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Subset method for one-dimensional QCD

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1 Introduction

- Simulation of QCD at $\mu \neq 0 \Rightarrow$ Sign Problem
  i.e. fluctuating sign of $\det(D)$
- Subset method
  - rearrange the path-integral over gauge configurations to that over subsets.
  - weight of subset (sum of fermion determinants) are real and positive.
- Using a toy model $0+1$d QCD
  - subset construction and proof of "no sign problem"
  - numerical simulation and comparison with analytic results
  - focus on $N_f = 1$ case (paper discusses larger N_f, too)

2 $0 + 1$ d QCD

- SU(3) gauge + quark with mass m on $1^3 \times N_t$ lattic

$$aD = \begin{pmatrix} am & e^{a\mu}U_1/2 & 0 & \cdots & 0 \\ -e^{-a\mu}U_1^\dagger/2 & am & e^{a\mu}U_2/2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & am \\ -e^{a\mu}U_{N_t}/2 & 0 & 0 & \cdots & -e^{-a\mu}U_{N_t-1}^\dagger/ \end{pmatrix}$$

- After gauge transformation, Polyakov line P is the only dynamical varibale

$$\langle O \rangle = \frac{1}{Z} \int dP \det[aD(P)]\, O(P)$$

- reduction and decomposition of $\det(aD)$

$$\det(aD) = \frac{1}{2^{3N_t}} \det(A\, I_3 + e^{\mu/T} P + e^{-\mu/T} P^\dagger),$$

where $A = 2\cosh(\mu_c/T)$ and $a\mu_c = \mathrm{arsinh}(am)$

$$\det D(P) = \sum_{q=-3}^{3} D_q(P)\, e^{q\mu/T}$$

($D_q(P)$ are canonical determinants, explicit form is known, but not necessary except $D_{-3} = D_3 = 1$ and D_0 (see below))

- Sign problem in terms of $\det D(P)$
  - $\det D(P)$ is in general complex
  - $\det D(P) + \det D(P^*)$ is real, but without a definite sign
  - for $\mu = 0$, $\det D(P)$ is real and positive

3 Subset Method

- Aim

  to gather configurations of the ensemble into small subsets such that the sum of their weights contributing to the partition function is real and nonnegative

- subset, weight, Z, $\langle O \rangle$

$$\Omega_P = \{P, e^{2\pi i/3}P, e^{4\pi i/3}P\} \equiv \{P_0, P_1, P_2\}$$

$$\sigma(\Omega_P) = \frac{1}{3}\sum_{k=0}^{2} \det D(P_k)$$

$$Z = \int dP\, \sigma(\Omega_P)$$

(Haar measure is Z_3 invariant)

$$\langle O \rangle = \frac{1}{Z} \int dP\, \sigma(\Omega_P)\, \langle O \rangle_{\Omega_P}$$

$$\langle O \rangle_{\Omega_P} = \frac{1}{3\sigma(\Omega_P)} \sum_{k=0}^{2} \det D(P_k)\, O(P_k)$$

(P_k can have different values of the observable)

Note that the measure $dP\sigma(\Omega_P)$ above indicates that subsets of configurations, rather than individual configurations, will be generated in the numerical simulations,

- Proof of “sign problem free”

$$\sigma(\Omega_P) = \frac{1}{3}\sum_{k=0}^{2}\sum_{q=-3}^{3} D_q(P)\, e^{q(\mu/T+2\pi i k/3)}$$

$$= \sum_{b=-1}^{1} D_{3b}(P)\, e^{3b\mu/T}$$

$$= \sigma(\Omega_P) = D_0(P) + 2\cosh(3\mu/T)$$

$$D_0(P) = A^3 + A(|\mathrm{tr}P|^2 - 3)$$

with $A = 2\cosh(\mu_c/T) \geq 2$ and $|\mathrm{tr}P| \in [0,3]$.
$\Rightarrow$ $\sigma(\Omega_P)$ is real and positive

4 Numerical tests

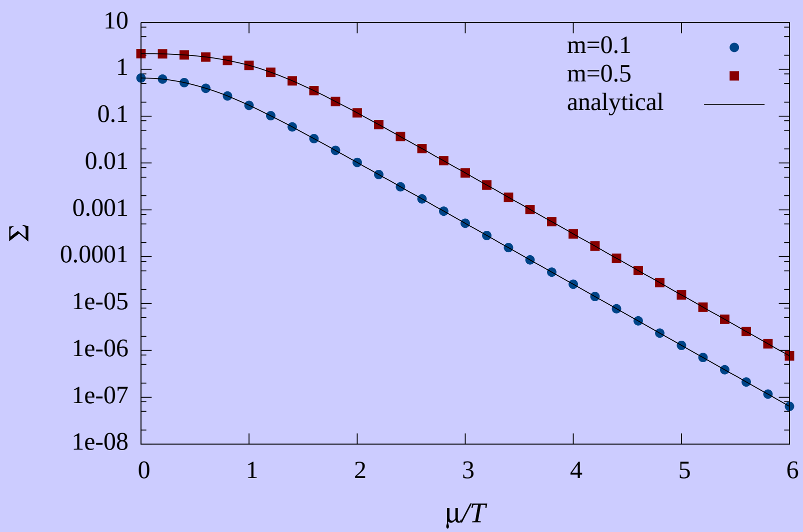
(In the paper, chiral condensate Σ, quark number density n and Polyakov loop $\langle \mathrm{tr} P \rangle$ are calculated analytically)

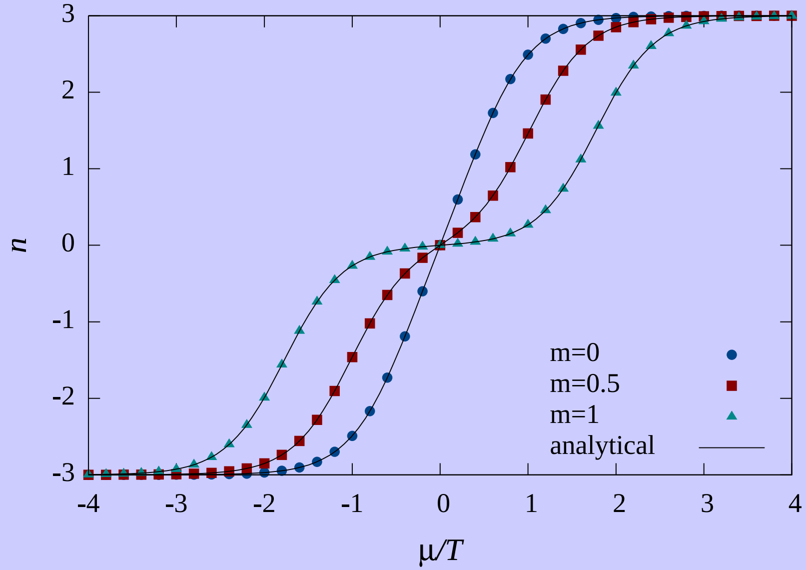
- $N_t = 2$ (most quantities are functions of μ/T and μ_c/T)
- generate Markov chains of subsets using Metropolis
  - an SU(3) link P' is generated accoring to the Haar measure
  - accept the new link with prob.

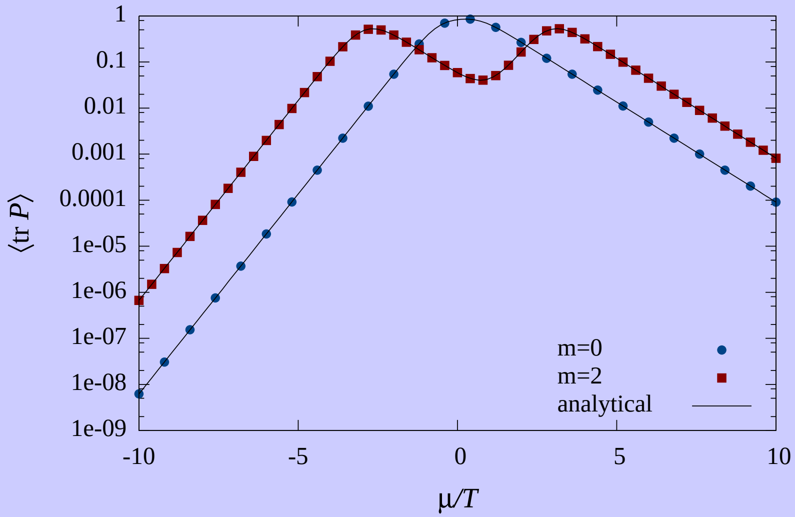
$$p = \min\left\{1, \frac{\sigma(\Omega_{p'})}{\sigma(\Omega_p)}\right\}$$

  - O(100,100) subsets
  -

$$O_\Sigma = \frac{1}{N_t}\mathrm{tr}\left[D^{-1}\right], \qquad O_n = \frac{1}{N_t}\mathrm{tr}\left[D^{-1}\frac{\partial D}{\partial \mu}\right]$$







Agree with analytical results

5 Summary for larger N_f

- expansion of the determinant

$$\det{}^{N_f} D(P) = \sum_{q=-3N_f}^{3N_f} D_q^{(N_f)}(P)\, e^{q\mu/T}$$

- subsets

$$\Omega_P = \{P, e^{2\pi i/3}P, e^{4\pi i/3}P, P^*, e^{2\pi i/3}P^*, e^{4\pi i/3}P^*\}$$

- sign problem
  - $\sigma(\Omega_P)$ is real and nonnegative for $N_f = 2 \sim 5$
  - for $N_f = 6$, $\sigma(\Omega_P)$ is negative for some P

6 Comment

Idea is interesting, but it seems difficult to apply it to more complicated systems