The scaling of Exact and Approximate Ginsparg-Wilson Fermions

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[C.Gattringer et al.(BGR collaboration):Nucl.Phys. B677,3(2004)]

[C.Gattringer et al., Phys. Rev. D79, 054501(2008)]

[G.P.Engel, C.B.Lang, D.Mohler, A.Schaefer Phys. Rev D87 074504(2013)]

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[S.Dürr, G.Koutsou, Phys. Rev D.83 114512(2011)]

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lattice fermions

- Naive fermions -> doubling
- Wilson fermions -> Chiral symmetry ×
- Overlap fermions -> no doubler, Chiral
- discretization effect -> dispersion relation
- Numerical cost

Chiral fermions on the Lattice

- Ginsparg-Wilson Relation(GWR)

$$\{D, \gamma_5\} = 2aD\gamma_5 RD$$

- Neuberger's solution(Overlap fermions)

$$D_{ov} = \frac{1}{Ra}\left[1 - \frac{A}{\sqrt{A^\dagger A}}\right] \qquad A = \mu - D_0$$

D_0: massless Wilson Dirac-operator

Hypercubic approximate Ginsparg-Wilson fermions(HF)

$$D_{x,y} = \gamma_\mu \rho_\mu (x-y) + \lambda (x-y)$$

$$\rho_\mu (x-y) = \rho_1 \left(\delta_{x+\hat{\mu},y} - \delta_{x-\hat{\mu},y}\right) + \rho_2 \sum_{\nu} \left(\delta_{x+\hat{\mu}+\hat{\nu},y} - \delta_{x-\hat{\mu}+\hat{\nu},y}\right) + \rho_3 \sum_{\nu,\rho} \left(\delta_{x+\hat{\mu}+\hat{\nu}+\hat{\rho}y} - \delta_{x-\hat{\mu}+\hat{\nu}+\hat{\rho},y}\right)$$
$$+ \rho_4 \sum_{\nu,\rho,\sigma} \left(\delta_{x+\hat{\mu}+\hat{\nu}+\hat{\rho}+\hat{\sigma},y} - \delta_{x-\hat{\mu}+\hat{\nu}+\hat{\rho}+\hat{\sigma},y}\right)$$

$$\lambda (x-y) = \lambda_0 \delta_{x,y} + \lambda_1 \sum_{\mu} \left(\delta_{x+\hat{\mu},y} + \delta_{x-\hat{\mu},y}\right) + \lambda_2 \sum_{\mu,\nu} \left(\delta_{x+\hat{\mu}+\hat{\nu},y} + \delta_{x-\hat{\mu}+\hat{\nu},y}\right)$$
$$+ \lambda_3 \sum_{\mu,\nu,\rho} \left(\delta_{x+\hat{\mu}+\hat{\nu}+\hat{\rho}y} + \delta_{x-\hat{\mu}+\hat{\nu}+\hat{\rho},y}\right) + \lambda_4 \sum_{\mu,\nu,\rho,\sigma} \left(\delta_{x+\hat{\mu}+\hat{\nu}+\hat{\rho}+\hat{\sigma},y} + \delta_{x-\hat{\mu}+\hat{\nu}+\hat{\rho}+\hat{\sigma},y}\right)$$

- Truncated Perfect HF(TP-HF)

  Perfect action[Nucl.Phys. B464,319]

- Chirally Optimized HF(CO-HF)

$$\nu_{st} = \sum \left[D\gamma_5 D - \{D, \gamma_5\}\right]^2$$

- Scaling optimized HF(SO-HF)

  tune in the free case

	TP-HF	CO-HF	SO-HF
$\rho^{(1)} := \rho_1(1,0)$	0.30938846	0.30583220	0.334
$\rho^{(2)} := \rho_1(1,1)$	0.09530577	0.09708390	0.083
$\lambda_0 := \lambda(0,0)$	1.48954496	1.49090692	1.5
$\lambda_1 := \lambda(1,0)$	−0.24477248	−0.24771369	−0.25
$\lambda_2 := \lambda(1,1)$	−0.12761376	−0.12501304	−0.125
$\mathcal{V}_{st}$	$3.008 \cdot 10^{-4}$	$1.006 \cdot 10^{-4}$	$65.518 \cdot 10^{-4}$

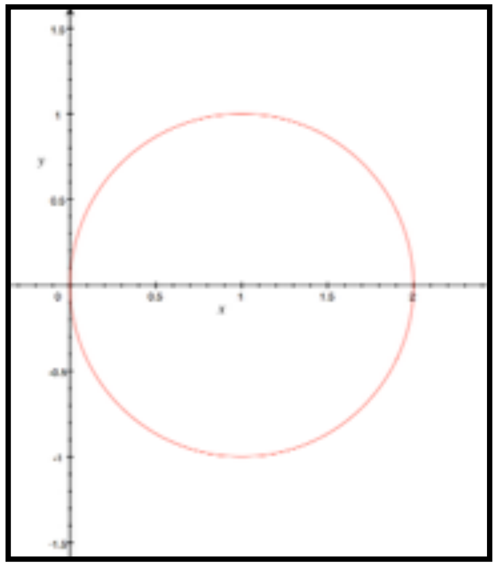
$$D(p) = \sum_{\mu} i p_\mu \gamma_\mu + \mathcal{O}\left(p^2\right) \quad \Rightarrow \quad \rho_1 + 2\rho_2 = \frac{1}{2} \qquad \lambda_0 + 4(\lambda_1 + \lambda_2) = 0$$

Eigenvalue spectra(free)

GWR

$$\{D, \gamma_5\} = 2aD\gamma_5 RD$$

$$\gamma_5 D \gamma_5 = D^\dagger \ \ (\gamma_5 - hermiticity)$$



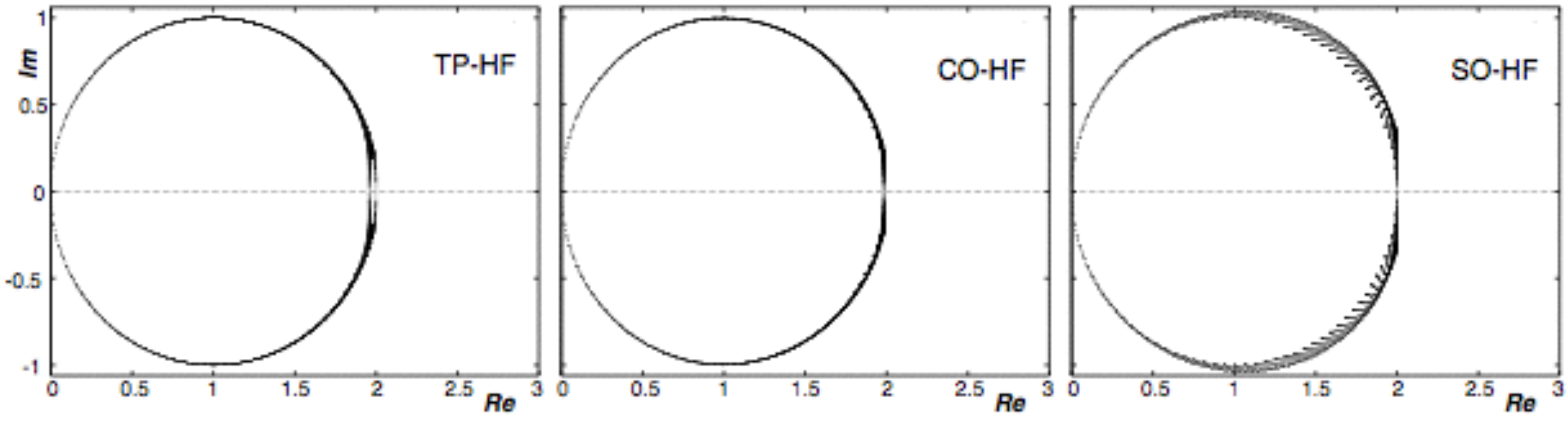


Figure 1: *The free spectra of 3 hypercube fermions: truncated perfect (left), chirally optimized (center) and scaling optimized (right), on a* 100×100 *lattice.*

dispersion relation(free)

evaluate a pole of $D^{-1}(p)$

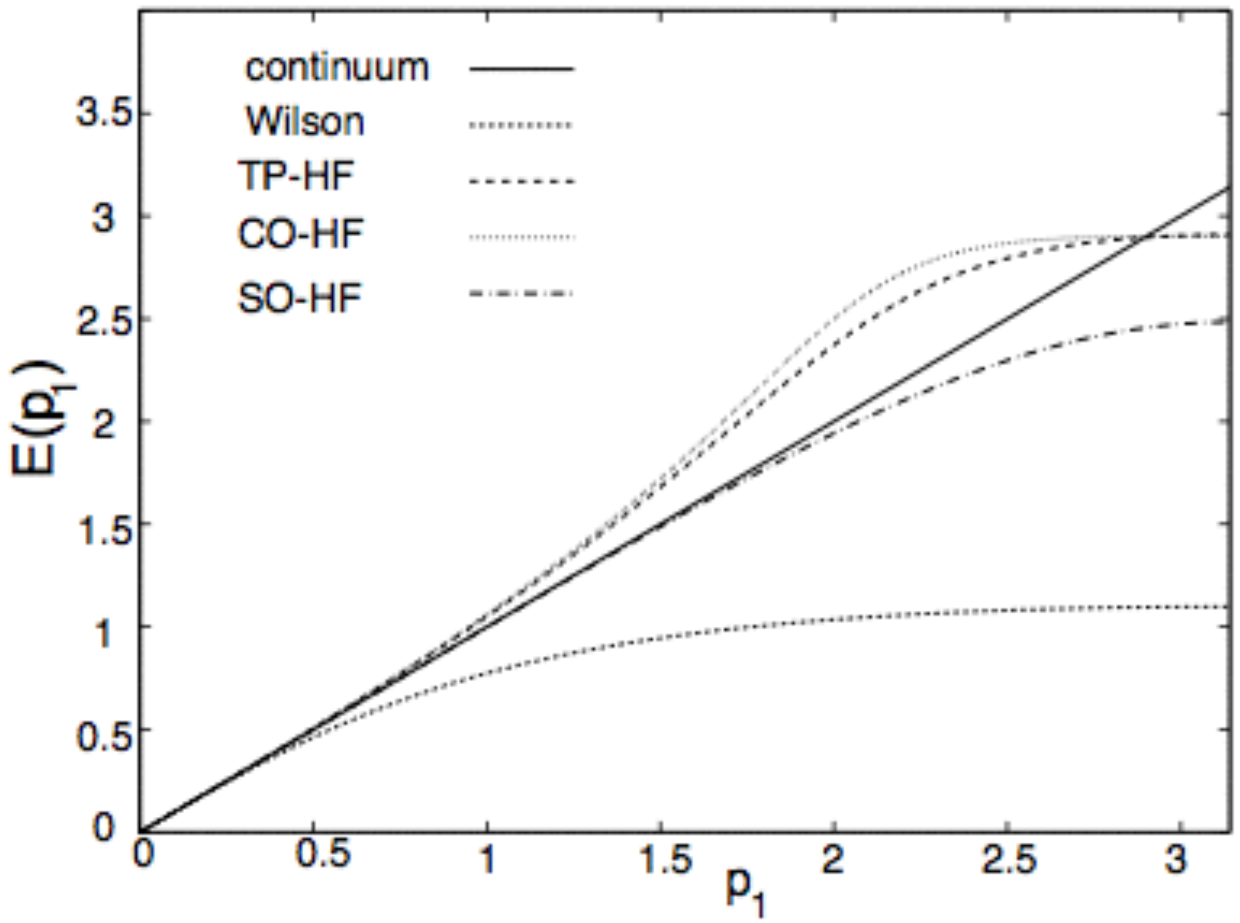


Figure 2: *The dispersion relation for free hypercube fermions compared to the Wilson fermion and to the continuum. (The energies of all higher branches keep above 4.4.)*

Applications to (lattice) Schwinger model

- 2-dimensional QED(abelian gauge), 2 flavor
- Wilson or Hypercube + clover term

$$S_W = \sum_x \bar{\psi}_x \psi_x - \kappa \sum_{x,\mu} \left[\bar{\psi}_{x+\mu} \left(1+\gamma_\mu\right) \psi_x + \bar{\psi}_x \left(1-\gamma_\mu\right) \psi_{x-\mu} \right]$$

$$\kappa = \frac{1}{2\left(m+D\right)} \qquad \gamma_1 = \sigma_1,\ \gamma_2 = \sigma_2,\ \gamma_3 = -i\gamma_1\gamma_2 = \sigma_3$$

$$S_{SW} = -\kappa c_{SW} \frac{i}{2} \sum_x F_{\mu\nu} \sigma_{\mu\nu} \bar{\psi}_x \psi_x$$

- quenched configurations $V=16^2$,Nconf=5000,$c_{SW}=1$

Meson Dispersion relation

$$\pi(p,t) = \sum_n e^{ipn}\left(\bar{u}(n,t)\,\sigma_1 u(n,t) - \bar{d}(n,t)\,\sigma_1 d(n,t)\right)$$

$$\eta(p,t) = \sum_n e^{ipn}\left(\bar{u}(n,t)\,\sigma_1 u(n,t) + \bar{d}(n,t)\,\sigma_1 d(n,t)\right)$$

$$\left\langle C(p,t)\bar{C}(0,0)\right\rangle \to Ae^{-E(p)}\left(1+\mathcal{O}\left(e^{-at\triangle E}\right)\right)$$

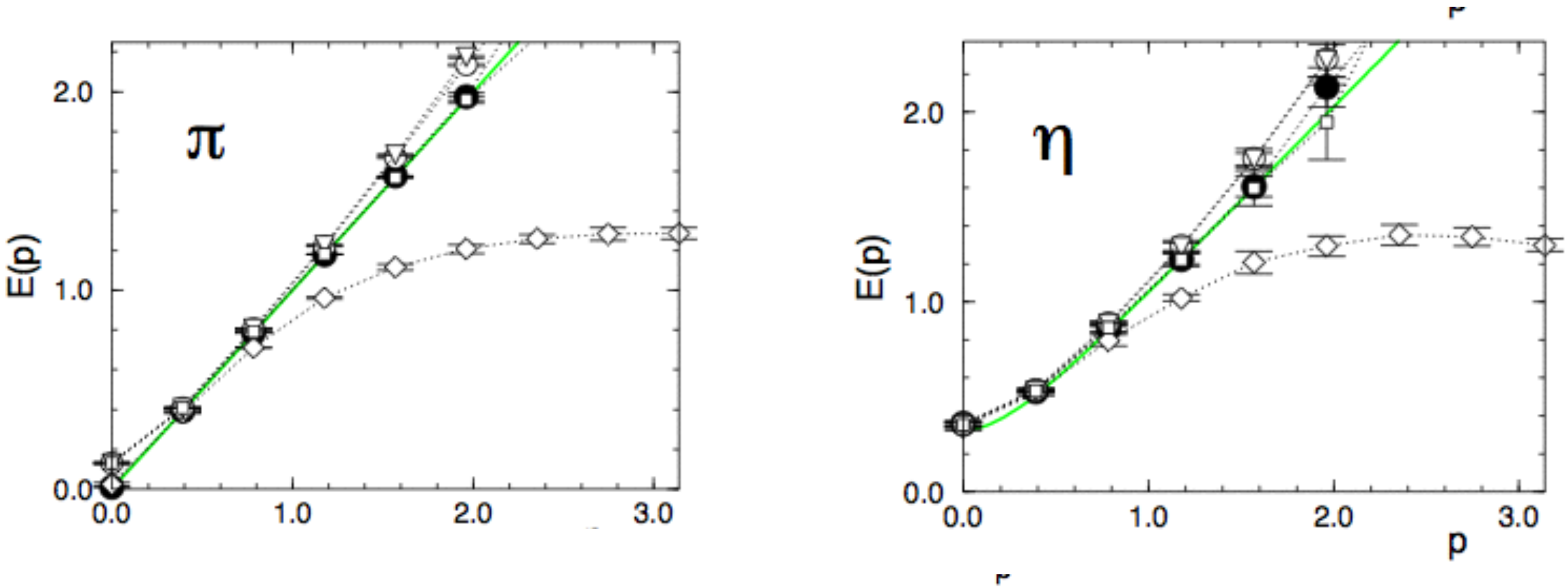


Figure 6: *Meson dispersion relations at $\beta = 6$: Wilson fermion (diamonds), TP-HF (empty circles), CO-HF (triangles) and SO-HF (little boxes) — all the HFs with a clover term — compared to the FPA (filled circles) and the continuum (solid line).*

Spectra(β=6)

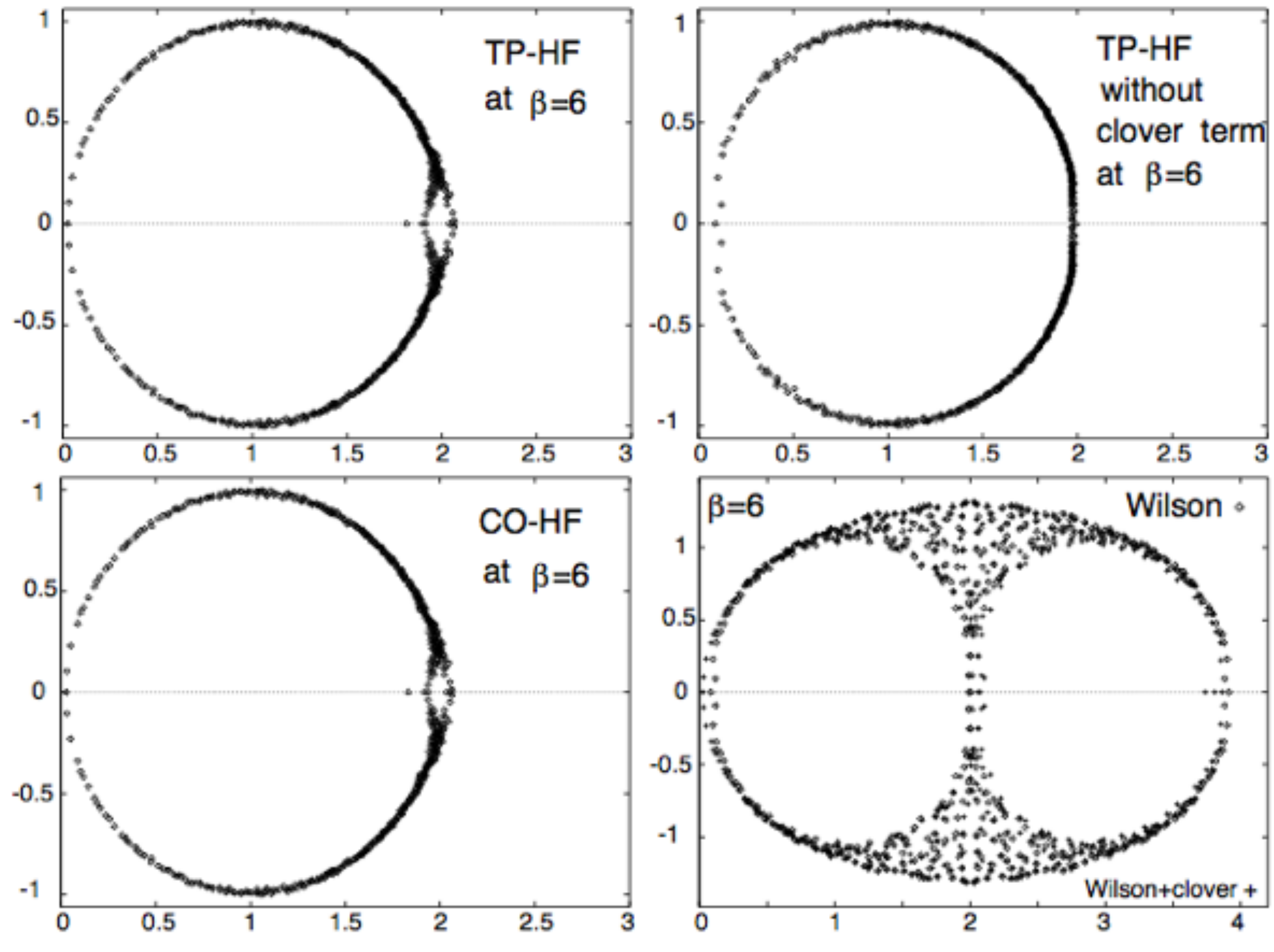


Figure 4: *Spectra of different HFs for a typical configuration at $\beta = 6$, compared to the Wilson fermion with and without clover term. (For the SO-HF with the same configuration, see Fig. 10.) The clover term is always included in the HF actions if not stated differently. It has a negative impact in the region around 2, but it improves the more important region close to 0.*

Spectra(β=4, 2)

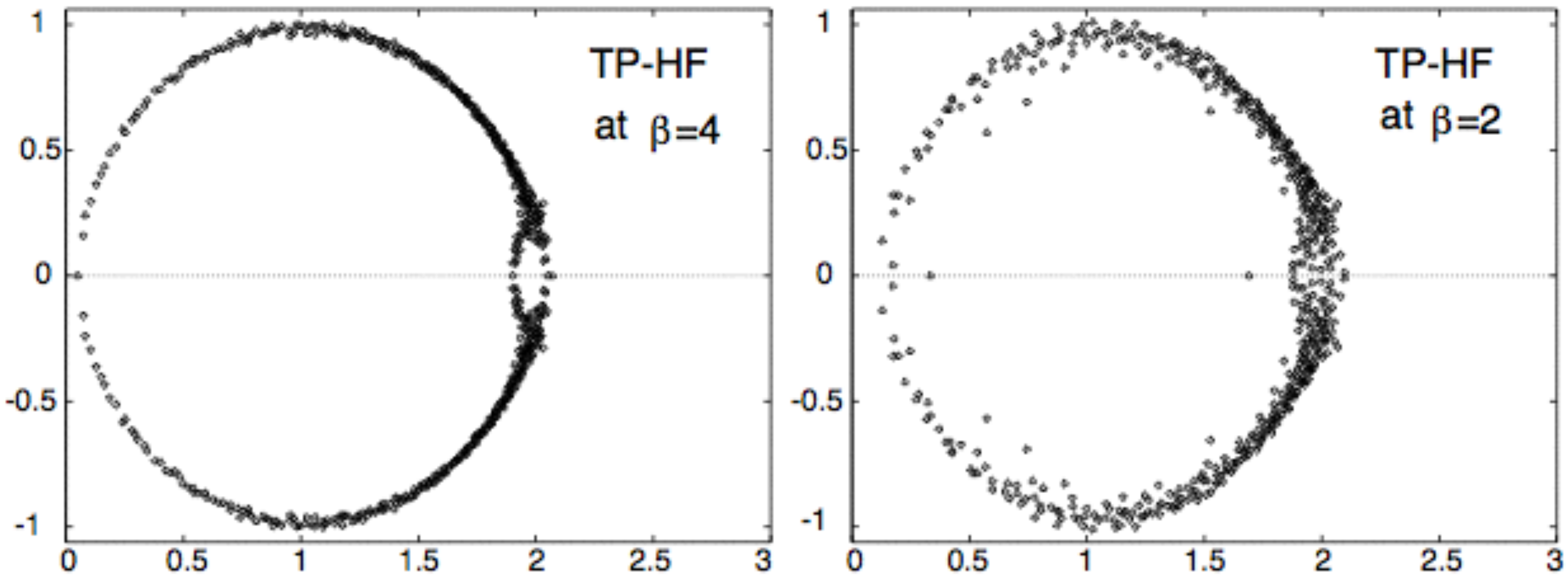


Figure 5: *Typical spectra of the TP-HF at $\beta = 4$ and $\beta = 2$. The results for the CO-HF and SO-HF are similar. Strong coupling amplifies the deviation form the unit circle, and in particular the mass renormalization.*

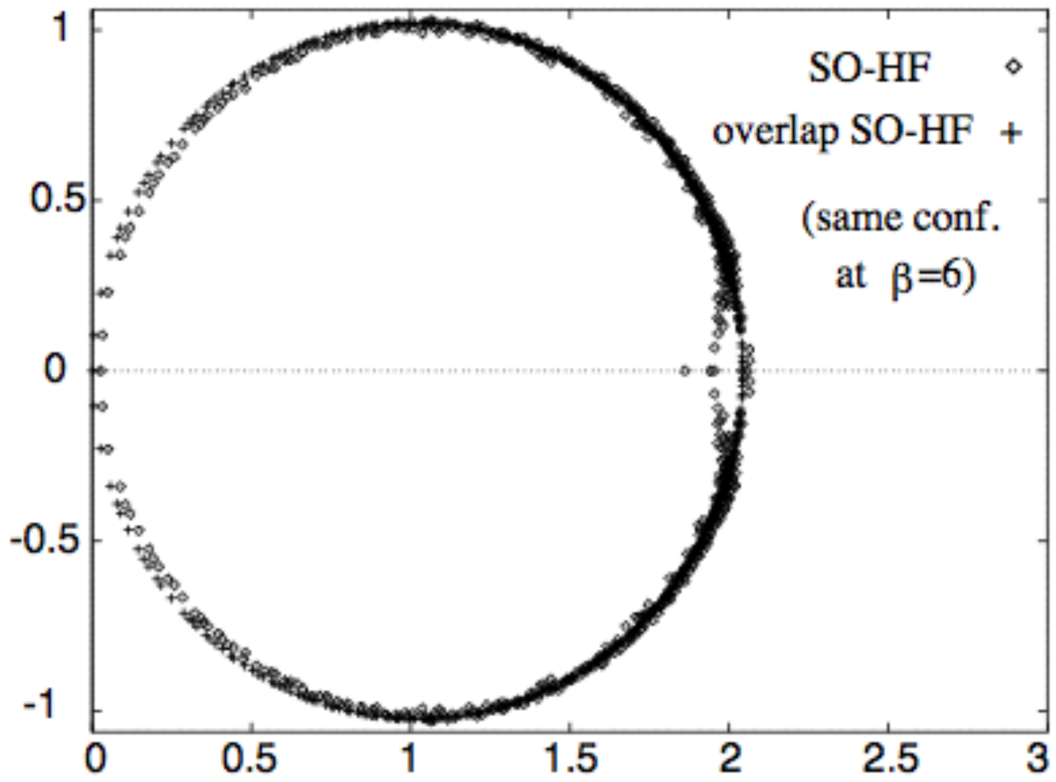
Overlap formula

- reduce deviation from GWR
- HF -> Overlap formula with HF kernel

$$A = \mu - D_0 \to A_{HF} = \mu - D_{HF}$$

$$D_{ov} = \frac{1}{Ra}\left[1 + \frac{A}{\sqrt{A^\dagger A}}\right] \to D_{HFov} = \frac{1}{Ra}\left[1 + \frac{A_{HF}}{\sqrt{A^\dagger_{HF} A_{HF}}}\right]$$

$$\{D_{HFov}, \gamma_5\} = 2aD_{HFov}\gamma_5 R D_{HFov}$$



overlap vs HF

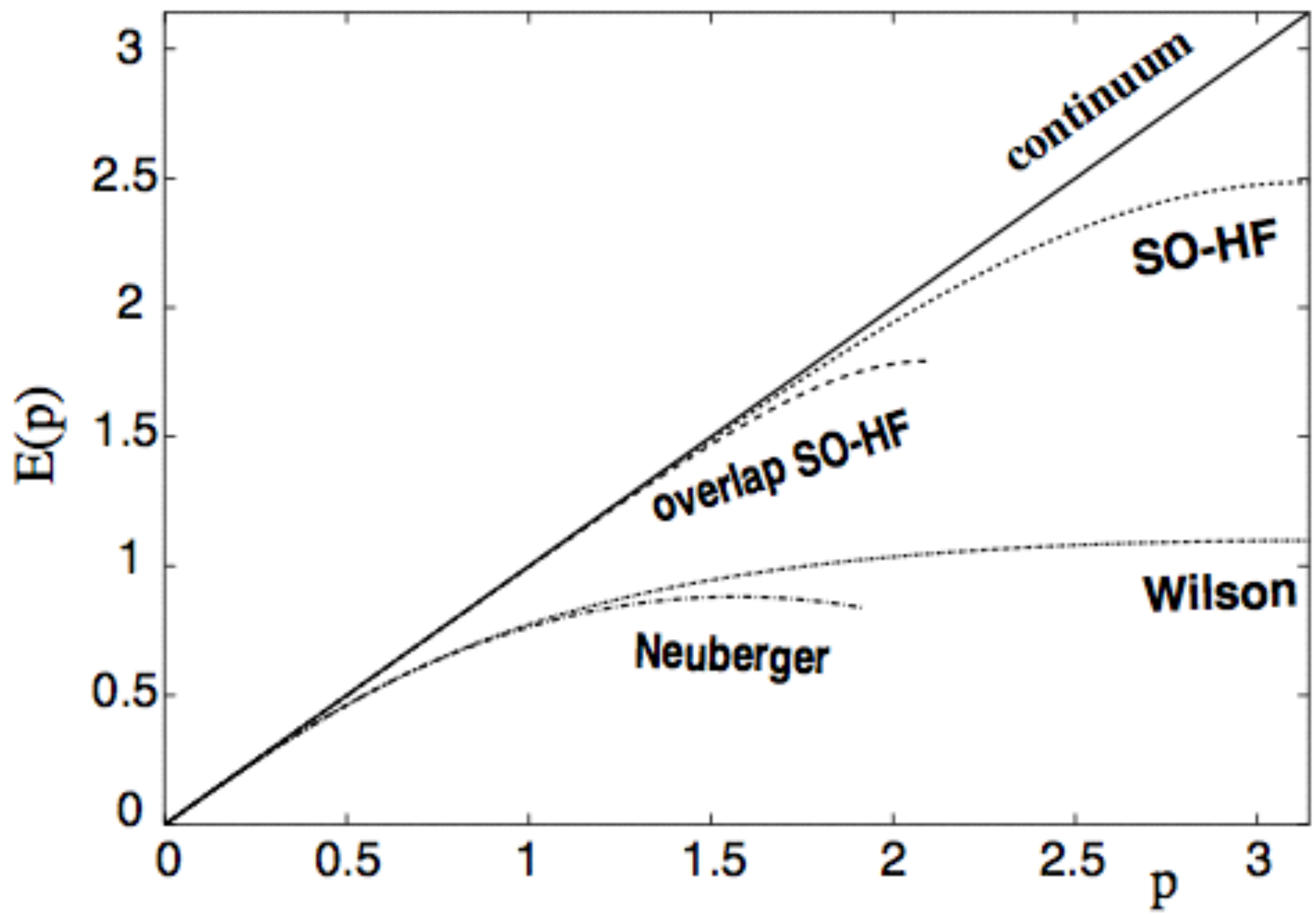


Figure 1. *Free fermion dispersion relations.*

Meson Dispersion relation using overlap Dirac op

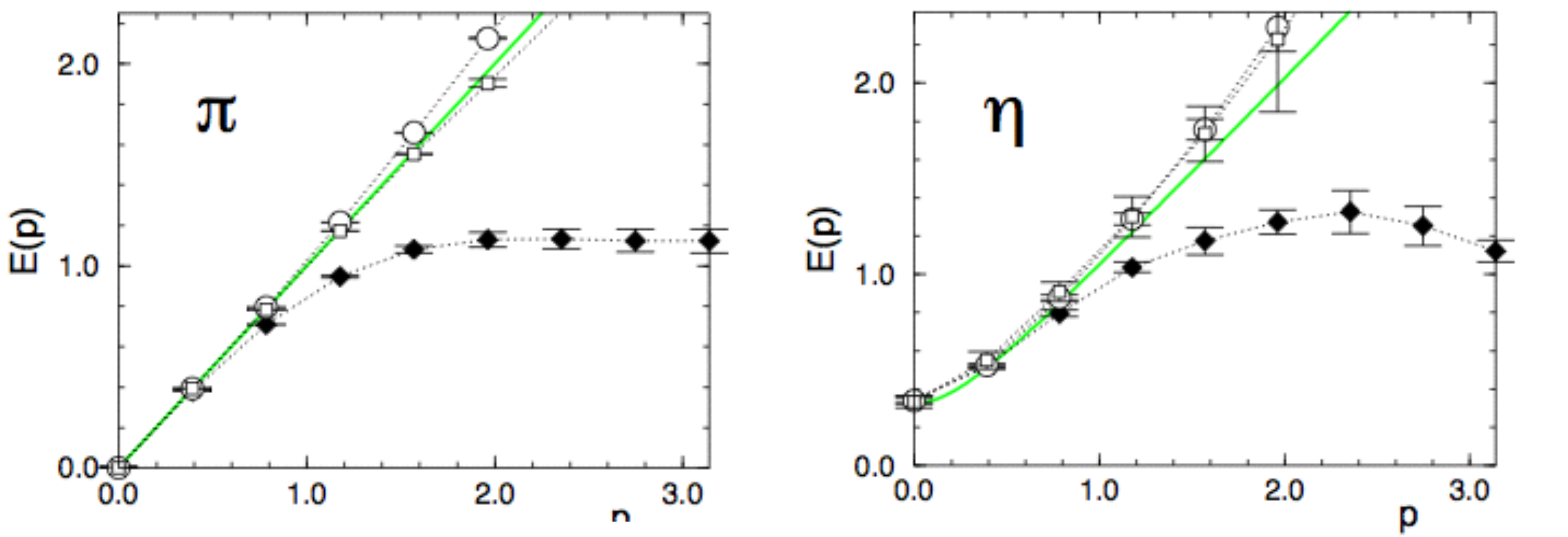


Figure 11: *Meson dispersion relations at $\beta = 6$: Neuberger fermion (diamonds), overlap TP-HF (open circles) and the overlap SO-HF (little boxes) compared to the continuum (solid line).*

Perturbative chiral correction

- overlap procedure is expensive $\left(A^\dagger A\right)^{-\frac{1}{2}}$

- consider approximation method

- assume that D_0 is close to GW fermion ~ HF

$$D_{ov} = \mu \left[1 - \frac{A}{\sqrt{A^\dagger A}}\right]$$

$$A = \mu - D_0$$

$$\begin{aligned}\varepsilon &= A^\dagger A - \mu^2 \\ &= \left(\mu - D_0\right)^\dagger \left(\mu - D_0\right) - \mu^2 \\ &= \left(\mu - \gamma_5 D_0 \gamma_5\right)\left(\mu - D_0\right) - \mu^2 \\ &= -\mu D_0 - \mu \gamma_5 D_0 \gamma_5 + \gamma_5 D_0 \gamma_5 D_0 \\ &= -\mu \gamma_5 \left[\{D_0, \gamma_5\} - 2 D_0 \gamma_5 R D_0\right]\end{aligned}$$

$$D_{ov} \simeq D_{pcp} = \mu - AY$$

$$Y = \frac{1}{2}\left[3 - \frac{1}{\mu^2} A^\dagger A\right]$$

$$\{D_{pcp}, \gamma_5\} = D_{pcp} \gamma_5 D_{pcp} + \mathcal{O}\left(\varepsilon\right)$$

$$Y = \frac{1}{8}\left[15 - \frac{10}{\mu^2} A^\dagger A + \frac{3}{\mu^4}\left(A^\dagger A\right)^2\right]$$

Spectrum(PCP)

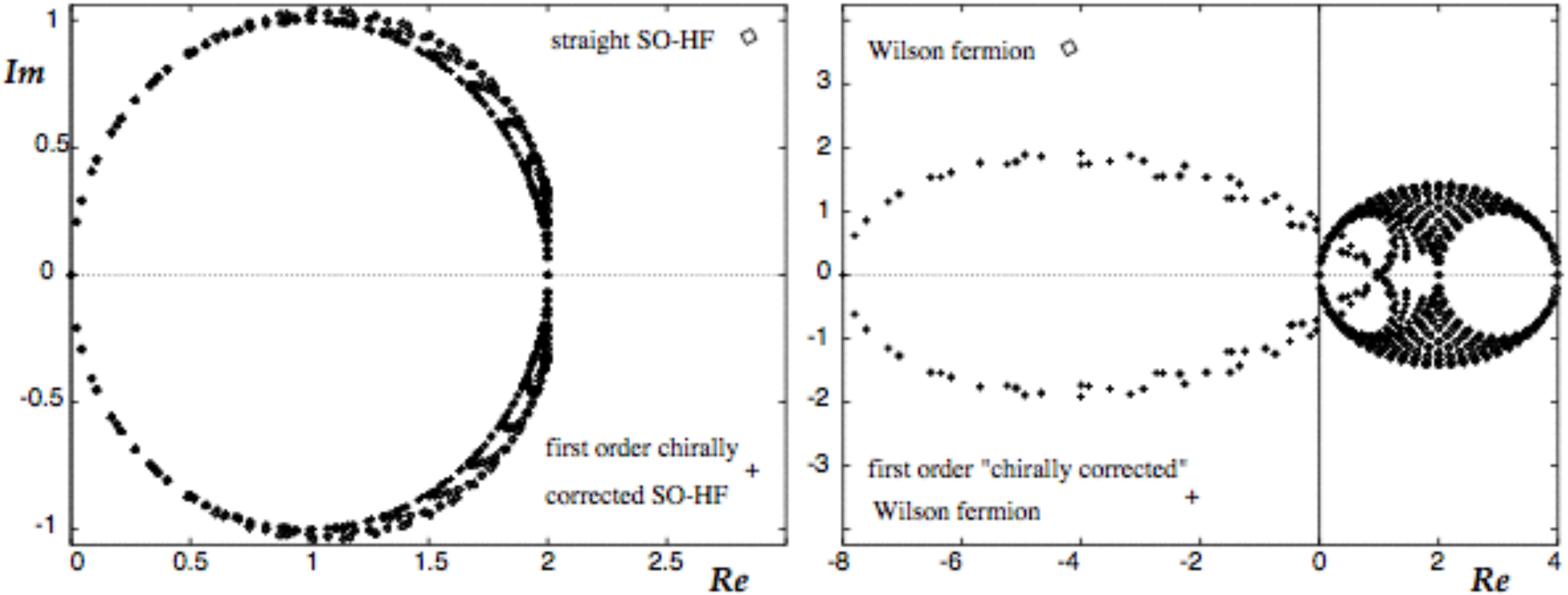


Figure 16: *The initial and the first order chirally corrected free spectrum (using $\mu = 1$) for the SO-HF (left) and for the Wilson fermion (right) on a 30×30 lattice.*

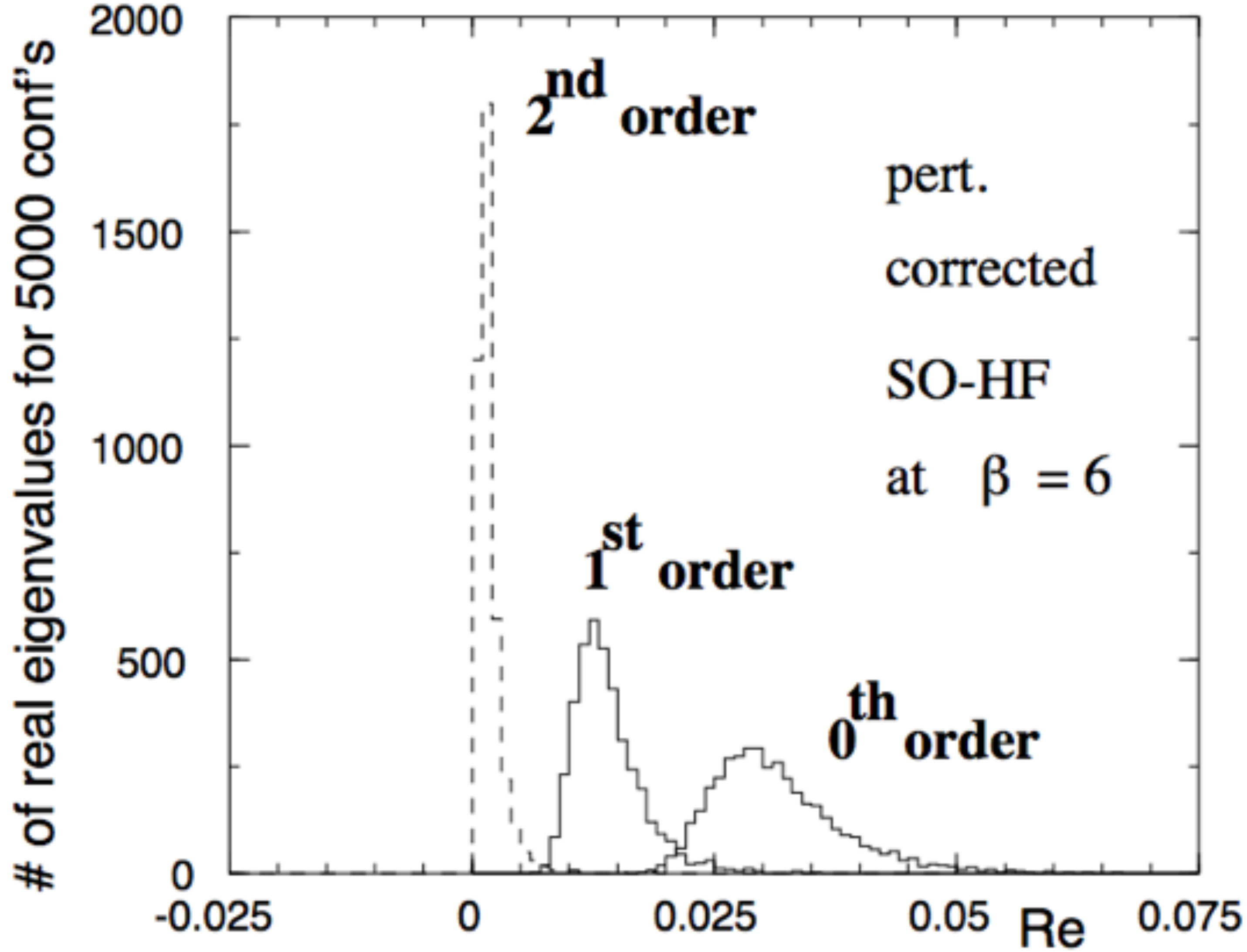


Figure 6. *Histogram of the small real eigenvalues of* $D_{p\chi c}$ *– based on* $D_0 = D_{SO-HF}$ *– showing that the mass renormalization vanishes quickly under perturbative chiral correction.*

Summary

- HF actions have good properties, but deviate from GWR as increasing coupling.
- overlap formula reduces deviations ,but expensive
- If start from HF, one can use perturbative chiral correction instead of exact overlap.

dispersion relation(free)

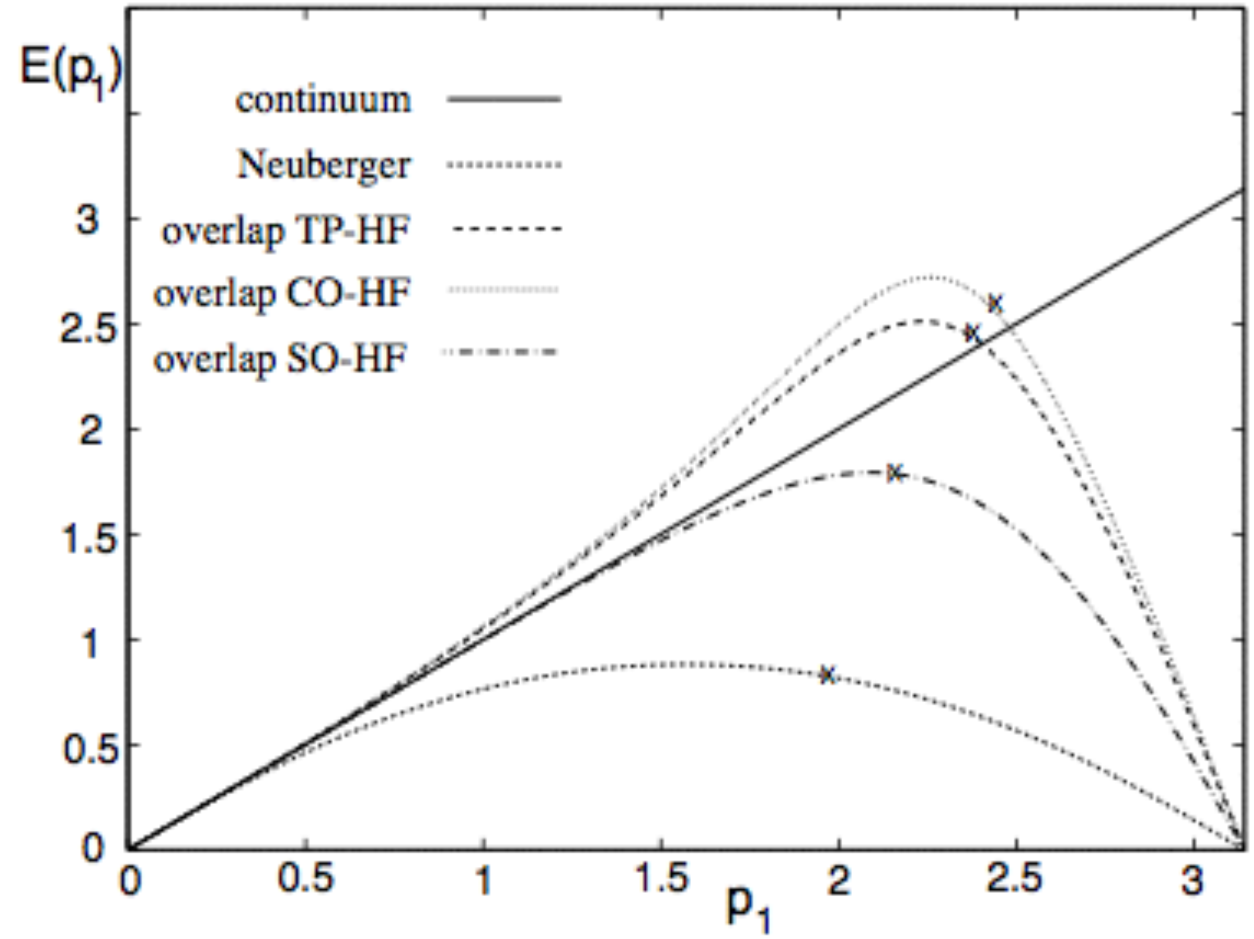


Figure 8: *The free dispersion relation for the Neuberger fermion (inserting $D_0 = D_W$ in the overlap formula) and for various improved overlap fermion (inserting $D_0 = D_{HF}$). We mark the end-points of the curves for the mass parameter $\mu = 1$.*