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Recursive approach to determine correlation functions in multibaryon systems

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We propose a recursive algorithm for the calculation of multibaryon correlation functions that combines the advantages of a recursive approach with those of the recently proposed unified contraction algorithm. The independent components of the correlators are built recursively by adding the baryons one after the other in a given order. The list of nonzero independent components is also constructed in a recursive manner, significantly reducing the resources required for this step. We computed the number of operations required to calculate the correlators up to ^{8}Be, and observed a significant speedup compared to other techniques. For the calculation of ^{4}He and ^{8}Be correlation functions in the fully relativistic case $\mathcal{O}(10^8)$ operations are required, whereas for nonrelativistic operators this number can be reduced to, e.g., $\mathcal{O}(10^4)$ in the case of ^{4}He.

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The purpose of this paper is to propose an efficient method for the calculation of baryonic correlation functions of the form

$$[C^{(N)}]^{\alpha'_1\alpha'_2\ldots\alpha'_N}_{\alpha_1\alpha_2\ldots\alpha_N}(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_N, t) = \left\langle \prod_{k=1}^{N} B_{\alpha_k}(\vec{x}_k, t) \prod_{l=1}^{N} \bar{B}^{\alpha'_l}(\vec{0}, 0) \right\rangle \tag{1.1}$$

that combines the advantages of the recursive approach with those of the algorithm introduced in [9]. The interpolating baryon operators are of the form

$$B_\alpha = \varepsilon_{abc}(\Gamma_1)_{\alpha\beta}(q_1)_{\beta;a}[(q_2)_{\gamma;b}(\Gamma_2)_{\gamma\delta}(q_3)_{\delta;c}], \tag{1.2a}$$

$$\bar{B}^\alpha = \varepsilon^{abc}(\Gamma_1)^{\alpha\beta}(\bar{q}_1)^{\beta;a}[(\bar{q}_2)^{\gamma;b}(\Gamma_2)^{\gamma\delta}(\bar{q}_3)^{\delta;c}], \tag{1.2b}$$

where the quark operators $q_n \in \{u, d, s\}$ and $\bar{q}_n \in \{\bar{u}, \bar{d}, \bar{s}\}$ are all taken at the same spacetime point. Here and throughout in the paper latin indices correspond to color indices correspond to quark operators at the sink. The choice of $\Gamma_1 = \mathbb{1}$ and $\Gamma_2 = C\gamma_5$ yields the interpolating operators for the proton with $(q_1, q_2, q_3) = (u, u, d)$ and for the neutron with $(q_1, q_2, q_3) = (d, u, d)$.

force [4–6] have been performed recently. However, the enormous amount of Wick contractions necessary for the evaluation of correlation functions of atomic nuclei is a severe problem in such calculations.

The number of Wick contractions for the naïve evaluation of correlation functions of multibaryon systems scales as $n_u!n_d!n_s!$, where n_u, n_d and n_s are the number of u, d and s quarks in the system, respectively. Furthermore, for each Wick contraction one has to evaluate the sum over all color and spin indices. These sums scale exponentially with the number of baryons in the system. As a consequence, the total number of required operations scales as $n_u!n_d!n_s!6^A4^A$, where A is the atomic mass number. The introduction of more complicated spatial

correlation function. The blocks are generally defined as

$$f_B^{q_1,q_2,q_3}(t,\delta;\alpha,\beta,\gamma;a,b,c) = \sum_{\vec{x}} s(\vec{x}) \langle B_\delta(\vec{x},t) \cdot \bar{q}_1^{\alpha;a} \bar{q}_2^{\beta;b} \bar{q}_3^{\gamma,c} \rangle. \tag{2.1}$$

Here δ is the spin index of the baryon B; α, β and γ are the spin indices of the three quarks $\bar{q}_1$, $\bar{q}_2$ and $\bar{q}_3$; and a, b, c are the corresponding color indices. The forms of all three quark source operators are taken to be the same. The function $s(\vec{x})$ characterizes the form of the baryon sink. A common choice is the projection to zero momentum $s(\vec{x}) \propto 1$, which is often needed, e.g., when the mass of a bound state is to be extracted from a correlation function. and three color degrees of freedom associated with a quark of a given flavor can be combined to form spinor-color indices $\xi^{(q)}$, which can take the values $1, 2, \ldots, 12$. Using these combined indices a block can be written as

$$f_B^{q_1,q_2,q_3}(t,\delta;\xi^{(q_1)},\xi^{(q_2)},\xi^{(q_3)}) = \sum_{\vec{x}} s(\vec{x}) \langle B_\delta(\vec{x},t) \cdot \bar{q}_1^{\xi^{(q_1)}} \bar{q}_2^{\xi^{(q_2)}} \bar{q}_3^{\xi^{(q_3)}} \rangle. \tag{2.2}$$

$$
\begin{aligned}
&[C^{(N)}]^{\alpha_1,\alpha_2,\ldots,\alpha_N}_{\delta_1,\delta_2,\ldots,\delta_N}(t) \\
&\quad = \sum_{\sigma\in\Sigma} f_{B_1}^{q_1,q_2,q_3}(t,\delta_1;\xi_1^{(q_1)},\xi_2^{(q_2)},\xi_3^{(q_3)})\ldots \\
&\qquad \times f_{B_N}^{q_1,q_2,q_3}(t,\delta_N;\xi_{3N-2}^{(q_1)},\xi_{3N-1}^{(q_2)},\xi_{3N}^{(q_3)}) \\
&\qquad \cdot G^{B_1}(\alpha_1;\xi_{\sigma(1)}^{(q_1)},\xi_{\sigma(2)}^{(q_2)},\xi_{\sigma(3)}^{(q_3)})\ldots \\
&\qquad \times G^{B_N}(\alpha_N;\xi_{\sigma(3N-2)}^{(q_1)},\xi_{\sigma(3N-1)}^{(q_2)},\xi_{\sigma(3N)}^{(q_3)})\mathrm{sgn}(\sigma), \qquad (2.3)
\end{aligned}
$$

$$
\begin{aligned}
&G^B(\alpha;\xi^{(q_1)},\xi^{(q_2)},\xi^{(q_3)}) \\
&\quad := (\Gamma_1)^{\alpha\beta(\xi^{(q_1)})}(\Gamma_2)^{\beta(\xi^{(q_2)})\beta(\xi^{(q_3)})}\varepsilon^{c(\xi^{(q_1)})c(\xi^{(q_2)})c(\xi^{(q_3)})}.
\end{aligned}
\qquad (2.4)
$$

Here $\beta(\xi)$ is the spin-index part of ξ and $c(\xi)$ is the color-index part and Σ is the set of all permutations that permute the indices associated with the different quark flavors q_k separately. The product of blocks $f_B^{q_1,q_2,q_3}$ does not depend on the permutations σ and hence the correlation function can be written in the form

$$[C^{(N)}]^{\alpha_1,\alpha_2,\ldots,\alpha_N}_{\delta_1,\delta_2,\ldots,\delta_N}(t) = f^{q_1,q_2,q_3}_{B_1}(t,\delta_1;\xi^{(q_1)}_1,\xi^{(q_2)}_2,\xi^{(q_3)}_3)\cdots f^{q_1,q_2,q_3}_{B_N}(t,\delta_N;\xi^{(q_1)}_{3N-2},\xi^{(q_2)}_{3N-1},\xi^{(q_3)}_{3N})$$
$$\cdot L^{(N)}(\alpha_1,\ldots,\alpha_N;\xi^{(q_1)}_1,\xi^{(q_2)}_2,\xi^{(q_3)}_3,\ldots,\xi^{(q_1)}_{3N-2},\xi^{(q_2)}_{3N-1},\xi^{(q_3)}_{3N}) \tag{2.5}$$

with the tensor

$$L^{(N)}(\alpha_1,\ldots,\alpha_N;\xi^{(q_1)}_1,\xi^{(q_2)}_2,\xi^{(q_3)}_3,\ldots,\xi^{(q_1)}_{3N-2},\xi^{(q_2)}_{3N-1},\xi^{(q_3)}_{3N})$$
$$= \sum_{\sigma\in\Sigma} G^{B_1}(\alpha_1;\xi^{(q_1)}_{\sigma(1)},\xi^{(q_2)}_{\sigma(2)},\xi^{(q_3)}_{\sigma(3)})\ldots G^{B_N}(\alpha_N;\xi^{(q_1)}_{\sigma(3N-2)},\xi^{(q_2)}_{\sigma(3N-1)},\xi^{(q_3)}_{\sigma(3N)})\mathrm{sgn}(\sigma). \tag{2.6}$$

The object L has a high degree of symmetry which reduces the number of its independent components. From the definition it is straightforward to see that L is antisymmetric under the exchange of two indices ξ as long as they belong to the same quark flavor. It also possesses a number of spin indices $\alpha_1,\ldots,\alpha_N$ corresponding to baryons of types $B_1, B_2, \ldots, B_N$. From the Pauli principle it follows that the correlator $[C^{(N)}]^{\alpha_1,\alpha_2,\ldots,\alpha_N}_{\delta_1,\delta_2,\ldots,\delta_N}(t)$ has to be antisymmetric under the exchange of any two indices α corresponding to the same type of baryon. Hence the same property has to hold for L.

A component of a tensor $X(\xi_1, \xi_2, \ldots, \xi_l)$, which is antisymmetric in the indices $\xi_1, \xi_2, \ldots, \xi_l$, each ranging from 1 to k, can be uniquely defined by a k-tuple $\boldsymbol{A}\{\xi\} = (n(1), n(2), \ldots, n(k))$, where $n(i)$ denotes how often the value i occurs amongst the l indices in the set $\{\xi\} = \{\xi_1, \xi_2, \ldots, \xi_l\}$. As a consequence of the antisymmetry

the permutation into account. For example, if X is a tensor with three antisymmetric indices, each ranging from one to four, the tuple $\boldsymbol{A}\{\xi\} = (1, 0, 1, 1)$ corresponds to the component $X(1, 3, 4)$. If a tensor is antisymmetric in

If X is an antisymmetric tensor with k indices and Y is an antisymmetric tensor with l indices, then their antisymmetric product $Z = X \bullet Y$ is a tensor with $k + l$ antisymmetric indices, whose components are defined as[2]

$$(X \bullet Y)(z) := Z(z) = \sum_{z=x+y} X(x)Y(y)\mathrm{sgn}(x|y), \tag{3.1}$$

where the tuples

$$z = A\{\xi_1, \ldots, \xi_{k+l}\}, \tag{3.2}$$

$$x = A\{\xi_1, \ldots, \xi_k\}, \tag{3.3}$$

$$y = A\{\xi_{k+1}, \ldots, \xi_{k+l}\}, \tag{3.4}$$

identify the antisymmetric components and

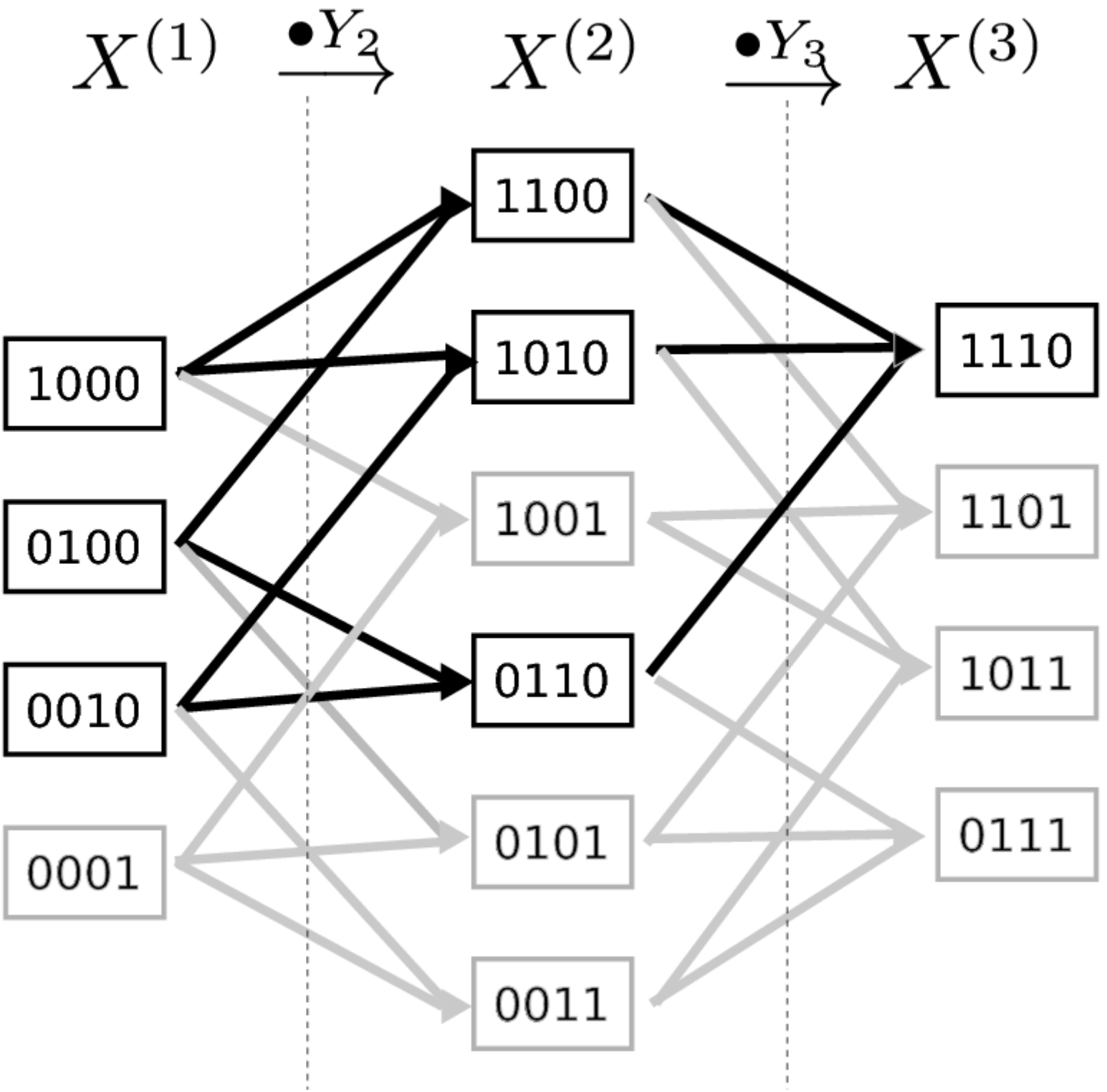
$$\mathrm{sgn}(x|y) = \prod_{\substack{i>j \\ y_j=1}} (-1)^{x_i} \tag{3.5}$$

In the following it will be often required to antisymmetrize only the subset of the quark spinor-color indices ξ that corresponds to a given quark flavor q. In this case it will be useful to write $A^{(q)}\{\xi_1, \xi_2, \ldots, \xi_n\}$ for the tuple of indices associated with the respective quark flavor. In the case of the spinor indices α and δ of the baryons a similar notation is adopted: here $A^{(B)}\{\alpha_1, \alpha_2 \ldots \alpha_n\}$ is the tuple associated with the antisymmetrization of only those spinor indices that correspond to the baryon type B.

In the special case where an antisymmetric tensor can be written as $X^{(n)} = Y_1 \bullet Y_2 \bullet \cdots \bullet Y_n$, a recursion relation $X^{(i)} = X^{(i-1)} \bullet Y_i$ can be set up with the starting condition $X^{(1)} = Y_1$. The usage of this recursion relation is often much more efficient than the direct evaluation of the product $Y_1 \bullet Y_2 \bullet \cdots \bullet Y_n$. Assuming that there are r groups of antisymmetric indices, each index in the pth group can take values from 1 to m_p, and in the pth group at the stage $X^{(i)}$ there are n_p indices; then the number of components at the intermediate step is

$$P^{(i)} = \prod_{p=1}^{r} C(n_p, m_p - n_p), \tag{3.7a}$$

where $C(n_1, n_2, \ldots, n_m) = (n_1 + n_2 + \cdots + n_m)!/n_1!n_2!\ldots n_m!$ are the multinomial coefficients. The num-


$X^{(1)}$
$\bullet Y_2$
$X^{(2)}$
$\bullet Y_3$
$X^{(3)}$
1000
0100
0010
0001
1100
1010
1001
0110
0101
0011
1110
1101
1011
0111

$$L^{(N)}(\alpha_1,\ldots,\alpha_N;\xi_1^{(q_1)},\xi_2^{(q_2)},\xi_3^{(q_3)},\ldots,\xi_{3N-2}^{(q_1)},\xi_{3N-1}^{(q_2)},\xi_{3N}^{(q_3)})$$
$$=\sum_{\sigma\in\Sigma}G^{B_1}(\alpha_1;\xi_{\sigma(1)}^{(q_1)},\xi_{\sigma(2)}^{(q_2)},\xi_{\sigma(3)}^{(q_3)})\ldots G^{B_N}(\alpha_N;\xi_{\sigma(3N-2)}^{(q_1)},\xi_{\sigma(3N-1)}^{(q_2)},\xi_{\sigma(3N)}^{(q_3)})\mathrm{sgn}(\sigma). \quad (2.6)$$

Using the notation from the previous section the tensor L can be written in the form[3]

$$L(A^{(B_a)}\{\alpha\},A^{(B_b)}\{\alpha\},\ldots,A^{(u)}\{\xi\},A^{(d)}\{\xi\},A^{(s)}\{\xi\}). \quad (4.1)$$

$$G^B(\alpha;\xi^{(q_1)},\xi^{(q_2)},\xi^{(q_3)})$$
$$:=(\Gamma_1)^{\alpha\beta(\xi^{(q_1)})}(\Gamma_2)^{\beta(\xi^{(q_2)})\beta(\xi^{(q_3)})}\varepsilon^{c(\xi^{(q_1)})c(\xi^{(q_2)})c(\xi^{(q_3)})}.$$

(2.4)

system. Using a similar argumentation the object G^B can be written as

$$G^B(\alpha,A^{(u)}\{\xi\},A^{(d)}\{\xi\},A^{(s)}\{\xi\}). \quad (4.2)$$

$$L^{(n+1)}=L^{(n)}\bullet G_{B_{n+1}} \quad (4.3)$$

on the permutations σ and hence the correlation function can be written in the form

with the starting condition

$$L^{(1)}=G_{B_1}. \quad (4.4)$$

$$[C^{(N)}]^{\alpha_1,\alpha_2,\ldots,\alpha_N}_{\delta_1,\delta_2,\ldots,\delta_N}(t) = f_{B_1}^{q_1,q_2,q_3}(t,\delta_1;\xi_1^{(q_1)},\xi_2^{(q_2)},\xi_3^{(q_3)})\cdots f_{B_N}^{q_1,q_2,q_3}(t,\delta_N;\xi_{3N-2}^{(q_1)},\xi_{3N-1}^{(q_2)},\xi_{3N}^{(q_3)})$$
$$\cdot L^{(N)}(\alpha_1,\ldots,\alpha_N;\xi_1^{(q_1)},\xi_2^{(q_2)},\xi_3^{(q_3)},\ldots,\xi_{3N-2}^{(q_1)},\xi_{3N-1}^{(q_2)},\xi_{3N}^{(q_3)}) \tag{2.5}$$

$$F^{(N)}(\delta_1,\ldots,\delta_N;t;\xi_1^{(q_1)},\xi_2^{(q_2)},\xi_3^{(q_3)},\ldots,\xi_{3N-2}^{(q_1)},\xi_{3N-1}^{(q_2)},\xi_{3N}^{(q_3)})$$
$$:= f_{B_1}^{q_1,q_2,q_3}(t,\delta_1;\xi_1^{(q_1)},\xi_2^{(q_2)},\xi_3^{(q_3)})\ldots f_{B_N}^{q_1,q_2,q_3}(t,\delta_N;\xi_{3N-2}^{(q_1)},\xi_{3N-1}^{(q_2)},\xi_{3N}^{(q_3)}). \tag{4.5}$$

$$F_{-}^{(n)}(A^{(B_a)}\{\delta\}, A^{(B_b)}\{\delta\},\ldots, A^{(u)}\{\xi\}, A^{(d)}\{\xi\}, A^{(s)}\{\xi\}). \tag{4.6}$$

Since F_- is composed of the independent factors $f_B^{q_1,q_2,q_3}$, a similar recursion relation

$$F_{-}^{(n+1)} = F_{-}^{(n)} \bullet f_{B_{n+1}}^{q_1,q_2,q_3} \tag{4.7}$$

with the starting condition $F_{-}^{(1)} = f_{B_1}^{q_1,q_2,q_3}$ can be defined.

$$C^{(N)}(t; A^{(B_a)}\{\delta\},\ldots, A^{(B_a)}\{\alpha\},\ldots)$$
$$= \frac{1}{\mathcal{N}} \sum_{\substack{A^{(q_i)}\{\xi\} \\ i\in\{a,b,c\}}} F_{-}^{(N)}(A^{(B_a)}\{\delta\},\ldots, A^{(u)}\{\xi\},\ldots)$$
$$\cdot L^{(N)}(A^{(B_a)}\{\alpha\},\ldots, A^{(u)}\{\xi\},\ldots) \tag{4.8}$$

$$P_\alpha = \varepsilon_{abc}(\Gamma_1)_{\alpha\beta} u_{\beta;a}[u_{\gamma;b}(\Gamma_2)_{\gamma\delta} d_{\delta;c}], \quad (7.1a)$$

$$N_\alpha = \varepsilon_{abc}(\Gamma_1)_{\alpha\beta} d_{\beta;a}[u_{\gamma;b}(\Gamma_2)_{\gamma\delta} d_{\delta;c}] \quad (7.1b)$$

are used at the sink and the operators

$$\bar{P}^\alpha = \varepsilon^{abc}(\Gamma_1)^{\alpha\beta} \bar{u}^{\beta;a}[\bar{u}^{\gamma;b}(\Gamma_2)^{\gamma\delta} \bar{d}^{\delta;c}], \quad (7.1c)$$

$$\bar{N}^\alpha = \varepsilon^{abc}(\Gamma_1)^{\alpha\beta} \bar{d}^{\beta;a}[\bar{u}^{\gamma;b}(\Gamma_2)^{\gamma\delta} \bar{d}^{\delta;c}] \quad (7.1d)$$

are used at the source. There are two common choices for the set of matrices (Γ_1, Γ_2). The first choice, $\Gamma_1 = \mathbb{1}$ and $\Gamma_2 = C\gamma_5$, where C is the charge conjugation matrix, gives fully relativistic nucleon operators. In case of the second choice, $\Gamma_1 = P_{\text{nr}}$ and $\Gamma_2 = C\gamma_5 P_{\text{nr}}$, the projection $P_{\text{nr}} = (1 + \gamma_4)/2$ to the "nonrelativistic" spinor components is inserted. In this case, if the Dirac representation of the

Introducing the variables n_P and n_N to denote the number of protons and neutrons in the system, the recursion relations of adding one proton or one neutron can be written as

$$L^{(n_P+1,n_N)} = L^{(n_P,n_N)} \bullet G_P, \tag{7.2a}$$

$$L^{(n_P,n_N+1)} = L^{(n_P,n_N)} \bullet G_N, \tag{7.2b}$$

$$F_{-}^{(n_P+1,n_N)} = F_{-}^{(n_P,n_N)} \bullet f_P^{u,u,d}, \tag{7.2c}$$

$$F_{-}^{(n_P,n_N+1)} = F_{-}^{(n_P,n_N)} \bullet f_N^{d,u,d}, \tag{7.2d}$$

with the starting conditions either

$$L^{(0,1)} = G_N \quad \text{and} \quad F_{-}^{(0,1)} = f_N^{d,u,d} \tag{7.3a}$$

or

$$L^{(1,0)} = G_P \quad \text{and} \quad F_{-}^{(1,0)} = f_P^{u,u,d}. \tag{7.3b}$$

The upper bound for the number of components of F_- at each stage is

$$\begin{aligned} P(n_P, n_N) = {} & C(n_P, D - n_P)C(n_N, D - n_N) \\ & \times C(2n_P + n_N, 3D - 2n_P - n_N) \\ & \times C(n_P + 2n_N, 3D - n_P - 2n_N), \end{aligned} \tag{7.4}$$

where D denotes the effective number of spinor components. For relativistic operators $D = 4$ and for nonrelativistic operators $D = 2$. The upper bounds for the number

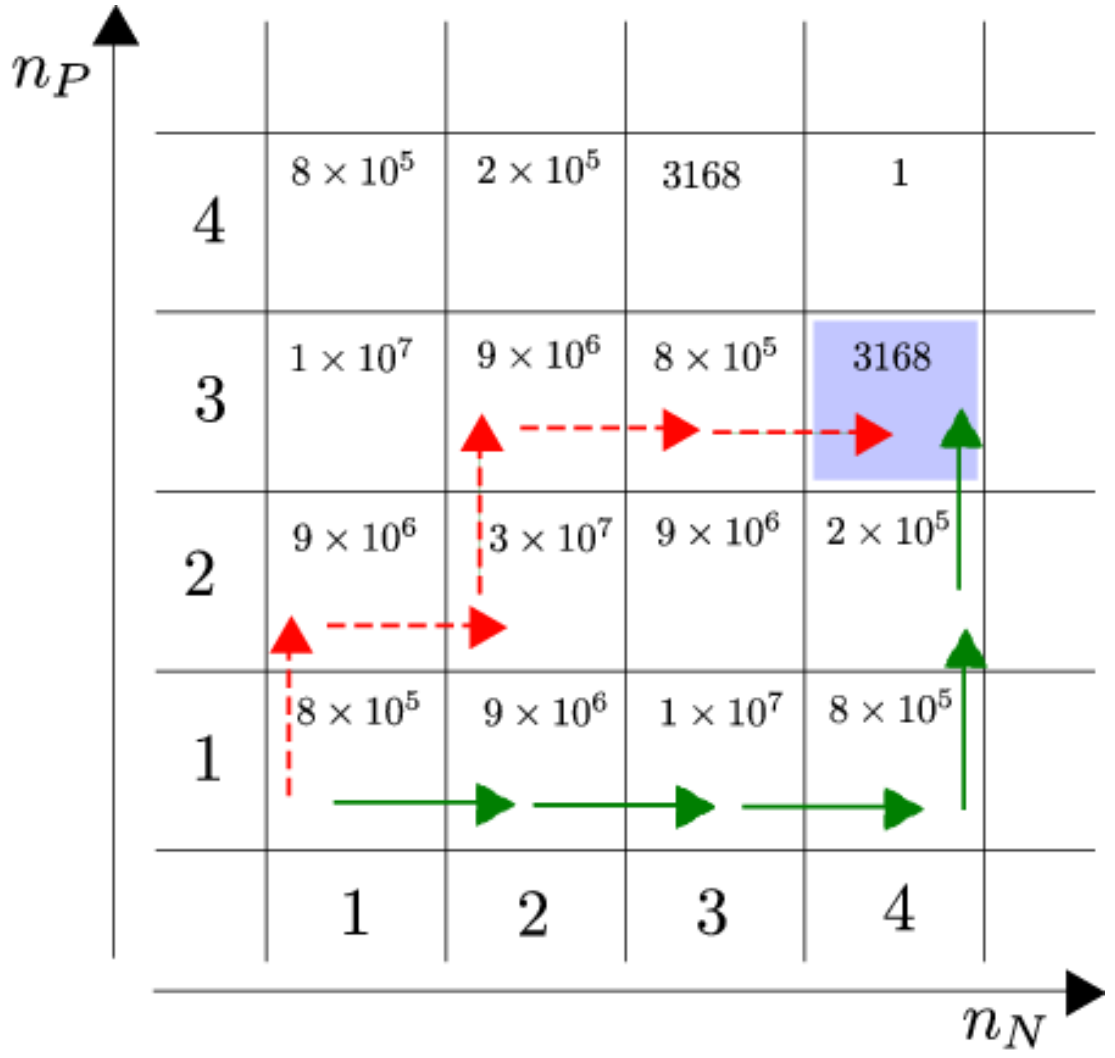

n_P
4
3
2
1
8×10^5
2×10^5
3168
1
1×10^7
9×10^6
8×10^5
3168
9×10^6
3×10^7
9×10^6
2×10^5
8×10^5
9×10^6
1×10^7
8×10^5
1
2
3
4
n_N

TABLE I. Number of operations (each operation is a complex multiplication and addition) required to compute all independent spinor components of the correlation function with N_P protons and N_N neutrons with *one quark source* and *relativistic operators*. Both the naïve number and the number using the recursive approach are given. η is the gain factor, that is, the ratio of the naïve and the recursive numbers of operations.

N_P	N_N	Number of operations	Naïve number of operations	η
0	2	199584	15925248	79.8
0	3	5825088	8.3×10^{11}	1.4×10^{5}
0	4	54768672	1.1×10^{17}	1.9×10^{9}
1	1	474048	11943936	25.2
1	2	19241280	5.5×10^{11}	2.9×10^{4}
1	3	109789200	6.7×10^{16}	6.1×10^{8}
1	4	179769600	1.7×10^{22}	9.2×10^{13}
2	2	531321120	5.7×10^{16}	1.1×10^{8}
2	3	756897264	1.3×10^{22}	1.7×10^{13}
2	4	291957888	5.3×10^{27}	1.8×10^{19}
3	3	2905079520	4.9×10^{27}	1.7×10^{18}
3	4	404946240	3.0×10^{33}	7.5×10^{24}
4	4	448496928	2.8×10^{39}	6.2×10^{30}

TABLE II. Number of operations (each operation is a complex multiplication and addition) required to compute all independent spinor components of the correlation function with N_P protons and N_N neutrons with *one quark source* and *nonrelativistic operators*. Both the naïve number and the number using the recursive approach are given. η is the gain factor, that is, the ratio of the naïve and the recursive numbers of operations.

N_P	N_N	Number of operations	Naïve number of operations	η
0	2	504	995328	1974.9
1	1	2664	746496	280.2
1	2	6048	8.6×10^9	1.4×10^6
2	2	10980	2.2×10^{14}	2.0×10^{10}

TABLE III. Number of operations (each operation is a complex multiplication and addition) required to compute all independent spinor components of the correlation function with N_P protons and N_N neutrons with *two quark sources* and *nonrelativistic operators*. Both the naïve number and the number using the recursive approach are given. η is the gain factor, that is, the ratio of the naïve and the recursive numbers of operations.

N_P	N_N	Number of operations	Naïve number of operations	η
0	2	3024	995328	329.1
0	3	1052136	1.3×10^{10}	1.2×10^{4}
0	4	18881568	4.2×10^{14}	2.2×10^{7}
1	1	10656	746496	70.1
1	2	42768	8.6×10^{9}	2.0×10^{5}
1	3	6329016	2.6×10^{14}	4.1×10^{7}
1	4	67720680	1.6×10^{19}	2.4×10^{11}
2	2	103680	2.2×10^{14}	2.1×10^{9}
2	3	10038672	1.3×10^{19}	1.3×10^{12}
2	4	81850128	1.3×10^{24}	1.6×10^{16}
3	3	338263368	1.3×10^{24}	3.5×10^{15}
3	4	287427384	1.9×10^{29}	6.5×10^{20}
4	4	448496928	4.2×10^{34}	9.5×10^{25}

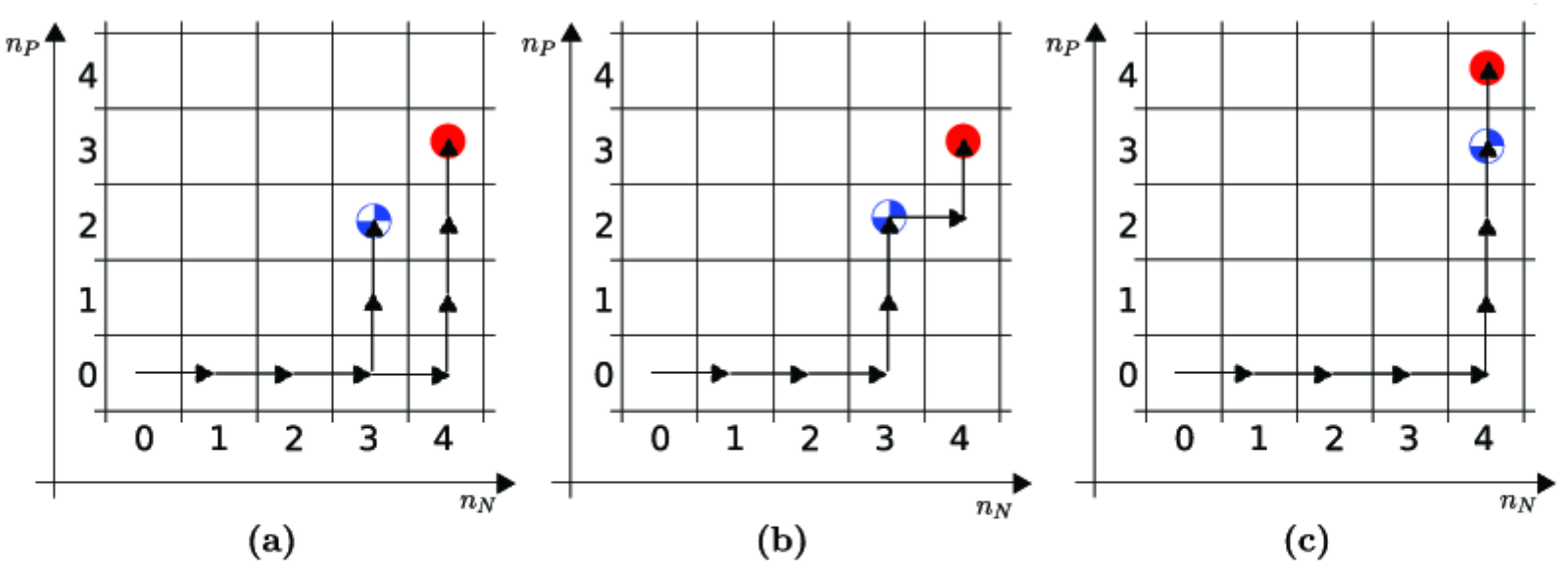


FIG. 3 (color online). Three representative cases for the combined calculation of two atomic nuclei. The red (filled) and blue (half filled) dots represent the nuclei that are to be calculated and the arrows indicate the order in which protons and neutrons are added. In the case (a) a speedup of about 10% can be reached. In case (b) more operations are required than for the separate calculation of the indicated nuclei. In case (c) the blue (half filled) nucleus can be calculated without additional effort. Therefore in this case a speedup of about 47% can be observed.

TABLE IV. The efficiency of the presented algorithm for the calculation of individual spin components with *nonrelativistic operators*. N_Λ is the number of operations (complex multiplications and additions) required for the construction of F_-. N_L is the number of independent components of the tensor L. N_{list} and N_{contr} are the number of entries in the unified contraction list and the number of independent contractions from Ref. [9], respectively. (Numbers that are not presented in [9] are from our own calculations.) $AN_{\text{contr}}/N_\Lambda$ is approximately the ratio of the number of operations required for the unified contraction algorithm and for the algorithm presented in this paper if additions are taken as much faster than multiplications.

N_P	N_N	Spin state	N_Λ	N_L	N_{list}	N_{contr}	$AN_{\text{contr}}/N_\Lambda$
0	2	(0,1)	504	21	1008	252	1
1	1	(0,0)	189	21	756	189	2
1	1	(1,0)	252	28	1008	252	2
1	1	(0,1)	252	28	1008	252	2
1	1	(1,1)	189	21	756	189	2
1	2	(0,0,1)	4662	9	25920	3240	2.1
1	2	(1,0,1)	4662	9	25920	3240	2.1
2	2	(0,1,0,1)	10980	1	518400	32400	11.8

TABLE V. The efficiency of the presented algorithm for the calculation of individual spin components with *relativistic operators*. N_Λ is the number of operations (complex multiplications and additions) required for the construction of F_-. N_L is the number of independent components of the tensor L. N_{list} and N_{contr} are the number of entries in the unified contraction list and the number of independent contractions from Ref. [9], respectively. (Numbers that are not presented in [9] are from our own calculations.) $AN_{\text{contr}}/N_\Lambda$ is approximately the ratio of the number of operations required for the unified contraction algorithm and for the algorithm presented in this paper if additions are taken as much faster than multiplications.

N_P	N_N	Spin state	N_Λ	N_L	N_{list}	N_{contr}	$AN_{\text{contr}}/N_\Lambda$
0	2	(0,1)	5544	231	11088	2772	1
0	3	(0,1,2)	1360098	1110	4795200	599400	1.3
0	3	(0,1,3)	1360098	1110	4795200	599400	1.3
0	4	(0,1,2,3)	54768672	1845	1785369600	111585600	8.1
1	1	(0,0)	2079	231	8316	2079	2
1	1	(1,0)	2358	262	9432	2358	2
1	1	(0,1)	2358	262	9432	2358	2
1	1	(1,1)	2079	231	8316	2079	2
1	2	(0,0,1)	381978	1311	3775680	471960	3.7
1	2	(1,0,1)	381978	1311	3775680	471960	3.7
1	3	(0,0,1,2)	11717937	2232	1349913600	84369600	28.8
1	3	(1,0,1,2)	11717937	2232	1349913600	84369600	28.8
1	3	(0,0,1,3)	11717937	2232	1349913600	84369600	28.8
1	3	(1,0,1,3)	11717937	2232	1349913600	84369600	28.8
1	4	(0,0,1,2,3)	141103602	1110	290013696000	9062928000	321.1
1	4	(1,0,1,2,3)	141103602	1110	290013696000	9062928000	321.1
2	2	(0,1,0,1)	8541864	2716	1407974400	87998400	41.2
2	3	(0,1,0,1,2)	44343561	1311	266411980800	8325374400	938.7
2	3	(0,1,0,1,3)	44343561	1311	266411980800	8325374400	938.7
2	4	(0,1,0,1,2,3)	214572144	231	33798352896000	528099264000	14767
3	3	(0,1,2,0,1,2)	163007703	231	30418517606400	475289337600	17494.5
3	3	(0,1,3,0,1,2)	181280493	262	34500656332800	539072755200	17842.2
3	3	(0,1,2,0,1,3)	181280493	262	34500656332800	539072755200	17842.2
3	3	(0,1,3,0,1,3)	163007703	231	30418517606400	475289337600	17494.5
3	4	(0,1,2,0,1,2,3)	293717796	21	3041851760640000	23764466880000	566364
3	4	(0,1,3,0,1,2,3)	293717796	21	3041851760640000	23764466880000	566364
4	4	(0,1,2,3,0,1,2,3)	448496928	1	229442532802560000	896259893760000	1.6×10^7

TABLE VI. The efficiency of the presented algorithm for the calculation of individual spin components with *nonrelativistic operators using two quark sources*. N_Λ is the number of operations (complex multiplications and additions) required for the construction of T_-. N_L is the number of independent components of the tensor L. N_{list} and N_{contr} are the number of entries in the unified contraction list and the number of independent contractions from Ref. [9], respectively. (Numbers that are not presented in [9] are from our own calculations.) $AN_{\text{contr}}/N_\Lambda$ is approximately the ratio of the number of operations required for the unified contraction algorithm and for the algorithm presented in this paper if additions are taken as much faster than multiplications. The combined spinor-source indices are of the form $2\alpha + s$ where α is the spinor part and s is the source part.

N_P	N_N	spinor-source indices	N_Λ	N_L	N_{list}	N_{contr}	$AN_{\text{contr}}/N_\Lambda$
0	2	(0,2)	504	21	1008	252	1
0	3	(0,1,2)	330291	189	816480	102060	0.9
0	3	(0,2,3)	330291	189	816480	102060	0.9
0	4	(0,1,2,3)	18881568	441	426746880	26671680	5.7
1	1	(0,0)	189	21	756	189	2
1	1	(2,0)	252	28	1008	252	2
1	1	(0,2)	252	28	1008	252	2
1	1	(2,2)	189	21	756	189	2
1	2	(0,0,2)	4662	9	25920	3240	2.1
1	2	(2,0,2)	4662	9	25920	3240	2.1
1	3	(0,0,1,2)	1100034	81	48988800	3061800	11.1
1	3	(2,0,1,2)	1100034	81	48988800	3061800	11.1
1	3	(0,0,2,3)	1100034	81	48988800	3061800	11.1
1	3	(2,0,2,3)	1100034	81	48988800	3061800	11.1
1	4	(0,0,1,2,3)	59747247	189	49380710400	1543147200	129.1
1	4	(2,0,1,2,3)	59747247	189	49380710400	1543147200	129.1
2	2	(0,2,0,2)	10980	1	518400	32400	11.8
2	3	(0,2,0,1,2)	1717569	9	1828915200	57153600	166.4
2	3	(0,2,0,2,3)	1717569	9	1828915200	57153600	166.4
2	4	(0,2,0,1,2,3)	80357088	21	3072577536000	48009024000	3584.7
3	3	(0,1,2,0,1,2)	31373721	21	2765319782400	43208121600	8263.3
3	3	(0,2,3,0,1,2)	40214061	28	3687093043200	57610828800	8595.6
3	3	(0,1,2,0,2,3)	40214061	28	3687093043200	57610828800	8595.6
3	3	(0,2,3,0,2,3)	31373721	21	2765319782400	43208121600	8263.3
3	4	(0,1,2,0,1,2,3)	225681807	9	1303650754560000	10184771520000	315902
3	4	(0,2,3,0,1,2,3)	225681807	9	1303650754560000	10184771520000	315902
4	4	(0,1,2,3,0,1,2,3)	448496928	1	229442532802560000	896259893760000	1.6×10^7