Dirac-Kähler Fermion form Clifford Product with Noncommutative Differential Form on a Lattice

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Contents

1 Introduction & Motivation

Ultimate purpose of elementary particle physics:
To unify all interactions, including gravity

candidates:
- string theory
- lattice

Features of lattice

- Succeeded in QCD & 2 dim. gravity
- Regularized
- Constructive(cf. Matrix Model)
- Corresponds with differential forms
- Lattice artifact $\cdots$ How to determine?

$$\Downarrow$$

Method :
- Use correspondence between lattice and form
- *Noncommutative differential form* gives systematic description on a lattice (It keeps properties of differential)

Differential forms

tools of differential geometry —— can be used even in curved space

point	●	$\Leftrightarrow$	0-form	1
line	——	$\Leftrightarrow$	1-form	dx^μ
area	△	$\Leftrightarrow$	2-form	$dx^\mu \wedge dx^\nu$
⋮			⋮	

differential forms

algebra
$$dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$$
$$df = (\partial_\mu f) dx^\mu = dx^\mu (\partial_\mu f)$$
$$d^2 f = 0$$

NC differential form on a lattice

- Noncommutativity between function and form
  $$\theta^{\pm\mu} f(x) = f(x \pm \mu)\theta^{\pm\mu} \quad (\theta^{\pm\mu} : \text{1-form})$$
- Leibniz's rule and nilpotency on a lattice
- continuum $\rightarrow$ lattice $\cdots$ determines lattice artifact (gauge field : Wilson action, Dimakis et al.,'94)
  boson

$\boxed{\text{Question Can we apply to Fermion ?}}$

$$\Downarrow$$

Apply to Dirac-Kähler Fermion

Why Dirac-Kähler Fermion?

- Described with differential forms
- Continuum limit of staggered fermion(in flat space)
- NC form relates to staggered fermion(Dai et.al, '01)
- Correspondence between form and lattice
  Fermion with Form(cont.):
  can be used in curved space
  Lattice Fermion:
  not known in curved space(except for 2 dim.)
  $\Rightarrow$ lattice fermion in curved space?
- D-K connects spinors and anti sym. tensors
  $\cdots$ Similar to Twisted SUSY in Topological Filed Theory (Kawamoto & Tsukioka '00)
  $\Rightarrow$ Lattice SUSY?

♣ We treat even D-dimensional Euclidean space-time

2 NC Differential Form on a Lattice

(Dimakis et.al.('93), Aschieri et.al('93), Woronowics('89) etc.)

Leibniz's rule & nilpotency

$$\boxed{\begin{array}{c} d(\psi^{(p)}\psi^{(q)}) = (d\psi^{(p)})\psi^{(q)} + (-)^p \psi^{(p)}\, d\psi^{(q)} \\ d^2 = 0 \end{array}}$$

are *not* kept in simple replacement of "differential" by "difference"

2.1 Universal differential algebra

Define "differential" for a set

Function(0-form) on a lattice

$f = \sum_x f_x \mathrm{e}^x$ basis of function : $\mathrm{e}^x \sim |x\rangle\langle x|$

$\boxed{\mathrm{e}^x \text{ is } \delta\text{-function on the point } x}$

cf. $f(y) = \int dx\, f(x)\delta(x-y)$

$$\begin{array}{ll} \mathrm{e}^x(y) = \delta^{x,y} & f(y) = f_y \\ \sum_x \mathrm{e}^x = 1 & \text{completeness} \\ \mathrm{e}^x \mathrm{e}^y = \delta^{xy}\mathrm{e}^x & \Big(fg\Big)(x) = f(x)g(x), \\ & \quad \text{commutative, pointwise} \end{array}$$

Derivative

Impose Leibniz's rule and linearity for 0-form

$$d(fg) = (df)g + f\,dg \qquad d(\alpha f + \beta g) = \alpha\,df + \beta\,dg$$

$$\begin{aligned} df &= \sum_x f_x d\mathrm{e}^x = \sum_{x,y} f_x(\mathrm{e}^y d\mathrm{e}^x - \mathrm{e}^x d\mathrm{e}^y) \\ &= \sum_{y,x} f_x(\mathrm{e}^{y,x} - \mathrm{e}^{x,y}) \end{aligned}$$

$\mathrm{e}^{x,y} = \mathrm{e}^x d\mathrm{e}^y =$ link: $\overset{x}{\bullet}\longrightarrow\overset{y}{\bullet}$ $(x \neq y)$ (1-form)

Using $\theta^x \equiv \sum_y \mathrm{e}^{y,y+x} = \sum_y \mathrm{e}^y d\mathrm{e}^{y+x} = \sum_y \overset{y}{\bullet}\longrightarrow\overset{y+x}{\bullet}$,

$$\begin{aligned} df &= \sum_{y,x} \left(f_x \mathrm{e}^{x-y}\, d\mathrm{e}^x - f_x \mathrm{e}^x\, d\mathrm{e}^{x+y} \right) \\ &= \sum_x \underbrace{(T_x f - f)}_{\partial_{+x} f} \theta^x \end{aligned}$$

Shift operator: $T_y \mathrm{e}^x = \mathrm{e}^{x-y}, (T_y f)(x) = f_{x+y}$

Noncommutativity

θ^y : from ${}^{\forall}z$ to $z+y$ $\qquad \sum_z \overset{z}{\bullet} \longrightarrow \overset{z+y}{\bullet}$

$e^x\theta^y$: form x to $x+y$ $\qquad \overset{x}{\bullet} \longrightarrow \overset{x+y}{\bullet}$

$\theta^y e^x$: form $x-y$ to x $\qquad \overset{x-y}{\bullet} \longrightarrow \overset{x}{\bullet}$

$$\Longrightarrow \theta^y e^x = e^{x-y}\theta^y \Longrightarrow \boxed{\theta^y f(x) = f(x+y)\theta^y}$$

The noncommutativity cares difference of start point and end point of the links.

Thanks to the noncommutativity, Leibniz's rule holds

$$\begin{aligned} d(fg) &= \sum_x (T_x(fg) - fg)\theta^x \\ &= \sum_x [(T_x f)\theta^x g - fg\theta^x - f\theta^x g + f(T_x g)\theta^x] \\ &= \sum_x (T_x f - f)\theta^x g + f\sum_x (T_x g - g)\theta^x \\ &= df\, g + f\, dg \end{aligned}$$

cf. $\quad \partial_{+\mu}(fg) = (\partial_{+\mu}f)T_{+\mu}g + f\partial_{+\mu}g$

Generalization to Higher Forms

Impose $d^2(\text{0-form}) = 0$

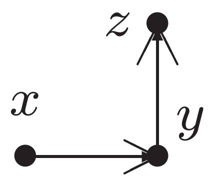
$$d(f\,dg) = df\,dg$$

$$d\mathrm{e}^x = \sum_y (\mathrm{e}^{y,x} - \mathrm{e}^{x,y}) \qquad \text{for 0-form}$$

$$d\mathrm{e}^{x,y} = \sum_z \left(\mathrm{e}^{z,x,y} - \mathrm{e}^{x,z,y} + \mathrm{e}^{x,y,z}\right) \qquad \text{for 1-form}$$

$$\vdots$$

$$\begin{aligned}\mathrm{e}^{x,y,z} &= \mathrm{e}^{x,y}\mathrm{e}^{y,z} = \mathrm{e}^x d\mathrm{e}^y d\mathrm{e}^z \\ &= \mathrm{e}^x \theta^{y-x}\theta^{z-y} \\ &\left(y \neq x \text{ and } y \neq z\right)\end{aligned}$$



$d \sim$ coboundary operator ($\Longrightarrow$ nilpotent)

p-form: $\mathrm{e}^{x_0,x_1,\cdots,x_p}$ or $\boxed{\mathrm{e}^y\theta^{y_1}\ldots\theta^{y_p}}$ are basis

2.2 Oriented Lattice

Too many 1-forms (θ^x for ${}^\forall x$)!! $\Longrightarrow$ Reduction

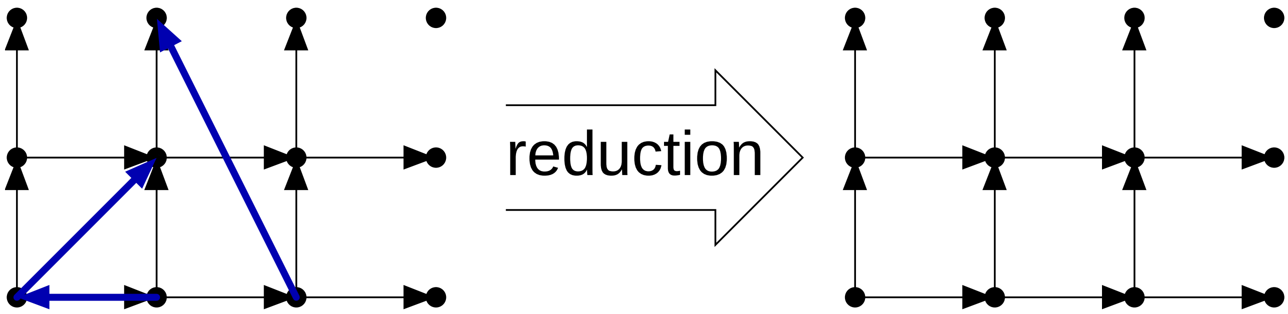
Oriented reduction (Dimakis et.al.,'93)

$e^{x,y} = 0 (y \neq x + \mu)$

$$\begin{aligned} df &= \sum_\mu (T_\mu f - f)\theta^\mu \\ &= \sum_\mu (\partial_{+\mu} f)\theta^\mu = \sum_\mu \theta^\mu \partial_{-\mu} f \end{aligned}$$

(only θ^μ are non-zero)

$$\boxed{dx^\mu = \sum_\nu \partial_{+\nu} x^\mu \theta^\nu = \theta^\mu}$$



Highly non-local or Point

oriented lattice

- ○ Simple correspondence to forms in cont.
- × Asym. difference, so difficult to construct Hermitian action

We can regard the "reduction" as "deconstruction"

2.3 Symmetric Lattice (Dimakis et.al '94)

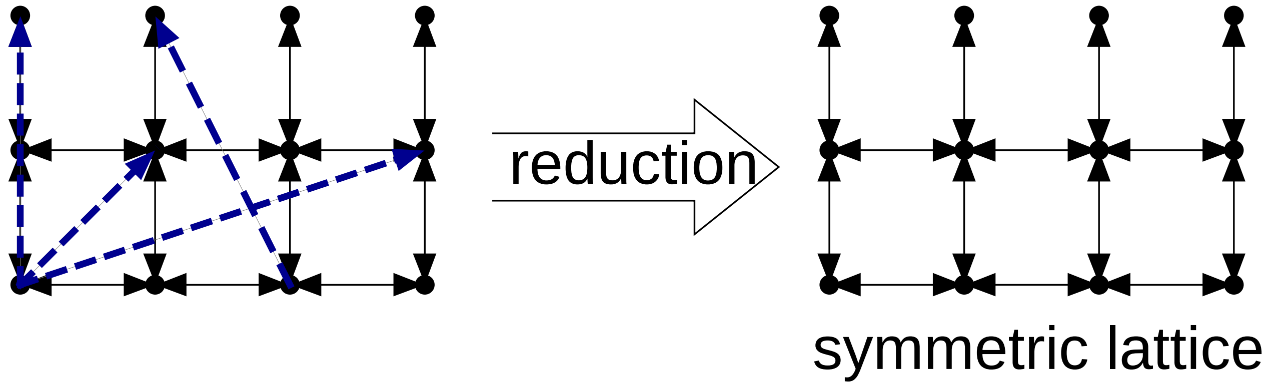
Symmetric reduction : $e^{x,y} = 0 (y \neq x \pm \mu)$

$$df = \sum_\mu \left\{ (T_\mu f - f)\theta^\mu + (T_{-\mu} f - f)\theta^{-\mu} \right\}$$
$$= \sum_\mu \left\{ (\partial_{+\mu} f)\theta^\mu - (\partial_{-\mu} f)\theta^{-\mu} \right\}$$
$$= \sum_\mu \left\{ \theta^\mu \partial_{-\mu} - \theta^{-\mu}\partial_{+\mu} \right\} f$$
$$= \frac{1}{2}\sum_\mu \left\{ (\partial_{+\mu} + \partial_{-\mu}) f dx^\mu + \underbrace{(\partial_{+\mu} - \partial_{-\mu}) f \tau^\mu}_{\text{vanish in cont. limit}} \right\}$$

(Only $\theta^{\pm\mu}$ are non-zero)

$$\boxed{dx^\mu = \sum_\nu (\partial_{+\nu} x^\mu \theta^\nu - \partial_{-\nu} x^\mu \theta^{-\mu}) = \theta^\mu - \theta^{-\mu}}$$

$$\tau^\mu \equiv \theta^\mu + \theta^{-\mu}$$



- 1-form ($\sim$ link) are sym., $\theta^\mu(\rightarrow)$ & $\theta^{-\mu}(\leftarrow)$

× How correspond forms in the cont.?(τ^μ??)

○ Sym. difference, so easy to construct Hermitian action

We Use Symmetric Lattice

Exterior Product

$$\theta^{\pm\mu}\theta^{\pm\nu} = -\theta^{\pm\nu}\theta^{\pm\mu} \qquad (\leftarrow \mathrm{e}^{x,x\pm 2\mu} = 0)$$

$$\theta^{\mu}\theta^{-\mu} = -\theta^{-\mu}\theta^{\mu} \quad (\Leftarrow \sum_{\mu}(\theta^{\mu}\theta^{-\mu} + \theta^{-\mu}\theta^{\mu}) = \underbrace{d\mathrm{e}^{x,x}}_{=0} = 0)$$

$*$-Conjugation(Hermite Conjugation)

$(df)^* = d(f)^*$ (Hermitian differential)

$$\Longrightarrow (\theta^{\pm\mu})^* = -\theta^{\mp\mu}$$

cf. link variables: $U_\mu, U_\mu^\dagger$

$$(dx^{\mu_1}\dots dx^{\mu_p})^* = dx^{\mu_1}\dots dx^{\mu_p}$$

$$\Longrightarrow (\omega_p\omega_q)^* = (-)^{pq}(\omega_q)^*(\omega_p)^*$$

Integral

$$\int \underbrace{f}_{\text{0-form}} vol = \sum_x f(x) \qquad (vol : \text{volume form})$$

- $2D$ 1-forms in D dim. , so we have to investigate vol (D-form? $2D$-form?)
- We, however, can prove Stokes' theorem with $\int \theta^{\mu}\theta^{-\mu} = 0$.

3 Dirac-Kähler fermion(cont.)

(Kähler '62, Becher & Joos '82)

$(d-\delta)^2 = \Box \;\Rightarrow\; d-\delta$ is a Dirac operator

We formulate in terms of *Clifford product*

Clifford Product(Cup Product, $\vee$)

- $dx^\mu \vee \; = dx^\mu + e^\mu \quad (e_\mu = \frac{\partial}{\partial dx^\mu})$

  $$dx^\mu \vee dx^\nu = dx^\mu dx^\nu + g^{\mu\nu}$$

  satisfies Clifford algebra:

  $$dx^\mu \vee dx^\nu + dx^\nu \vee dx^\mu = 2g^{\mu\nu}$$

  $$\Longrightarrow d\vee\,(= d-\delta) \text{ is Dirac operator}$$

- Generalization to higher forms(associative)

  $$(f\,dx^K)\vee(g\,dx^L)$$
  $$\equiv \sum_p \frac{1}{p!}(-)^{p(p-1)/2}\left(\eta^p e_{\mu_1}\ldots e_{\mu_p} f\,dx^K\right)\wedge\left(e^{\mu_1}\ldots e^{\mu_p} g\,dx^L\right)$$
  $$\simeq f\,dx^K \wedge g\,dx^L + (\text{1-forms are contracted}) + (\text{2-forms are contracted}) + \cdots$$

  [sign factor : $\eta\omega_p = (-)^p\omega_p$]

- Inner product

  $$(\Phi',\Phi) = \int (\zeta\Phi'^* \vee \Phi)\Big|_{\text{0-form}} vol$$
  $$= \int \sum_p \frac{1}{p!}\phi'^*_{\mu_1\cdots\mu_p}\phi^{\mu_1\cdots\mu_p}\, vol$$

  [sign factor: $\zeta\omega_p = (-)^{p(p-1)/2}\omega_p$]

$\boxed{\text{Operate Dirac op. } d\vee \text{ on inhomogeneous forms}}$

$$\Psi = \varphi + \varphi_\mu\, dx^\mu + \varphi_{\mu\nu}\, dx^\mu \wedge dx^\nu + \cdots$$
$$\psi_i{}^{(j)} = \left(\varphi \mathbb{1} + \varphi_\mu \gamma^\mu + \varphi_{\mu\nu}\gamma^\mu\gamma^\nu + \cdots\right)_i{}^{(j)}$$

Following base is useful ($\Psi = \psi_i{}^{(j)} Z^i{}_{(j)}$)

$$Z^i{}_{(j)} = \sum_{p=0}^{D} \frac{1}{p!}\left(\gamma_{\mu_1}{}^T \gamma_{\mu_2}{}^T \ldots \gamma_{\mu_p}{}^T\right)^i{}_{(j)}\, dx^{\mu_1}\ldots dx^{\mu_p}$$
$$= \sum_P \left(\gamma^T\right)_P{}^i{}_{(j)}\, dx^P$$

γ matrix representation of $\vee$:

$$dx^\mu \vee Z^i{}_{(j)} = \gamma^\mu{}_k{}^i Z^k{}_{(j)}$$
$$Z^i{}_{(j)} \vee dx^\mu = Z^i{}_{(l)} \gamma^\mu{}_{(k)}{}^{(l)}$$

Action

$$S = \int \left(\overline{\Psi} \vee (d+m) \vee \Psi\right) vol$$
$$= 2^{D/2} \int \sum_{(j)} \overline{\psi}_{(j)} (\gamma^\mu \partial_\mu + m) \psi^{(j)} dx^1 \cdots dx^D$$

(j) : flavor $(1 \sim 2^{D/2})$

$\boxed{\text{(Dirac fermion)} \times \text{(flavor)}}$

4 Dirac-Kähler fermion(Lattice)

We formulate D-K fermion on a lattice thorough Clifford product

Caution

- Noncommutativity can spoil the associativity
- Two-way
  (two forms par 1 dim. $dx^\mu \Rightarrow \theta^\mu$ & $\theta^{-\mu}$)

Criteria 1. *Associativity*
2. Positivity of the inner product

And more

3. Maintain two-way links

or

3'. Respect for spacetime dimension

We consider both cases, 3. and 3'.

4.1 Maintain two-way links

In continuum space:

$$(f\,dx^K)\vee(g\,dx^L)$$

$$\equiv \sum_{p=0}^{D} \frac{1}{p!}(-)^{\frac{p(p-1)}{2}} \Big(\eta^p \underbrace{e_{\mu_1}\dots e_{\mu_p}}_{(a)} f\,dx^K\Big)\wedge\Big(\underbrace{e^{\mu_1}\dots e^{\mu_p}}_{(b)} g\,dx^L\Big)$$

$$(a):\ e_\mu = \frac{\partial}{\partial dx^\mu} \ \Rightarrow\ \alpha\frac{\partial}{\partial\theta^{\epsilon\mu}}\overrightarrow{T}_{\epsilon\mu}+\beta\frac{\partial}{\partial\theta^{-\epsilon\mu}}\overrightarrow{T}_{-\epsilon\mu}$$

$$(b):\ e^\mu = \frac{\partial}{\partial dx_\mu} \ \Rightarrow\ \gamma\frac{\partial}{\partial\theta^{\epsilon\mu}}\overrightarrow{T}_{\epsilon\mu}+\delta\frac{\partial}{\partial\theta^{-\epsilon\mu}}\overrightarrow{T}_{-\epsilon\mu}$$

Positivity of inner product

Normalize to give Clifford algebra

Then parameters α,β,γ and δ are fixed

$$\boxed{\begin{aligned} & f\theta^K \vee g\theta^L \\ &= \sum_{p=0}^{2D}(-)^{p(p-1)/2}\left(-\frac{1}{2}\right)^p \sum_{\{\epsilon_i\mu_i\}}\Big\{ \\ &\quad \eta^p\Big(\prod_i \frac{\partial}{\partial\theta^{\epsilon_i\mu_i}}T_{\epsilon_i\mu_i}\Big) f\theta^K\Big\}\wedge\Big\{\prod_i \frac{\partial}{\partial\theta^{-\epsilon_i\mu_i}} g\theta^L\Big\}\end{aligned}}$$

Uniquely determined!

$$\left[\begin{array}{l} \theta^K = \theta^{\epsilon_1\mu_1}\dots\theta^{\epsilon_k\mu_k},\ \ \epsilon_i = 1 \text{ or } -1 \\ \frac{\partial}{\partial\theta^{\epsilon\mu}} f = (T_{-\epsilon\mu} f)\frac{\partial}{\partial\theta^{\epsilon\mu}} \\ \frac{\partial}{\partial\theta^{\epsilon\mu}}\overrightarrow{T}_{\epsilon\mu} f = f\frac{\partial}{\partial\theta^{\epsilon\mu}}\overrightarrow{T}_{\epsilon\mu} \end{array}\right]$$

Features

$\theta^K \vee \theta^L$ and $\theta^K \theta^L$ have same noncommutative properties

$$\begin{aligned}&(f\theta^K \vee g\theta^L)\\ &\quad = (fT_K g)(\theta^K \vee \theta^L) = (\theta^K \vee \theta^L)(T_{-L} f) g\end{aligned}$$

This is crucial to associativity.

$$\theta^\mu \vee = \theta^\mu - \frac{1}{2}\frac{\partial}{\partial \theta^{-\mu}} \qquad \theta^{-\mu} \vee = \theta^{-\mu} - \frac{1}{2}\frac{\partial}{\partial \theta^{\mu}}$$

$$\{\theta^\mu - \theta^{-\mu},\ \theta^\nu - \theta^{-\nu}\} = 2\delta^{\mu\nu}$$

Action(naive case)

Expand $\Psi, \overline{\Psi}$ with $dx^\mu = \theta^\mu - \theta^{-\mu}$

$$\Psi = \psi_i{}^{(j)} Z^i{}_{(j)}, \;\; \overline{\Psi} = Z^{(j)}{}_i \overline{\psi}_{(j)}{}^i = (Z^i{}_{(j)})^* \overline{\psi}_{(j)}{}^i$$

$$\psi_i{}^{(j)} = \left(\varphi \mathbb{1} + \varphi_\mu \gamma^\mu + \varphi_{\mu\nu} \gamma^\mu \gamma^\nu + \cdots \right)_i{}^{(j)}$$
$$\overline{\psi}_{(l)}{}^k = \left(\overline{\varphi} \mathbb{1} + \overline{\varphi}_\mu \gamma^\mu + \overline{\varphi}_{\mu\nu} \gamma^\mu \gamma^\nu + \cdots \right)_{(l)}{}^k$$

Action:

$$\int \left(\overline{\Psi} \vee (d+m) \vee \Psi\right)\Big|_{\text{0-form}} vol$$
$$= 2^{\frac{D}{2}} \sum_x \Big[\sum_\mu \Big\{ \overline{\psi}_{(j)}(x) \gamma^\mu \frac{1}{2} \Big(\psi(x+\mu) - \psi(x-\mu) \Big)^{(j)} \Big\}$$
$$+ m \overline{\psi}_{(j)}(x) \psi^{(j)}(x) \Big]$$

It has doublers ($\times$ flavors) $\cdots$ too naive!!

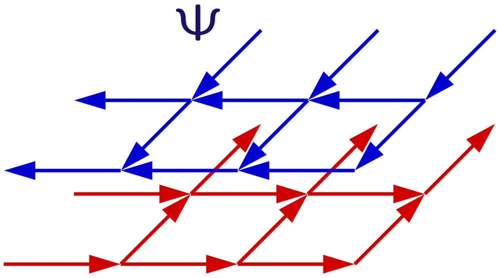
Note: We can regard this fermion as *Kogut-Susskind* fermion in the sense it reflects correspondence between lattice and forms.

Action(staggered case)

Expand Ψ with θ^{μ} , $\overline{\Psi}$ with $\theta^{-\mu}$:

$$\Psi = \varphi + \varphi_\mu \sqrt{2}\theta^\mu + \varphi_{\mu\nu}(\sqrt{2}\theta^\mu)(\sqrt{2}\theta^\nu) + \cdots$$
$$= \psi_i{}^{(j)} Z_+{}^i{}_{(j)}$$

$$\overline{\Psi} = \overline{\varphi} - \sqrt{2}\theta^{-\mu}\,\overline{\varphi}_\mu + (-\sqrt{2}\theta^{-\mu})(-\sqrt{2}\theta^{-\nu})\,\overline{\varphi}_{\mu\nu} + \cdots$$
$$= Z_-{}^{(l)}{}_k \overline{\psi}_{(l)}{}^k = (\zeta Z_+{}^k{}_{(l)})^* \overline{\psi}_{(l)}{}^k$$



Action:

$$\int \overline{\Psi} \vee (\sqrt{2}d + m) \vee \Psi \Big|_{\text{0-form}} vol$$
$$= 2^{D/2} \sum_x \Big[\frac{1}{2} \sum_\mu \Big\{ \overline{\psi}_{(j)}(x)\gamma_\mu \Big(\psi(x+\mu) - \psi(x-\mu)\Big)^{(j)}$$
$$+ \overline{\psi}_{(j)}(x)\gamma_5^\dagger \Big(\psi(x+\mu) + \psi(x-\mu) - 2\psi(x)\Big)^{(l)} \times (\gamma_5\gamma_\mu)_{(l)}{}^{(j)} \Big\}$$
$$+ m\overline{\psi}_{(j)}(x)\,\psi^{(j)}(x) \Big]$$

This is the staggered fermion action written in spinors, exactly.

Dirac-Kähler fermion → staggered fermion !!!

4.2 Respect For Spacetime Dimension

In continuum space:

$$(f\,dx^K) \vee (g\,dx^L)$$
$$\equiv \sum_{p=0}^{D} \frac{1}{p!} (-)^{\frac{p(p-1)}{2}} \left(\eta^p e_{\mu_1} \ldots e_{\mu_p} f\,dx^K\right) \wedge \left(e^{\mu_1} \ldots e^{\mu_p} g\,dx^L\right)$$

$\Big\downarrow$

Almost same as previous case except for the range of summation

$$\boxed{\begin{aligned} & f\theta^K \vee g\theta^L \\ & \quad = \sum_{p=0}^{D} (-)^{\frac{p(p-1)}{2}} (-)^p \sum_{\mu_i} \frac{1}{p!} \Big\{ \\ & \qquad \eta^p \prod_i \left(\frac{\partial}{\partial \theta^{-\mu_i}} T_{-\mu_i} \right) f\theta^K \Big\} \Big\{ \prod_i \left(\frac{\partial}{\partial \theta^{\mu_i}} \right) g\theta^L \Big\} \end{aligned}}$$

Uniquely determined!

$$\theta^\mu \vee\ = \theta^\mu \wedge \qquad \theta^{-\mu} \vee\ = \theta^{-\mu} \wedge - \frac{\partial}{\partial \theta^\mu}$$

Action

Expand $\Psi, \overline{\Psi}$ with $dx^\mu = \theta^\mu - \theta^{-\mu}$

$$\Psi = \psi_i{}^{(j)} Z^i{}_{(j)}, \quad \overline{\Psi} = Z^{(j)}{}_i \overline{\psi}_{(j)}{}^i = (Z^i{}_{(j)})^* \overline{\psi}_{(j)}{}^i$$

We also obtain *staggered fermion*:

$$\int \overline{\Psi} \vee (d+m) \vee \Psi \Big|_{\text{0-form}} vol$$
$$= 2^{D/2} \sum_x \Big[\frac{1}{2} \sum_\mu \Big\{ \overline{\psi}_{(j)}(x) \gamma_\mu \Big(\psi(x+\mu) - \psi(x-\mu) \Big)^{(j)}$$
$$+ \overline{\psi}_{(j)}(x) \gamma_5^\dagger \Big(\psi(x+\mu) + \psi(x-\mu) - 2\psi(x) \Big)^{(l)} \times (\gamma_5 \gamma_\mu)_{(l)}{}^{(j)} \Big\}$$
$$+ m \overline{\psi}_{(j)}(x)\, \psi^{(j)}(x) \Big]$$

Chiral

$\Gamma_5 = \theta^1 \theta^{-1} \ldots \theta^D \theta^{-D}$ generates following transformation:

$$\delta \psi_i{}^{(j)} = i\alpha (\gamma_5)_i{}^k \, \psi_k{}^{(l)} \, (\gamma_5)_{(l)}{}^{(j)},$$

and $\Gamma_5 \vee d + d \vee \Gamma_5 = 0$.

$$D = \frac{1}{a} \left[1 - H \vee \frac{1}{\sqrt{H^\dagger \vee H}} \right] \qquad H = 1 - a(d+A)$$

satisfies G-W relation. (cf. Overlap fermion)

5 Gauge Interaction

Gauge Transformation(Dimakis et al. '93,'94)

fermion: $\Psi \longrightarrow G\Psi, \ \bar{\Psi} \longrightarrow \bar{\Psi}G^{-1}$

gauge field: $A = \sum_\mu (A_\mu \theta^\mu + A_{-\mu}\theta^{-\mu})$

$$A \longrightarrow GAG^{-1} + G\, dG^{-1}$$

$$d + A \longrightarrow G(d + A)G^{-1}$$

Thus

$$\int \bar{\Psi} \vee ((d + A) + m) \vee \Psi\Big|_{\text{0-form}} vol$$

is gauge invariant (due to associativity of $\vee$)

Link Variables

$U_{\pm\mu} = 1 + A_{\pm\mu}$: Lie group valued,

$$U_\mu(x) \longrightarrow G(x)U_\mu(x)G^{-1}(x + \mu)$$

$$U_{-\mu}(x + \mu) = U_\mu^\dagger(x) \longrightarrow G(x + \mu)U_\mu^\dagger(x)G^{-1}(x)$$

$$\text{or } U = \sum_\mu (U_\mu\theta^\mu + U_{-\mu}\theta^{-\mu}) \longrightarrow GUG^{-1}$$

The action: (free action) + (link variables)

$$2^{D/2}\sum_x \Big[\frac{1}{2}\sum_\mu \Big\{\bar{\psi}_{(j)}(x+\mu)\gamma_\mu U_\mu^\dagger(x)\psi(x)^{(j)} - \bar{\psi}_{(j)}(x)\gamma_\mu U_\mu(x)\psi(x+\mu)^{(j)} + \bar{\psi}_{(j)}(x)\gamma_5^\dagger\big(U_\mu(x)\psi(x+\mu) + U_\mu^\dagger(x)\psi(x-\mu) - 2\psi(x)\big)^{(l)}(\gamma_5\gamma_\mu)^{(l)}{}_{(j)}\Big\} + m\bar{\psi}_{(j)}(x)\,\psi^{(j)}(x)\Big]$$

♣ $aA_\mu = \exp(aA'_\mu) - 1 = aA'_\mu + a^2(A'_\mu)^2/2 + \cdots$

Yang-Mills Action

Field strength: $F = dA + A^2 = U^2$

With both types of $\vee$, we have

$$\frac{1}{2g^2}(F, F)$$

$$= \frac{1}{2g^2} \int \mathrm{tr}[(\zeta F)^\dagger \vee F] \Big|_{\text{0-form}} \; vol$$

$$= \frac{1}{2g^2} \sum_x \sum_{\mu,\nu} \mathrm{tr}\Big\{1 - U_\mu(x) U_\nu(x+\mu) U_\mu^\dagger(x+\nu) U_\nu^\dagger(x)\Big\}$$

This is Wilson action !!

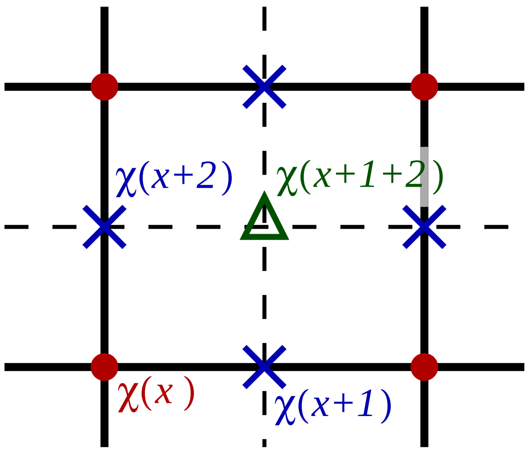
Other expression of the inner product gives the same action. (Dimakis et al. '94)

6 Conclusion

Summary

- Defined $\vee$ product on a lattice
- Dirac-Kähler fermion $\longrightarrow$ staggered fermion
- Gauge invariant action

Correspondence between staggered(K-S) fermion and Dirac-Kähler fermion:



Staggered fermion lives on sites of the small lattice.
In double spacing lattice,

point 0-form : $\varphi(x) = \chi(x)$

line 1-form : $\varphi_1(x) = \chi(x+1)$

$\varphi_2(x) = \chi(x+2)$

area 2-form : $\varphi_{12}(x)$

$= \chi(x+1+2)$.

This correspondence holds for NC form, too (with finite lattice spacing).

Future Work

- Path integral of NC form and D-K fermion
- Chiral anomaly
- Lattice SUSY with D-K fermion
  D-K formalism is same as “twist” in topological field theory which gives twisted SUSY.
- What is τ^μ ?

Far future...

- What can we use “flavors” of D-K fermion?
- Fermion in curved spacetime
  Can NC form be formulated in dynamically triangulated lattice?
  How about Clifford product?