_1, ..., z_n) \in \mathbb{C}^n \,|\, 1 - |z_1|^2 - \cdots - |z_n|^2 > 0\}$,

Kähler potential $K(\boldsymbol{z}, \bar{\boldsymbol{z}}) = -\log(1 - |\boldsymbol{z}|^2)$,

level $\ell \geq n + 1$ の内積

$$
\begin{aligned}
|\psi(\boldsymbol{z})|^2 &= \frac{1}{\pi^n} \int_{\mathbb{B}^n} \frac{(\ell k)^n}{n!} \exp(-\ell K(\boldsymbol{z}, \bar{\boldsymbol{z}})) |\psi(\boldsymbol{z})|^2 \\
&= \frac{\ell^n}{\pi^n} \int_{D_n} \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n (1 - |\boldsymbol{z}|^2)^{\ell - n - 1} |\psi(\boldsymbol{z})|^2, \qquad (71)
\end{aligned}
$$

は極座標を入れて計算出来て

$$
|\boldsymbol{z}^{\boldsymbol{a}}|^2 = \ell^n (\ell - n - 1)! \frac{\boldsymbol{a}!}{(|\boldsymbol{a}| + \ell - 1)!}, \qquad (72)
$$

Coherent states は

$$
\varphi_{\boldsymbol{w}}(\boldsymbol{z}) := \sum_{a_1, \ldots, a_n = 0}^{\infty} \overline{e_{\boldsymbol{a}}(\boldsymbol{w})} e_{\boldsymbol{a}}(\boldsymbol{z}) = \frac{\Gamma(\ell)}{\Gamma(\ell - n) \ell^n} \exp(\ell K(\boldsymbol{z}, \bar{\boldsymbol{w}})). \qquad (73)
$$

測地線距離と diastasis

$$
\cosh^2(d(\boldsymbol{z}, \boldsymbol{w})) = \frac{(1 - \boldsymbol{z} \cdot \bar{\boldsymbol{w}})(1 - \boldsymbol{w} \cdot \bar{\boldsymbol{z}})}{(1 - |\boldsymbol{z}|^2)(1 - |\boldsymbol{w}|^2)} = \exp(D(\boldsymbol{z}, \boldsymbol{w})). \qquad (74)
$$

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