The mild sign problem in QFT on Lefschetz thimbles

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1. Airy integral on Lefschetz thimbles

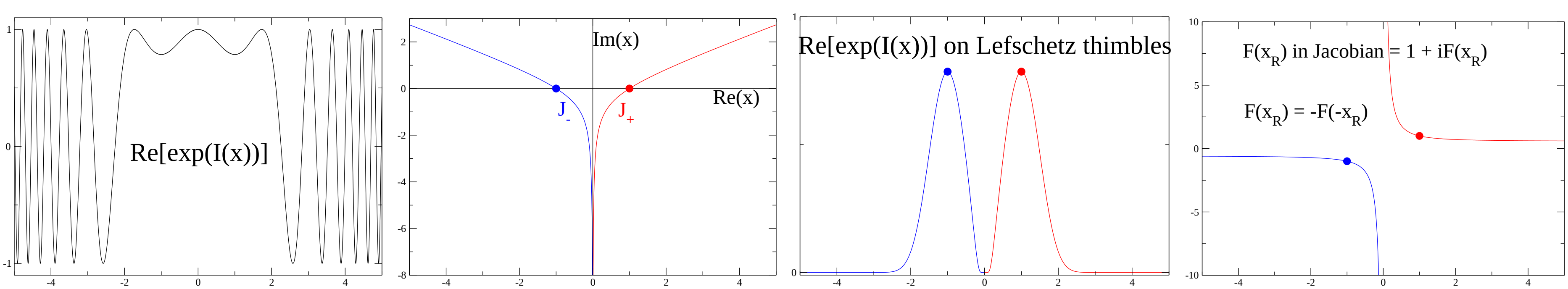
- Consider the Airy integral as a toy model with a sign problem,

$$Z = \int_{-\infty}^{\infty} \mathrm{d}x \exp(-S(x)), \quad S(x) = -i(x^3/3 - x).$$

- Express the integral in terms of Lefschetz thimbles,

1) real $x \longrightarrow$ complex $x = x_R + ix_I$.
2) Define Morse function $h = \mathrm{Re}S$, critical points $x = \pm 1$: $h' = 0, h'' \neq 0$.
3) Define downward flow associated to the critical points $\mathcal{J}_\pm$,
$\frac{\mathrm{d}x_\mathrm{R}}{\mathrm{d}t} = -\frac{\partial h}{\partial x_\mathrm{R}}, \frac{\mathrm{d}x_\mathrm{I}}{\mathrm{d}t} = -\frac{\partial h}{\partial x_\mathrm{I}}, x(t=\infty) = \pm 1. \quad \longrightarrow \frac{\mathrm{d}h}{\mathrm{d}t} = -(\frac{\partial h}{\partial x_\mathrm{R}})^2 - (\frac{\partial h}{\partial x_\mathrm{I}})^2 \leq 0, \ \frac{\mathrm{dIm}S}{\mathrm{d}t} = 0.$
4) Picking an orientation of $\mathcal{J}_\pm$, define Lefschetz thimbles.
5) Define the Airy integral on Lefschetz thimbles,

$$Z = \int_{\mathcal{C}=\mathcal{J}_- + \mathcal{J}_+} \mathrm{d}x \exp(-S(x)), \quad \mathrm{Im}S = \pm \frac{2}{3} \text{ on } \mathcal{J}_\pm.$$



2. QFT on Lefschetz thimbles

- Relativistic Bose gas at finite chemical potential μ in $d=4$:

$$\langle\mathcal{O}\rangle=\frac{1}{Z}\int_{\mathbb{C}^V}\prod_x \mathrm{d}\phi_x\ e^{-S[\phi]}\mathcal{O}[\phi],\quad Z=\int_{\mathbb{C}^V}\prod_x \mathrm{d}\phi_x\ e^{-S[\phi]},\quad \phi_x=\frac{1}{\sqrt{2}}(\phi_{1,x}+i\phi_{2,x}).$$

$$S[\{\phi_x\}] = \sum_x\left[\left(2d+m^2\right)\phi_x^*\phi_x+\lambda(\phi_x^*\phi_x)^2-\sum_{\nu=0}^{d-1}\phi_x^*e^{-\mu\delta_{\nu,0}}\phi_{x+\hat{\nu}}+\phi_{x+\hat{\nu}}^*e^{\mu\delta_{\nu,0}}\phi_x\right],$$

$$\begin{aligned}S[\{\phi_{a,x}\}] &= \sum_x\Bigg[\left(d+\frac{m^2}{2}\right)\sum_a\phi_{a,x}^2+\frac{\lambda}{4}(\sum_a\phi_{a,x}^2)^2\\ &\quad -\sum_a\sum_{\nu=1}^{d-1}\phi_{a,x}\phi_{a,x+\hat{\nu}}+\sum_{a,b} i\sinh\mu\,\varepsilon_{ab}\phi_{a,x}\phi_{b,x+\hat{0}}-\cosh\mu\,\delta_{a,b}\phi_{a,x}\phi_{b,x+\hat{0}}\Bigg],\end{aligned}$$

$$(\varepsilon_{12}=-\varepsilon_{21}=1,\varepsilon_{11}=\varepsilon_{22}=0).$$

- Silver Blaze phenomenon : independence of the physics ($\langle n\rangle$, $\langle|\phi|^2\rangle$) on the chemical potential up to some critical value.

2. QFT on Lefschetz thimbles (cont'd)

- To formulate the model on a Lefschetz thimble,

  1) complexify both the real and imaginary part of the original complex fields $\phi_x = \frac{1}{\sqrt{2}}(\phi_{1,x} + i\phi_{2,x})$,

$$\phi_{a,x} \rightarrow \phi_{a,x}^{(R)} + i\phi_{a,x}^{(I)}, \quad (a = 1, 2),$$

  2) take Morse function, $h = \mathrm{Re}S$ $\rightarrow$ critical point $=$ global minimum ϕ_0,

  3) define downward flow associated to the critical point $\mathcal{J}_0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_{a,x}^{(R)}(t) = -\frac{\delta h[\phi(t)]}{\delta \phi_{a,x}^{(R)}}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\phi_{a,x}^{(I)}(t) = -\frac{\delta h[\phi(t)]}{\delta \phi_{a,x}^{(I)}}, \quad \phi(t = +\infty) = \phi_0,$$

  4) picking an orientation of $\mathcal{J}_0$, define the Lefschetz thimble as an integration domain,

  5) rewrite QFT on the Lefschetz thimble,

$$Z_0 = \int_{\mathcal{J}_0} \prod_{a,x} \mathrm{d}\phi_{a,x}\, e^{-S[\phi]} = e^{-i\mathrm{Im}S} \int_{\mathcal{J}_0} \prod_{a,x} \mathrm{d}\phi_{a,x}\, e^{-\mathrm{Re}S[\phi]}.$$

- Although the oscillation part $e^{-i\mathrm{Im}S}$ is factorized, a residual phase factor remains in the path integral measure due to the curved space $\mathcal{J}_0$. $\rightarrow$ mild sign problem.
- Justification of this approach : the same symmetries, the same degree of freedom and also the same perturbative expansion as the standard formulation.

3. Aurora Monte Carlo algorithm

- In order to compute the integral through a Langevin algorithm, constrained in $\mathcal{J}_0$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\phi^{(R)}_{a,x}(\tau, t=0) &= -\frac{\delta h[\phi(\tau, t=0)]}{\delta \phi^{(R)}_{a,x}} + \eta^{(R)}_{a,x}(\tau, t=0), \\
\frac{\mathrm{d}}{\mathrm{d}\tau}\phi^{(I)}_{a,x}(\tau, t=0) &= -\frac{\delta h[\phi(\tau, t=0)]}{\delta \phi^{(I)}_{a,x}} + \eta^{(I)}_{a,x}(\tau, t=0), \\
\rightarrow \phi_j(\tau+\Delta\tau, t=0) &= \phi_j(\tau, t=0) - \Delta\tau\,\frac{\delta h[\phi]}{\delta\phi_j} + \sqrt{2\Delta\tau}\,\eta_j(\tau, t=0),
\end{aligned}$$

where $j = (R/I, a, x)$, Monte Carlo time τ, downward flow time t=[0,T]. The Gaussian noise $\eta_j(\tau, t=0)$ should be projected on the tangent space at $\phi_j(\tau, t=0)$ of $\mathcal{J}_0$.

i) Define the projection operator to the tangent space $\mathcal{G}_0$ at global minimum ϕ_0,

$$P = \frac{H}{\sqrt{H^2}} - 1, \qquad \text{and} \qquad H = \partial^2 S_R[\phi_0].$$

ii) make a path of the downward flow from $\phi_j(\tau, t=0)$ to $\phi_j(\tau, t=T) \sim \phi_0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_j(\tau, t) = -\frac{\delta h[\phi(\tau, t)]}{\delta \phi_j},$$

iii) Generate a Gaussian noise $\eta_j(\tau, t=T)$ on the tangent space $\mathcal{G}_0$ at global minimum ϕ_0.

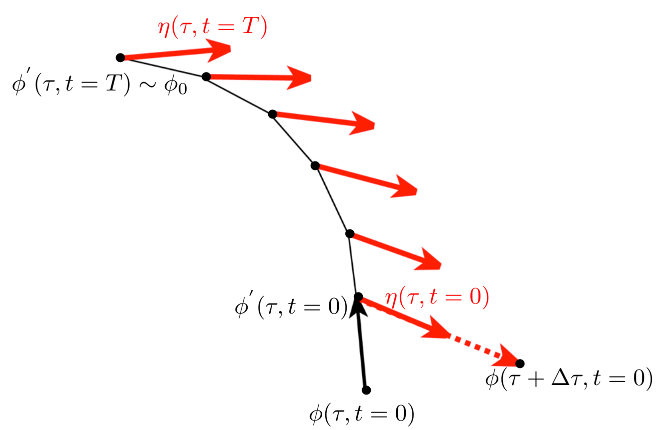
iv) Paralleltransporting the noise from $t = T$ along the path of upward flow to $t = 0$,

$$\frac{d}{dt}\eta_j(\tau, t) = \sum_k \eta_k(\tau, t)\partial_k\partial_j h(\tau, t),$$

get the noise $\eta_j(\tau, t = 0)$ on the tangent space to $\mathcal{J}_0$ at $\phi_j(\tau, t = 0)$.

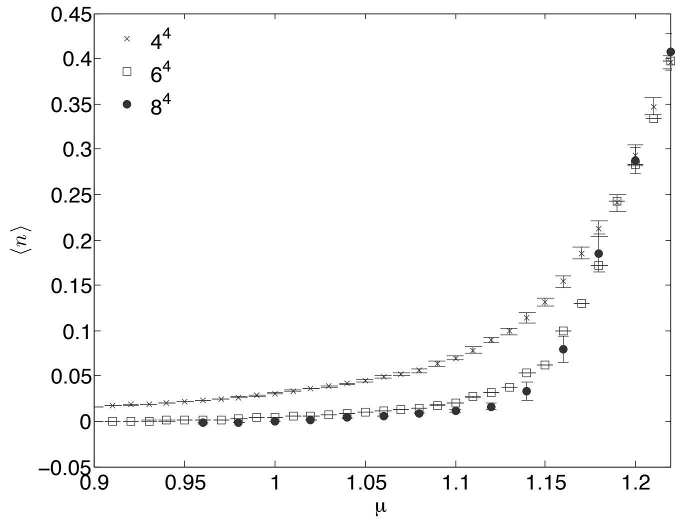
v) In the limit $\Delta\tau \to 0$ this simulates Langevin dynamics on the thimble. For $\Delta\tau > 0$, $\phi_i(\tau, t = 0)$ will move away from the thimble of order $(\Delta\tau)^2$. Correct this as follows,

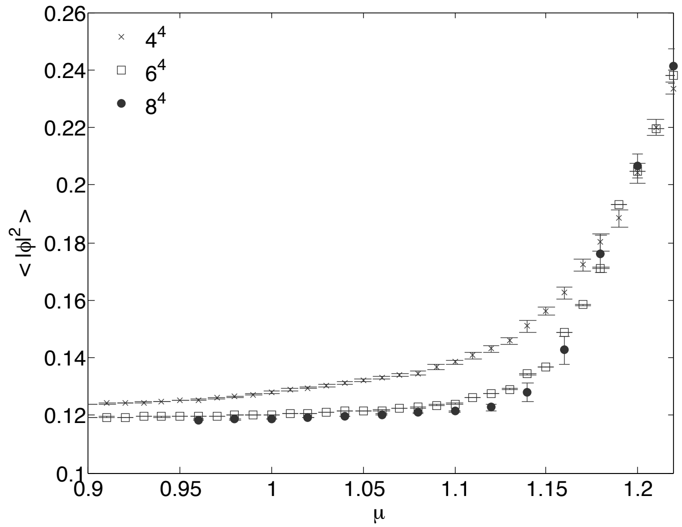
$$\begin{aligned}
\phi_i(\tau, t = 0) \quad &\longrightarrow_{\text{Langevin}} \quad \phi_i(\tau + \Delta\tau, t = 0) \\
&\longrightarrow_{\text{downward flow}} \quad \phi_i(\tau + \Delta\tau, t = T) \\
&\longrightarrow_{\text{projection}} \quad \phi_i^{\text{new}}(\tau + \Delta\tau, t = T) = P\phi_i(\tau + \Delta\tau, t = T) \\
&\longrightarrow_{\text{upwardflow}} \quad \phi_i^{\text{new}}(\tau + \Delta\tau, t = 0).
\end{aligned}$$

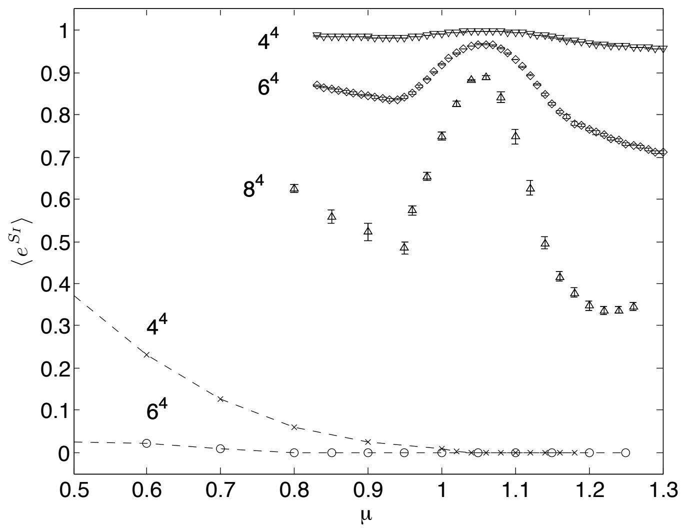


4. Results on $\mathcal{G}_0$ instead of $\mathcal{J}_0$

- $m = \lambda = 1$, $V = 4^4, 6^4, 8^4$. neglecte the computation of the residual phase.







μ dependence of $\langle n \rangle$, $\langle |\phi|^2 \rangle$ and $\langle e^{-i \mathrm{Im} S} \rangle$

5. Conclusions

- They have reported the first numerical application of the Lefschetz formulation to a nontrivial model with a hard sign problem.

- For the relativistic Bose gas model at finite chemical potential, agreement with the known results already on the crudest approximation of the thimble, i.e, the vector space $\mathcal{G}_0$, once the integral was regulated by removing the few diverging trajectories.

- Moreover, they showed that it is possible to improve the approximation of the thimble, by following the downward/upward flow equations.