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Numerical Analysis of Four Dimensional Quantum Gravity

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Mainly collaborated with H. S. Egawa (Tokai University)
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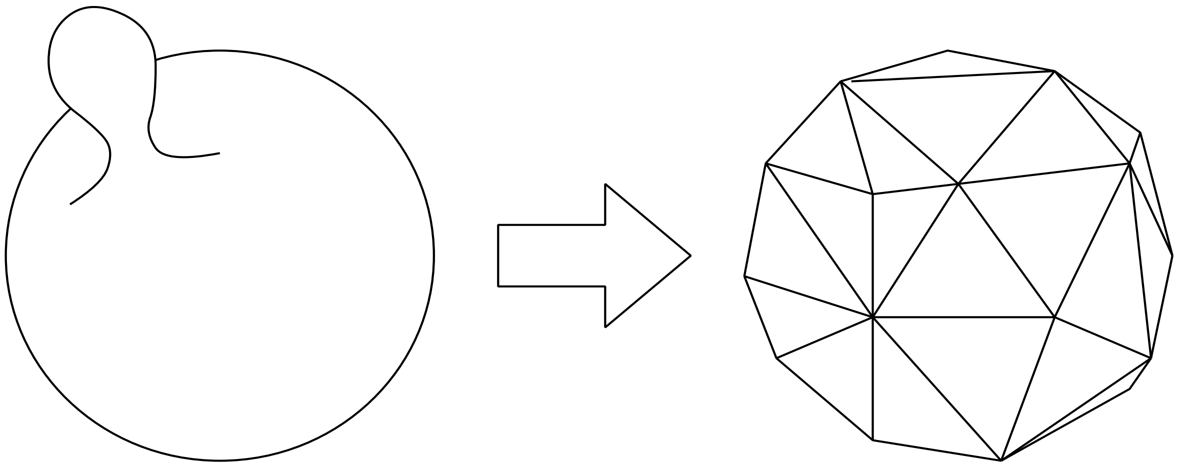


- Contents -

1. Introduction

Simplicial Quantum Gravity(SQG) is known as a powerful tools of studying the quantum geometries. SQG has been investigated from several points of view.

J. Ambjørn, B. Durhuus and T. Jonsson, "Quantum Geometry", Cambridge university press, 1997;

R. Loll, Class. Quant. Grav. **15** (1998), 799;

D. A. Johnston, Nucl. Phys. B (Proc. Suppl.) **53** (1997), 43;

G. Thorleifsson, Nucl. Phys. B (Proc. Suppl.) **73** (1998), 133.

Especially, in two-dimensions, the random surface constructed by SQG is equivalent to that of the random Matrix model. Then, 2D SQG has been studied with the numerical and the analytical method.

In this talk, we show the numerical results of 4D SQG.

Strategy of investigation of 4D SQG

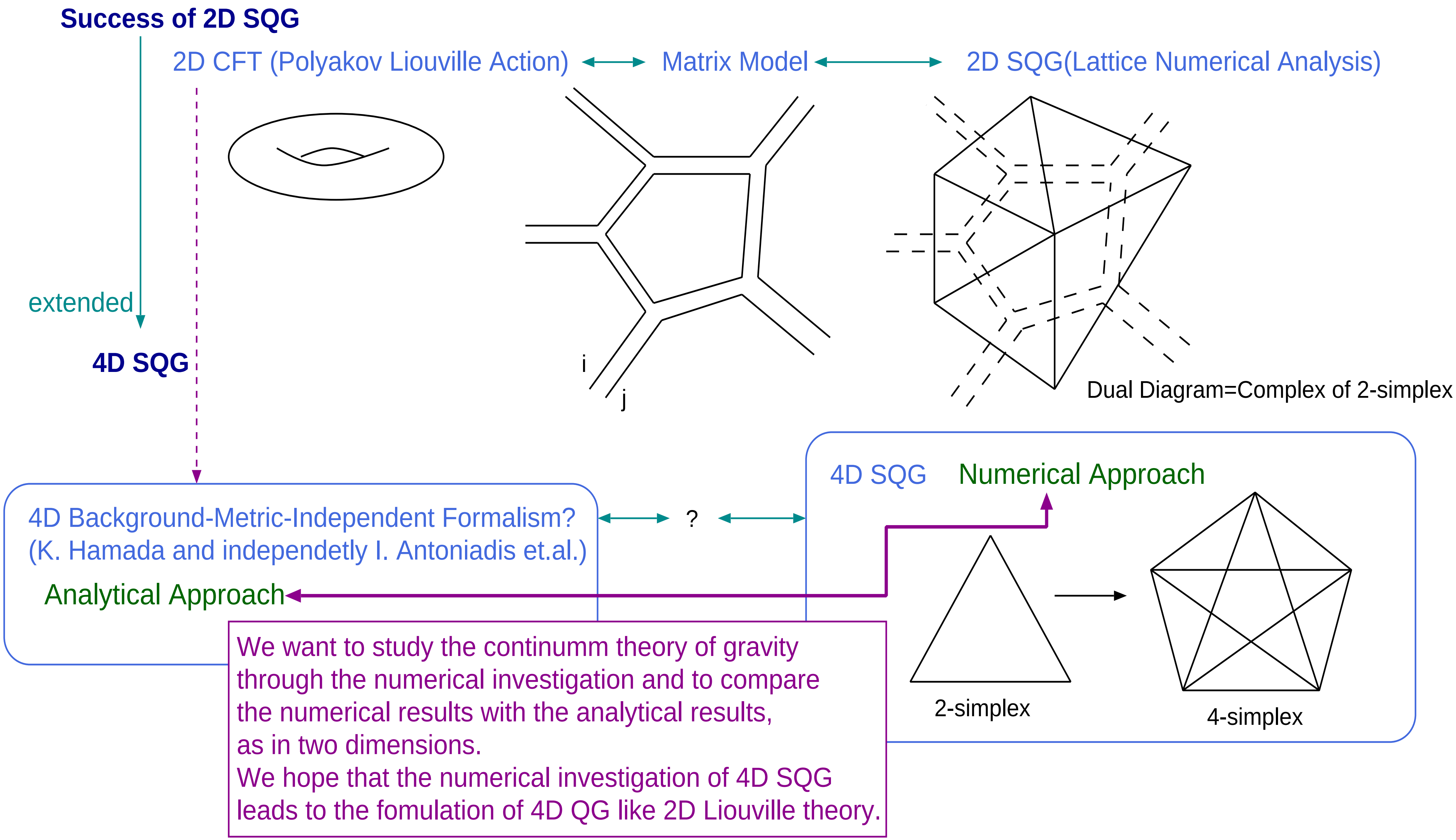


Figure 2: Our claim of relation between theories.

First, let us remember the reasons to study 4D SQG .

- Why is the SQG model studied?

  It is not yet known how to give a constructive definition of quantum gravity in 4D.

  Then, we consider 4D SQG based on the experience in 2D SQG.

  We hope that the continuum theory of 4D QG can be obtained through 4D SQG model.

• What should be investigated with SQG?

Since SQG is the lattice field theory of the gravity, the partition function is performed by replacing the path integral over the metric to the sum of the fragmentations of the discretized geometries $\mathcal{T}$,

$$\int \mathcal{D}g_{\mu\nu} \Longrightarrow \sum_{\mathcal{T}} .$$

While 2D SQG is a free theory, 4D SQG has the gravitational coupling.

$$\Downarrow$$

In order to exist the unique continuum limit, the phase transition between the strong coupling phase and the weak coupling phase should be of second-order.

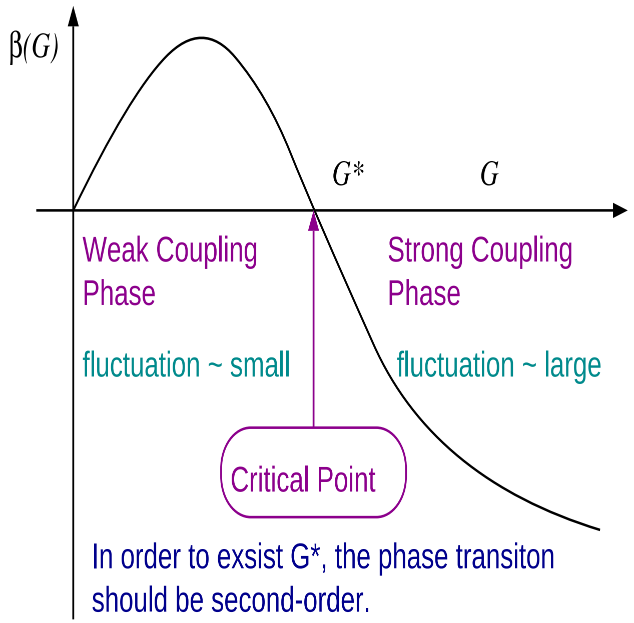


Figure 1: The context of the field theory and the phase diagram, where G is the coupling constant.

Overview of Developments of SQG

Let us review the developments of SQG briefly.

- 2D case

– String Susceptibility Exponent

The d-dimensional string susceptibility exponent $\gamma^{(d)}$ is defined from the partition function as,

$$Z(N_d) \sim N_d^{\gamma^{(d)}-3} e^{\mu N_d},$$

where N_d is the volume of d-dimensional surface.
In 2D, $\gamma^{(2)}$ is known to be given by the formula,

$$\gamma^{(2)} = \frac{1-h}{12}\left[c - 25 - \sqrt{(25-c)(1-c)}\right] + 2,$$

where c is the central charge and h is the number of handle.

V. G. Knizhnik, A. M. Polyakov and A. B. Zamolodchikov, Mod. Phys. Lett. **A3** (1988), 819;
J. Distler and H. Kawai, Nucl. Phys. **B321** (1989), 509; F. David, Mod. Phys. Lett. **A3** (1988), 1651.

– Instability for $\gamma^{(d)} < 0$

Estimate the partition function of the configuration with pinched spheres (Fig. 3),

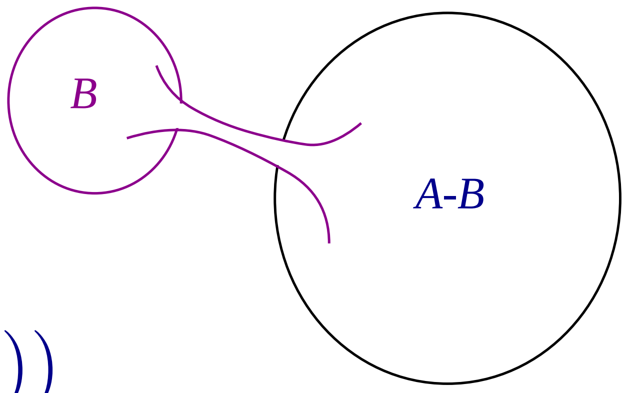


Figure 3: Sphere with a pinch.

$$\begin{aligned} Z &\sim \int dB\,(Z(B)B)\,(Z(A-B)(A-B)) \\ &\sim A^{\gamma^{2(d)}-3}e^{\mu A}, \end{aligned}$$

where B is the volume of the pinched sphere and A is the volume of surface.

By comparing between the sphere case and the pinched case, the configuration has many pinches for

$$\gamma^{(d)} > 0,$$

which means that it takes the shape of a branched polymer.

Let us show the comparison between the numerical results and the analytical results about the string susceptibility exponent $\gamma^{(2)}$.
The numerical simulations using Grand-Canonical ensemble method give excellent solution that is in agreement with the analytical ones.

S.Oda, N.Tsuda and T.Yukawa, Nucl. Phys. B (Proc.Suppl.) 63 (1998) 733.

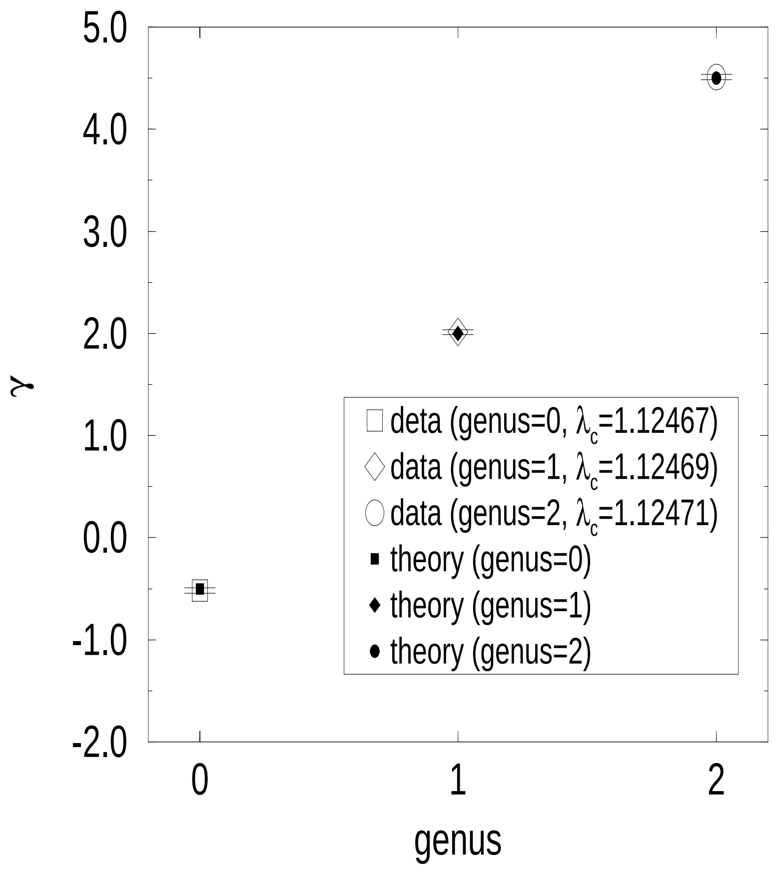


Figure 4: The String susceptibility exponents (γ) of a sphere, torus and double-torus for the $c = 0$ pure gravity case.

The discretized approach in 2D, implemented by the matrix model, exhibits the same behavior found in the Liouville theory.

– Loop Length Distribution

The distribution of boundary loops with the loop length L within a geodesic distance D is calculated with both numerical and analytical method.

H. Kawai, N. Kawamoto, T. Mogami and Y. Watabiki, Phys. Lett. **B306** (1993), 19;

N. Tsuda and T. Yukawa, Phys. Lett. **B305** (1993), 223.

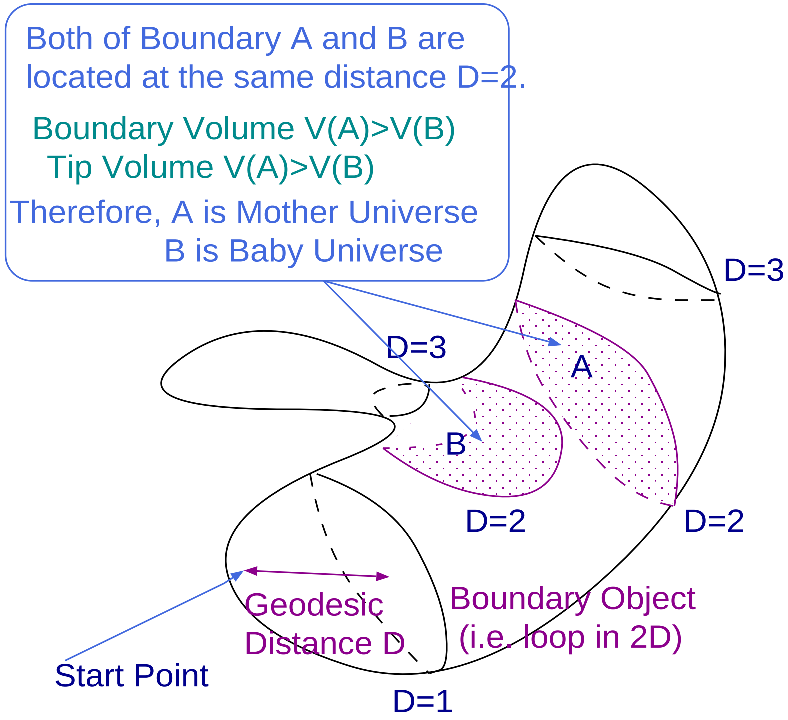


Figure 5: The schematic image of the boundary loops.

Geodesic distance D is defined as the shortest length on the simplex manifold between two simplices.

Excellent agreement between the numerical results and the analytical approaches has been obtained. (Fig.6)

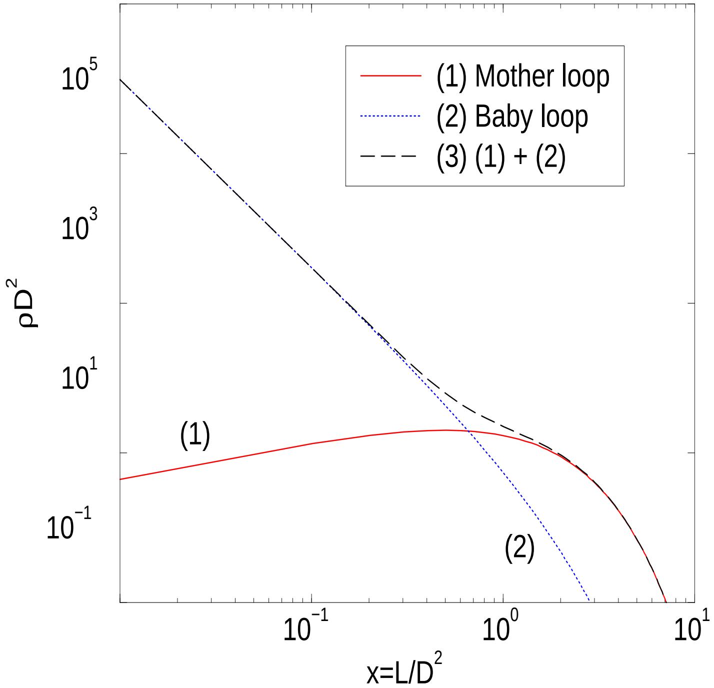


Figure 6: The loop length distribution function $\rho(x, D)$, consisting of the mother loop(1) and baby loops(2) distribution function.

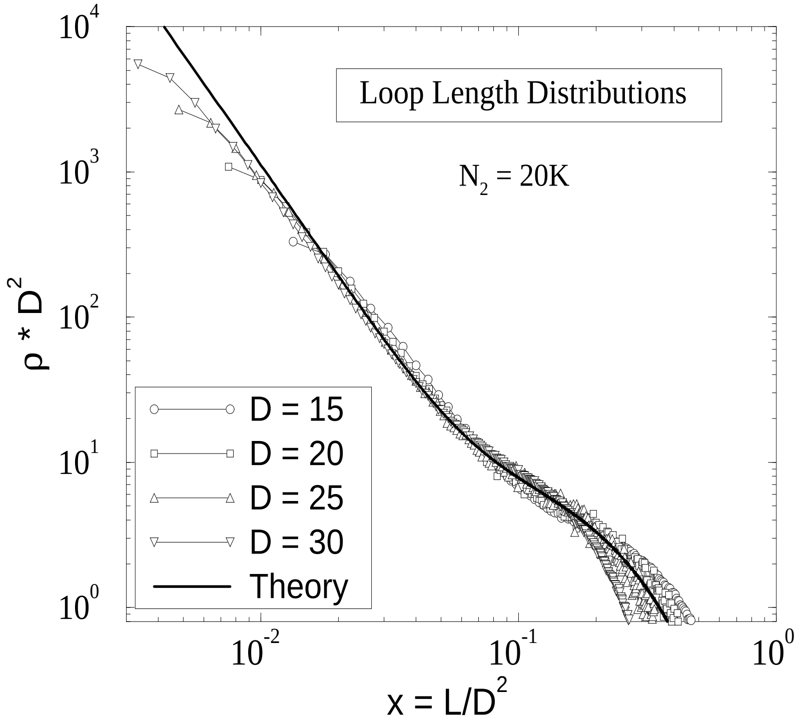


Figure 7: The distribution of boundary loops with the loop length L within a geodesic distance D. The volume is $N_2 = 20,000$.

The distribution function of the boundary loops can be divided into two parts, *i.e.* Mother Universe and Baby Universe.

- # 4D case

Pure Gravity Case

J. Ambjørn and J. Jurkiewicz, Phys. Lett. B **278** (1992), 42; Nucl. Phys. B **451** (1995), 643;
M. E. Agishtein and A. A. Migdal, Nucl. Phys. B **385** (1992), 395; Mod. Phys. Lett. A **7** (1992), 1039;
S. Catterall, J. Kogut and R. Renken, Phys. Lett. B **328** (1994), 277.
S. Catterall, J. Kogut, R. Renken and G. Thorleifsson, Nucl. Phys. B **468** (1996), 263.
P. Bialas, Z. Burda, A. Krzywicki and B. Petersson, Nucl. Phys. **B472** (1996), 293.
Bas V. de Bakker, Phys. Lett. **B348** (1995), 35.
H. S. Egawa, T. Hotta, T. Izubuchi, N. Tsuda and T. Yukawa, Prog. Theor. Phys. **97** (1997), 539; Nucl. Phys. B (Proc. Suppl.) **53** (1997), 760.

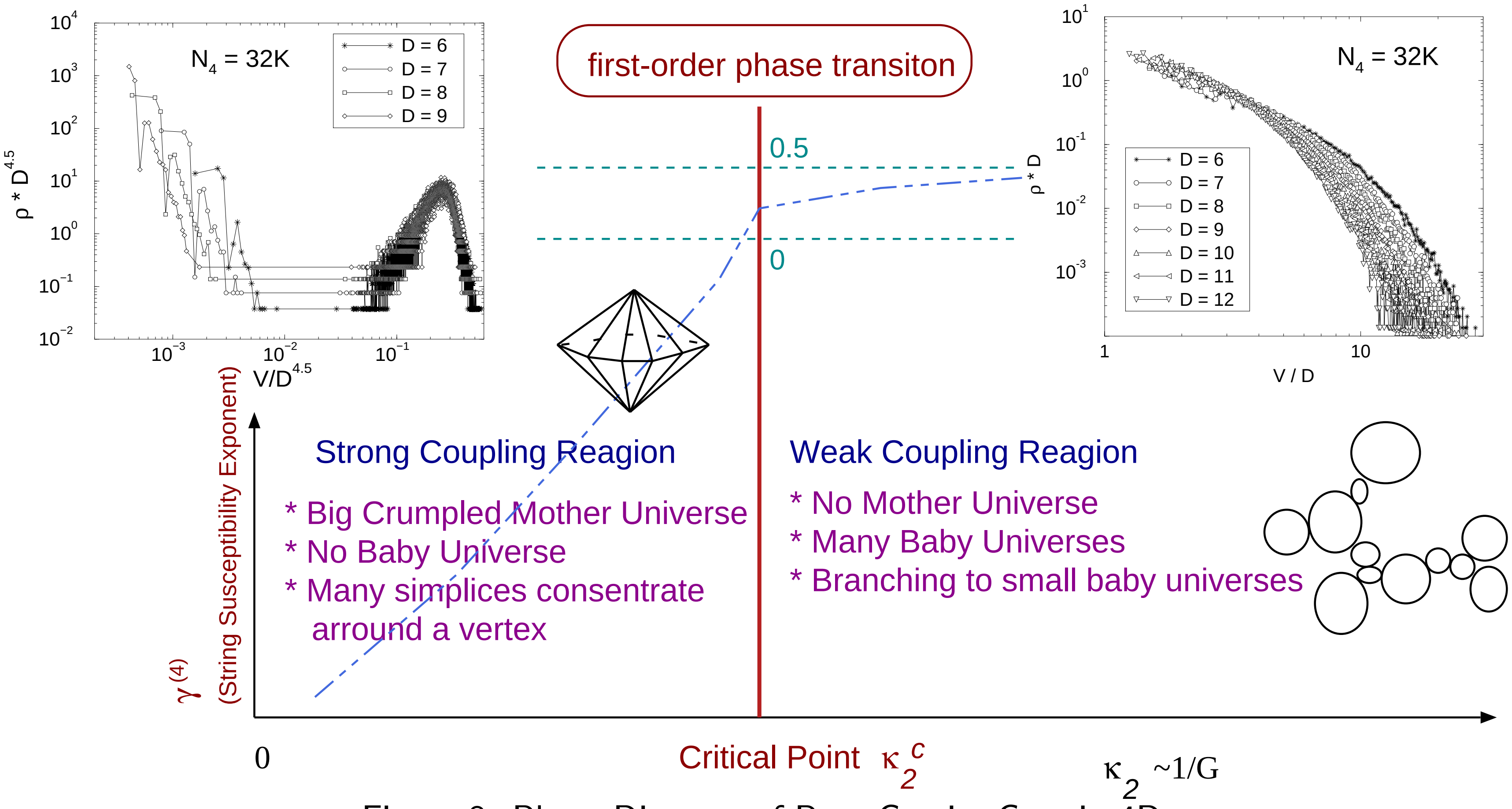


Figure 8: Phase Diagram of Pure Gravity Case in 4D

2. Model and Numerical Implementation

Our modification is to introduce the additional matter fields. We consider the system of simplicial gravity coupled to U(1) gauge fields and Massless scalar field, $S = S_{EH} + S_A + S_X$. We use Einstein-Hilbert action in 4D for gravity,

$$S_{EH}[\Lambda, G] = \int d^4x\sqrt{g}(\Lambda - \frac{1}{G}R),$$

where Λ is Cosmological constant, G is Newton's constant.

Then, Lattice action for Pure Gravity is given as,

$$S_{EH}[\kappa_2, \kappa_4] = \kappa_4 N_4 - \kappa_2 N_2,$$

where $\kappa_4 = \Lambda - \frac{2}{G}\cos^{-1}\left(\frac{1}{4}\right)$, $\kappa_2 = \frac{2\pi}{G}$ and N_i is number of i-simplices.

For matter part, we introduce $\mathrm{U}(1)$ gauge fields and Massless scalar fields. From the analysis of the p-form field L_p on lattice, each matter action is given by following form ,

Scalar Field X_i on vertex i (0-form)

$$S_X = \sum_{l_{ij}} o(l_{ij})(X_i - X_j)^2,$$

where $o(l_{ij})$ is number of 4-simplex sharing link l_{ij},

Vector Field A_{ij} on link l_{ij} (1-form)

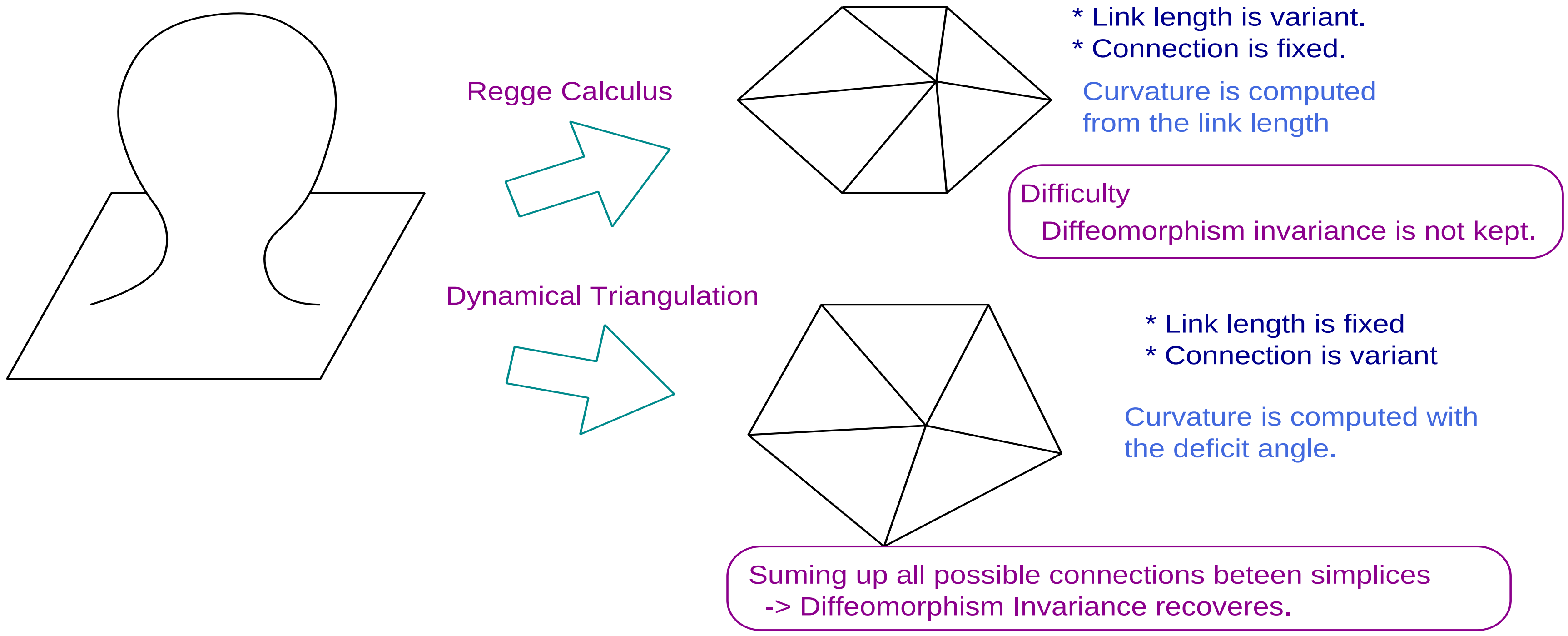
$$S_A = \sum_{t_{ijk}} o(t_{ijk})(A(l_{ij}) + A(l_{jk}) + A(l_{ki}))^2,$$

where $o(t_{ijk})$ is number of 4-simplex sharing triangle t_{ijk}.

It gives correct thermo-dynamical limit for large N_4 because of $S_X \propto N_4$ and $S_A \propto N_4$ in real (not dual) lattice space.

Numerical Implementation

In order to quantize using Feynmann's path integral method, we discretize Euclidean space filling with 4-simplices.
Let us show the schematic image of the discretization.



Since the diffeomorphism invariance of the discretized space is not trivial in the Regge calculus method, we use the dynamical triangulation method.

The path integral over the infinite dimensional space of metrics and matter fields is replaced by a sum over all possible triangulations $\mathcal{T}$ of space and matter configurations X as,

$$\int \mathcal{D}g_{\mu\nu}\mathcal{D}X \Longrightarrow \sum_{\mathcal{T}} \sum_{X_{\mathcal{T}}},$$

where $X_{\mathcal{T}}$ denotes the matter configuration on the triangulated manifold $\mathcal{T}$. In the dynamical triangulation approach, all possible configurations are generated by the set of (p, q)-moves.

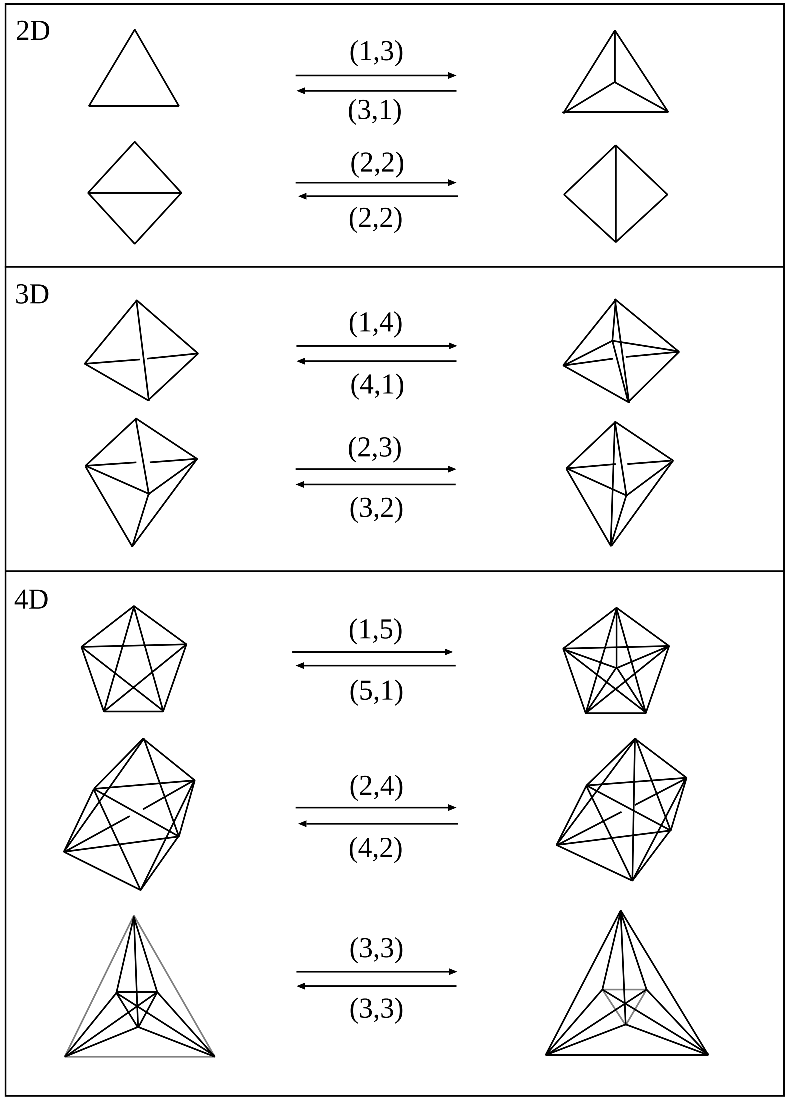


Figure 9: (p, q) moves in Euclidean 2, 3 and 4 dimensions.

3. Developments of Four-dimensional SQG

- Phase Diagram

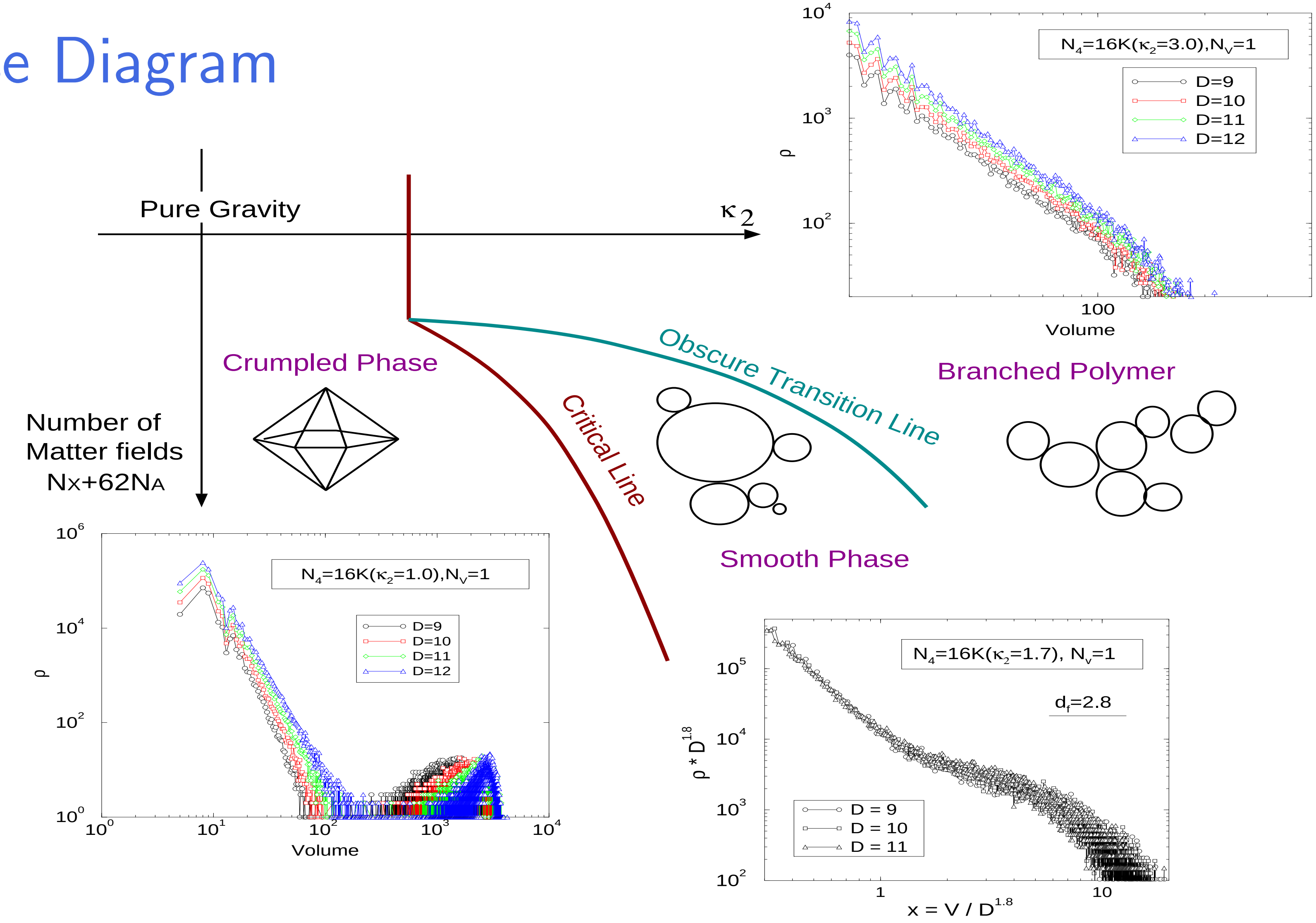


Figure 10: Phase diagram of 4D SQG coupled with matter fields. N_X is the number of scalar fields and N_A is the number of vector fields.

H.S.Egawa,S.Horata,N.Tsuda and T.Yukawa, Prog.Theor.Phys. 106 (2001);hep-lat/9809028,990848.

– String Susceptibility Exponent $\gamma^{(4)}$

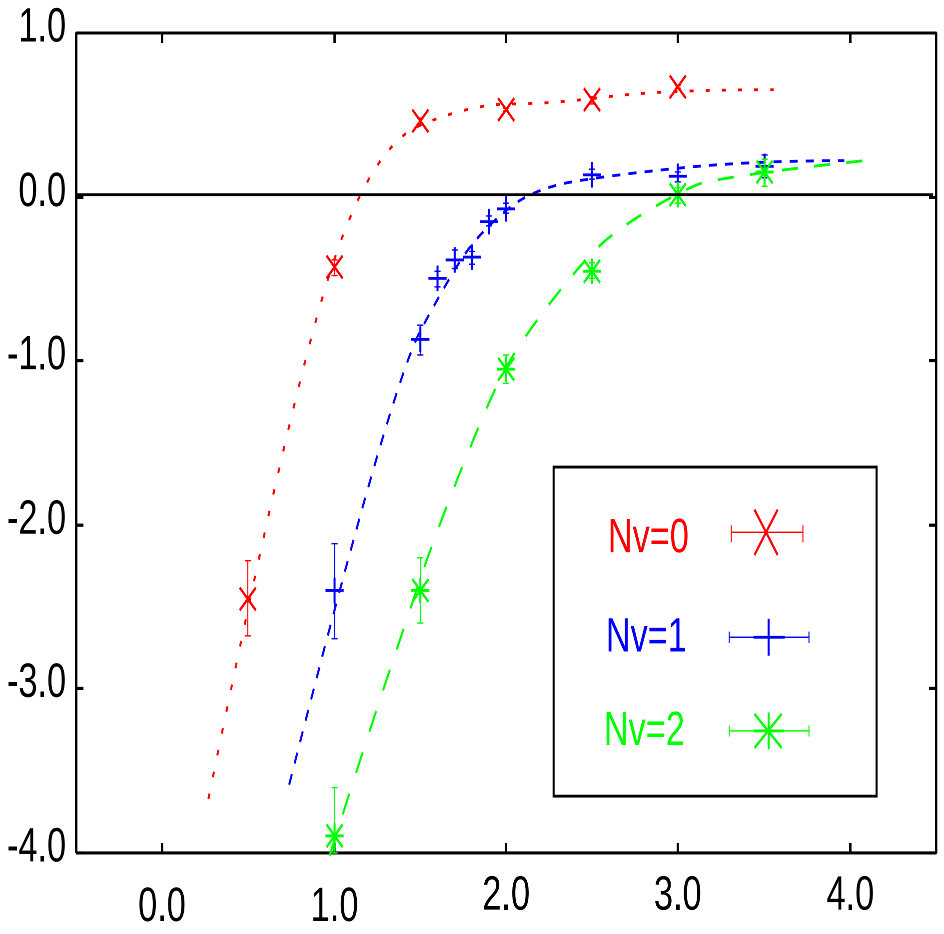


Figure 11: Numerical results of the case of adding vector fields. N_V denotes the number of the vector fields.

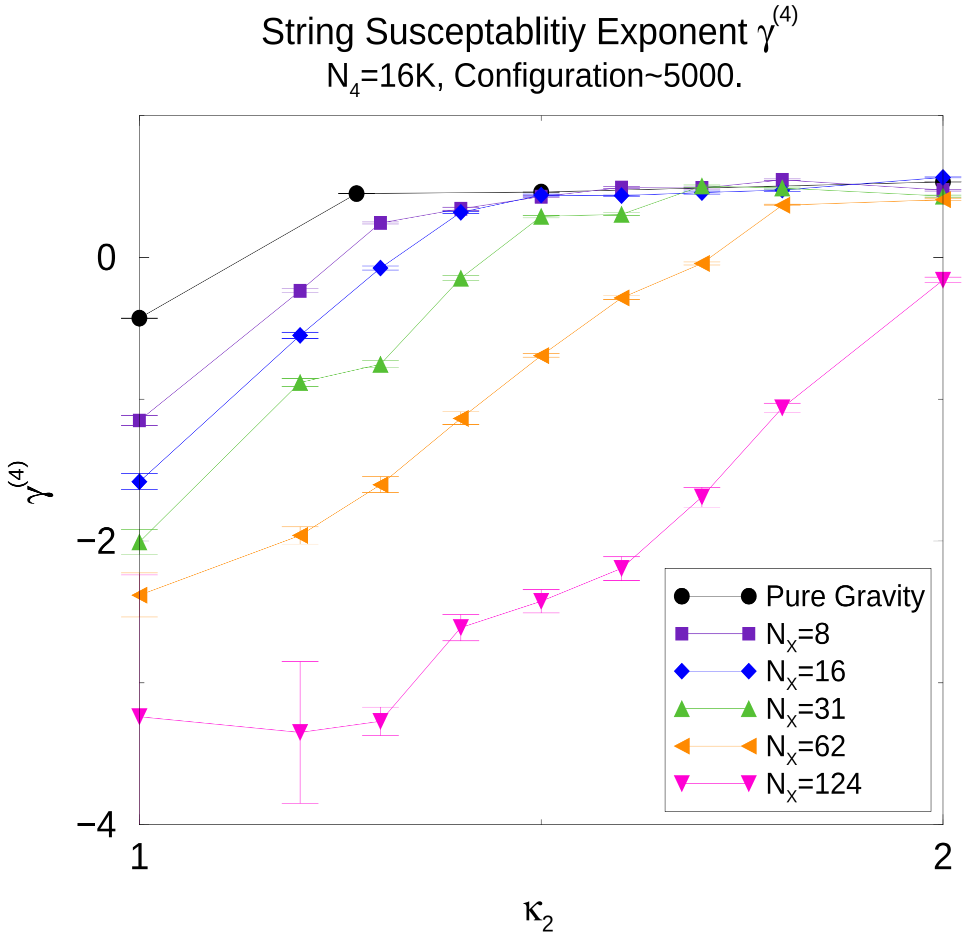


Figure 12: Numerical results of the case of adding scalar fields. N_X denotes the number of the scalar fields.

– Matter-Dependence of $\gamma^{(4)}$

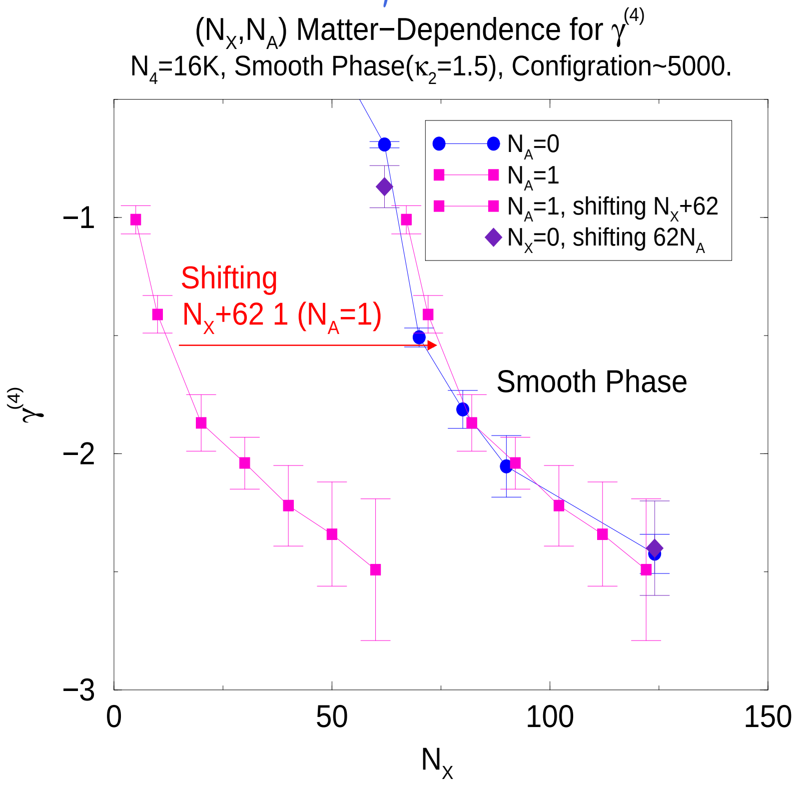


Figure 13: The String Susceptibility Exponent $\gamma^{(4)}$ vs $N_X + 62N_A$.

Numerical results suggest,

$$\gamma^{(4)} = \gamma^{(4)}(N_X + 62N_A).$$

- Second- or Higher-order Phase Transition

Finite Size Scaling of Node Susceptibility

$$\chi = (< N_0^2 > - < N_0 >^2)/N_4$$

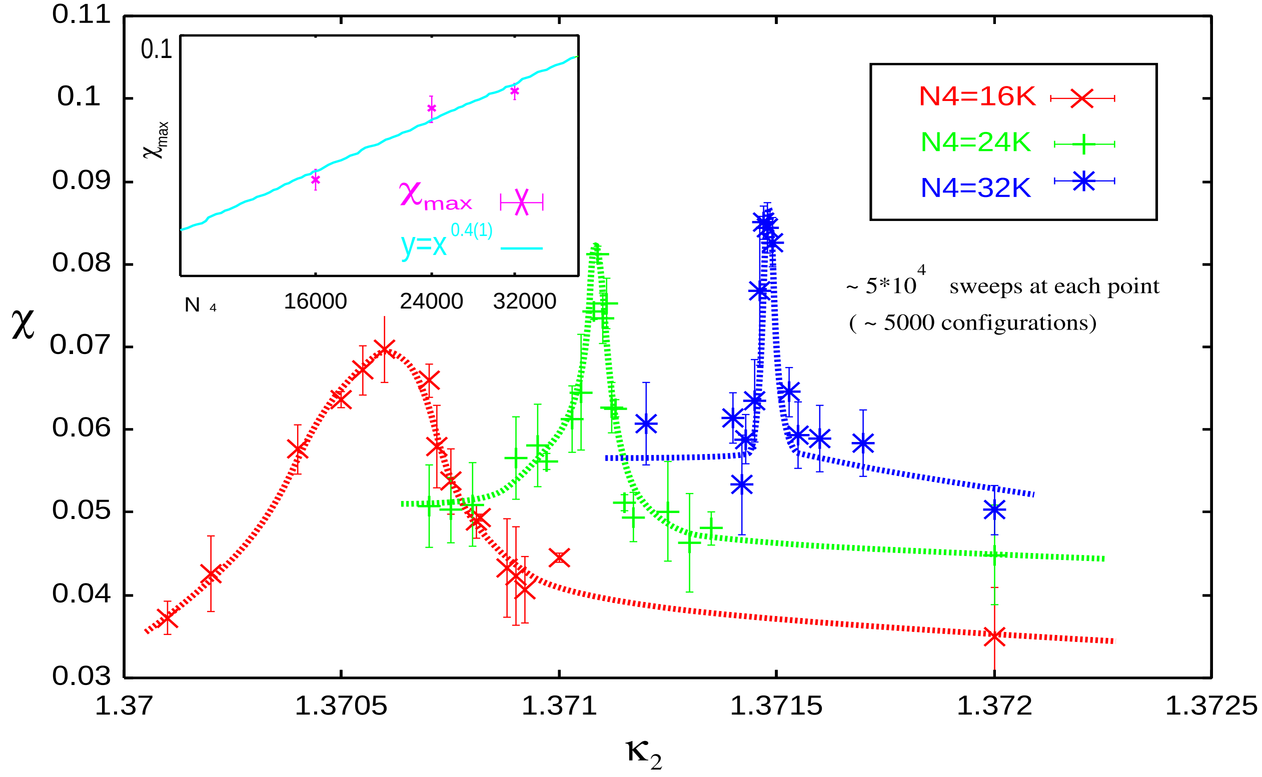


Figure 14: The node susceptibility (χ) plotted versus the coupling constant (κ_2) for $N_A = 1$.

Numerical Results;

$$\chi_{max} \sim N_4^{\Delta}, \quad \Delta \sim 0.4(1) < 1.$$

(Pure Gravity (first-order), $\Delta = 0.81$ (by Bas V. de Bakker, hep-lat/9603024))

Node Histogram

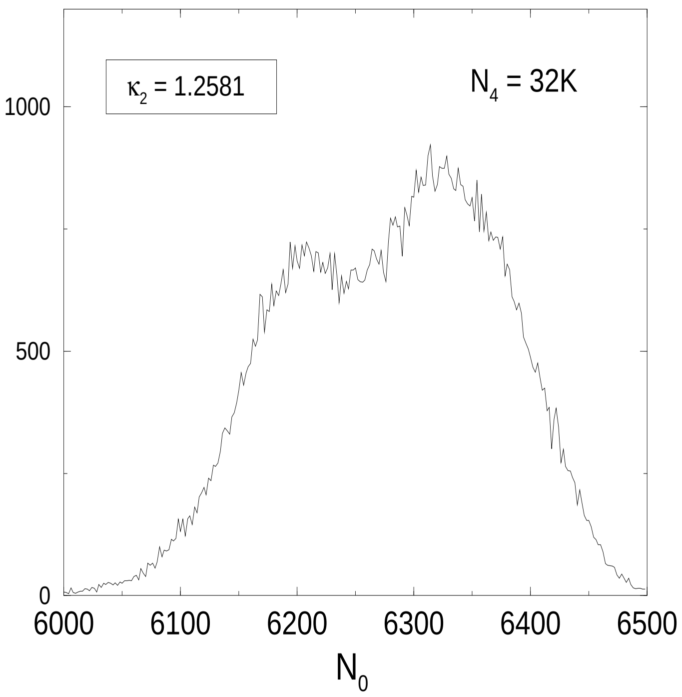


Figure 15: The histogram of node(N_0) near the critical point in the case of pure gravity.

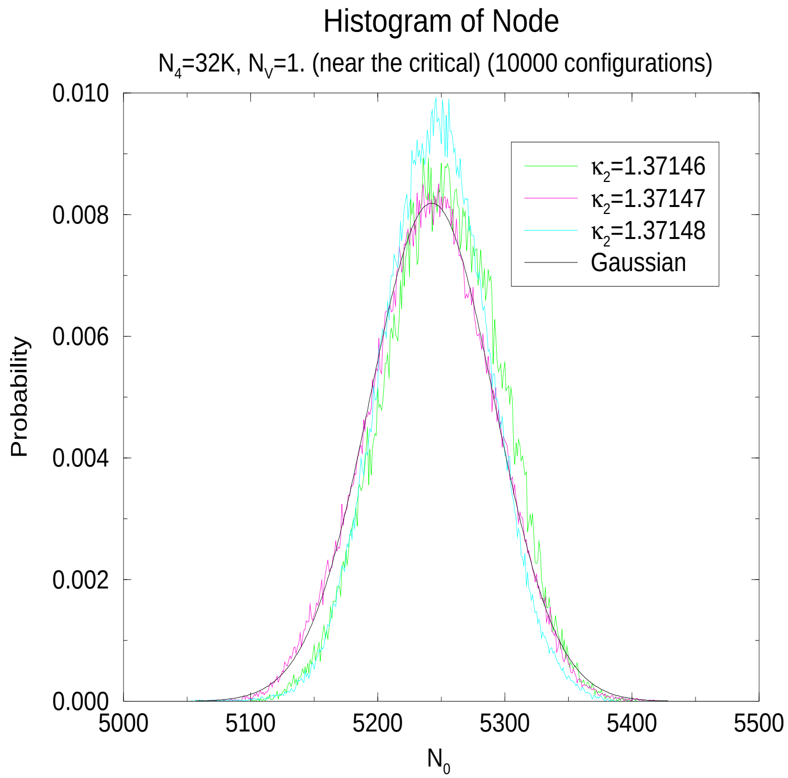


Figure 16: The histogram of node(N_0) near the critical point in the case of $N_A = 1$.

Numerical Results show the phase transition between Crumpled Phase and Smooth Phase is not first-order.

4. Improved Numerical Results

4D SQG with Matter fields
$\Longrightarrow$ Second- or Higher-order Phase Transition

The numerical results suggest that adding the matter fields is essential to cure the fatal situation of the pure gravity case of 4D SQG.

However, we consider that many problems are yet to be solved.

Unsolved Problems for 4D SQG

- How do the matter fields affect 4D SQG?
- What is continuum theory of SQG?

In 2D case, the matter fields play important role for the phase structure as the parameter for the central charge.

Measuring the string susceptibility exponent $\gamma^{(4)}$

The following ways to measure the string susceptibility exponent $\gamma^{(4)}$ are known.

- Canonical Ensemble Method with the Minimum Necked Baby Universe(MINBU) distributions
- Grand-Canonical Ensemble Method

Previously, to simplify the calculation, we perform the MINBU method to estimate the string susceptibility exponent.

It is expected that $\gamma^{(4)}$ computed by the Grand-Canonical method has smaller finite size effect than that by MINBU method.

- Canonical Ensemble Method and MINBU

Since 4D ergodic geometrical moves change volume, the term (δS) for fixing volume is introduced as,

$$Z = \sum_{4-\text{simplices}} e^{-\kappa_4 N_4 + \kappa_2 N_2 - \delta S(N_4, V_4)} \delta_{N_4, V_4},$$

where volume fixing term δS is,

$$\delta S(N_4, V_4) = \delta\kappa_4 \left(N_4 - V_4\right)^2,$$

where $\delta\kappa_4$ is free parameter and V_4 is size of target volume.Measurement timing is visited when $N_4 = V4$.

The MINBU method is to count the volume of baby universe connected to the mother universe with the 1 simplex (minimum neck). The probability of the baby universe is

$$P_A[B] \sim A^{\gamma^{(d)}-1} \left(\frac{B}{A}\right)^{\gamma^{(d)}-2} \left\{1 - \left(\frac{B}{A}\right)\right\}^{\gamma^{(d)}-2},$$

The $\gamma^{(d)}$ can be calculated from the slope of the distribution function of the MINBU.

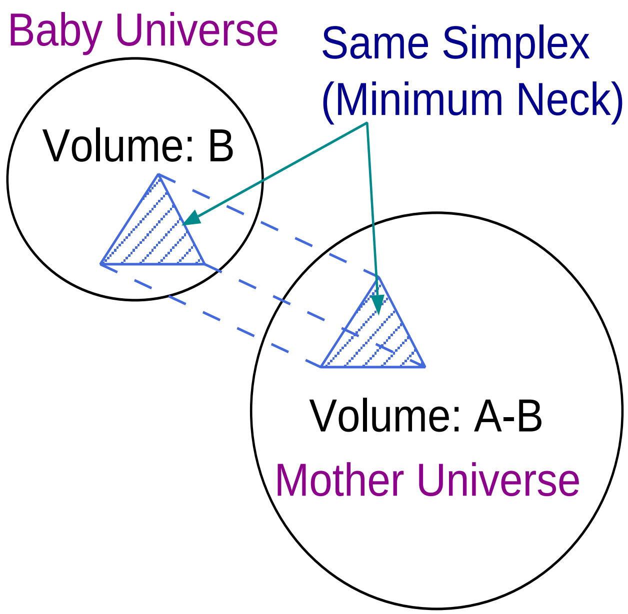


Figure 17: The Minimum Necked Baby Universe

However, for the MINBU method, the string susceptibility exponent has the finite effect from the volume of the baby universe.

- Grand-Canonical simulation and partition function

Otherwise, the string susceptibility exponent $\gamma^{(4)}$ is determined with the size dependence of the partition function in the Grand-Canonical ensemble.

Since the Grand-Canonical ensemble method does not require the target volume, we remove the term for fixing volume δS and partition function is,

$$Z(\kappa_2, \kappa_4, N_A, N_X) = \sum_T e^{-\kappa_4 N_4 + \kappa_2 N_2} \prod_{N_A, N_X} \left(\int \prod_{l \in T} dA_l e^{-S_A(A_l)} \int \prod_{i \in T} dX_i e^{-S_X(X_i)} \right),$$

where geometries and matter fields are updated with using Metropolis method.

In order to analyze partition function, we observe probability ($P(N_4)$) of a surface with volume N_4.

From the analogy of 2D, we expect that partition function obeys KPZ formula for large-N_4 limit as,

$$Z \approx \sum_{N_4} N_4^{\gamma^{(4)}(\kappa_2)-3} e^{-(\kappa_4-\kappa_4^c(\kappa_2))N_4}.$$

It gives that probability $P(N_4)$ for large N_4 behaves approximately,

$$P(N_4) \propto N_4^{\gamma^{(4)}-3} e^{-(\kappa_4-\kappa_4^c)N_4}.$$

If we tune the parameter κ_4 close to κ_4^c, we can evaluate string susceptibility exponent $\gamma^{(4)}$ by plotting $\ln P(N_4)$ as a function of $\ln N_4$.

How to tune parameter κ_4^c

- $\kappa_4 < \kappa_4^c \cdots$
  The system expand exponentially.
- $\kappa_4 > \kappa_4^c \cdots$
  The system shrink exponentially.

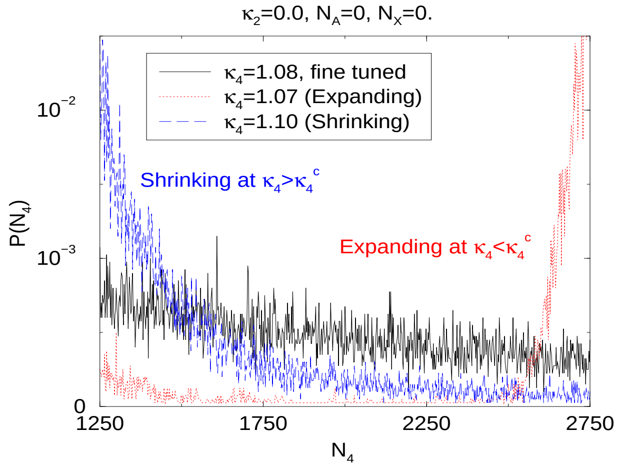


Figure 18: Shrinkage and Expansion

We set κ_4 near to κ_4^c and find precisely from

$$\frac{\ln P(N_4)}{\ln N_4} = (\gamma^{(4)} - 3) - (\kappa_4 - \kappa_4^c)\frac{N_4}{\ln N_4} + O(1/\ln N_4).$$

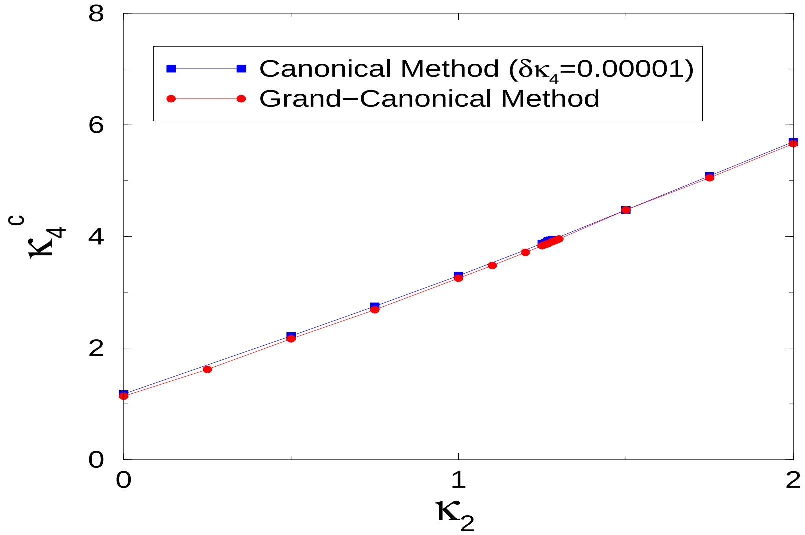


Figure 19: Plot κ_4^c versus κ_2

After tuning κ_4, we generate configurations constraining N_4 within a upper- and a lower-bound $(N_4^{min} \leq N_4 \leq N_4^{max})$.

Numerical result

We set N_4^{max} to 10000 and N_4^{min} to 1000.

In Fig. 20, we show $\ln P(N_4)$ versus $\ln N_4$.

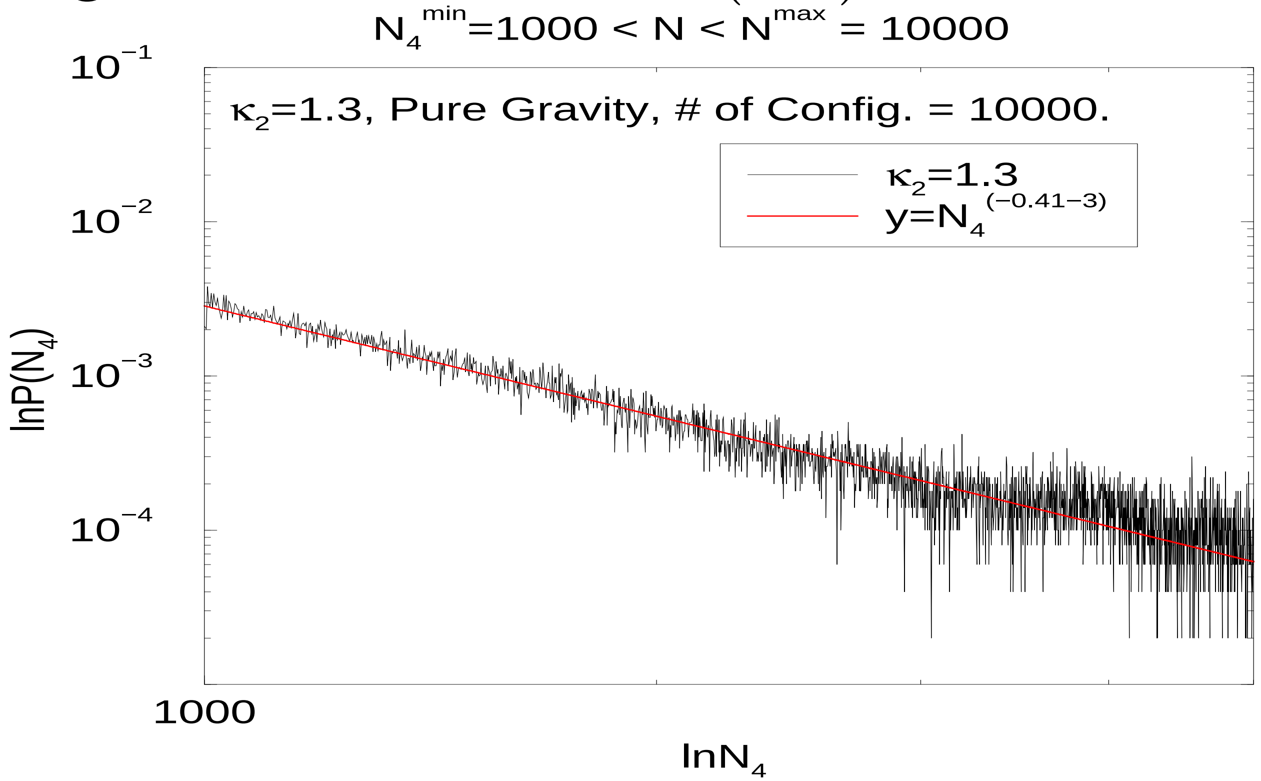


Figure 20: Plot $\ln P(N_4)$ versus $\ln N_4$ at $\kappa_4 \sim \kappa_4^c$

Numerical result suggests that partition function of 4D DT obeys KPZ type partition function as 2D DT case.

In the region where the counting of the MINBU is failed, the Grand-Canonical ensemble method give a correct value.

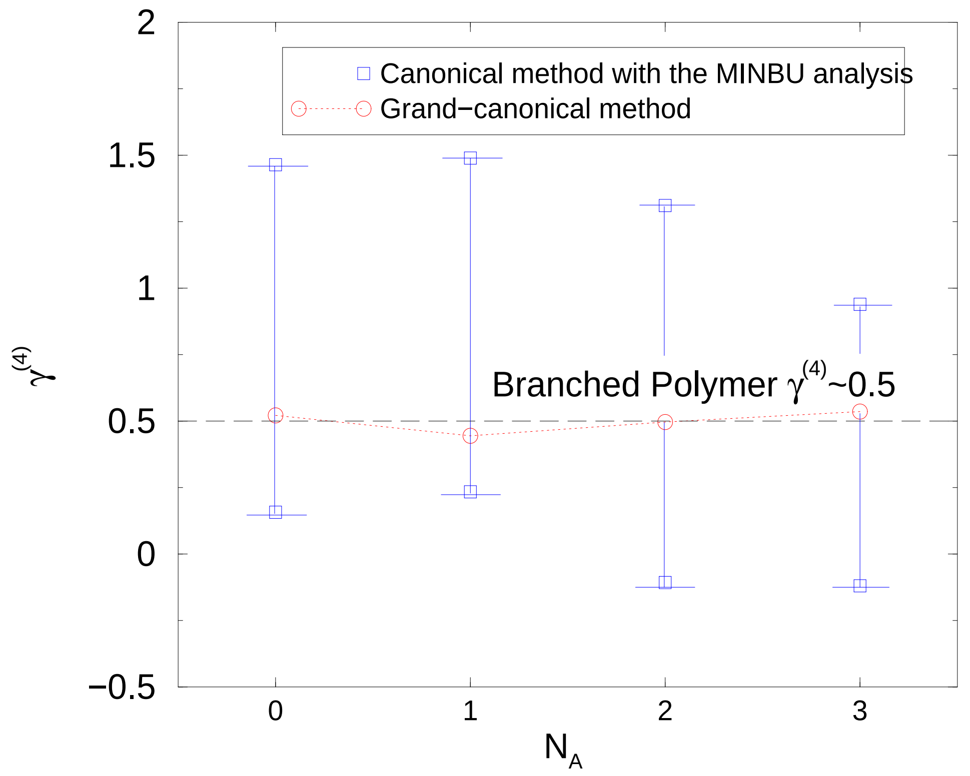


Figure 21: The string susceptibility exponent $\gamma^{(4)}$ at the weak coupling region, $\kappa_2 = 6.0$.

We expect that the grand-canonical ensemble method avoid a finite-size effect and that the grand-canonical ensemble method gives better ergodicity.

- Comparison with analytical one

In order to compare with the analysis such as conformal gravity in 4D, we notice string susceptibility exponent $\gamma^{(4)}$ near critical point κ_2^c.

From the analogy of 2D, we expect that $\gamma^{(4)}$ is given by,

$$\gamma^{(4)} = \frac{b}{8}\left(1 + \sqrt{1 - \frac{4}{b}}\right),$$

where b is similar to central charge in 4D and includes the contribution of graviton mode.

From conformal gravity analysis, $\gamma^{(4)}$ has been computed for the conformal limit that graviton self interaction is switched off.

$$\gamma^{(4)} = \frac{b_0}{8}\left(1 + \sqrt{1 - \frac{4}{b_0}}\right), \, b_0 = 0.0028(N_X + 62N_A) + 4.38.$$

In Fig. 22, we plot b computed by numerical results.

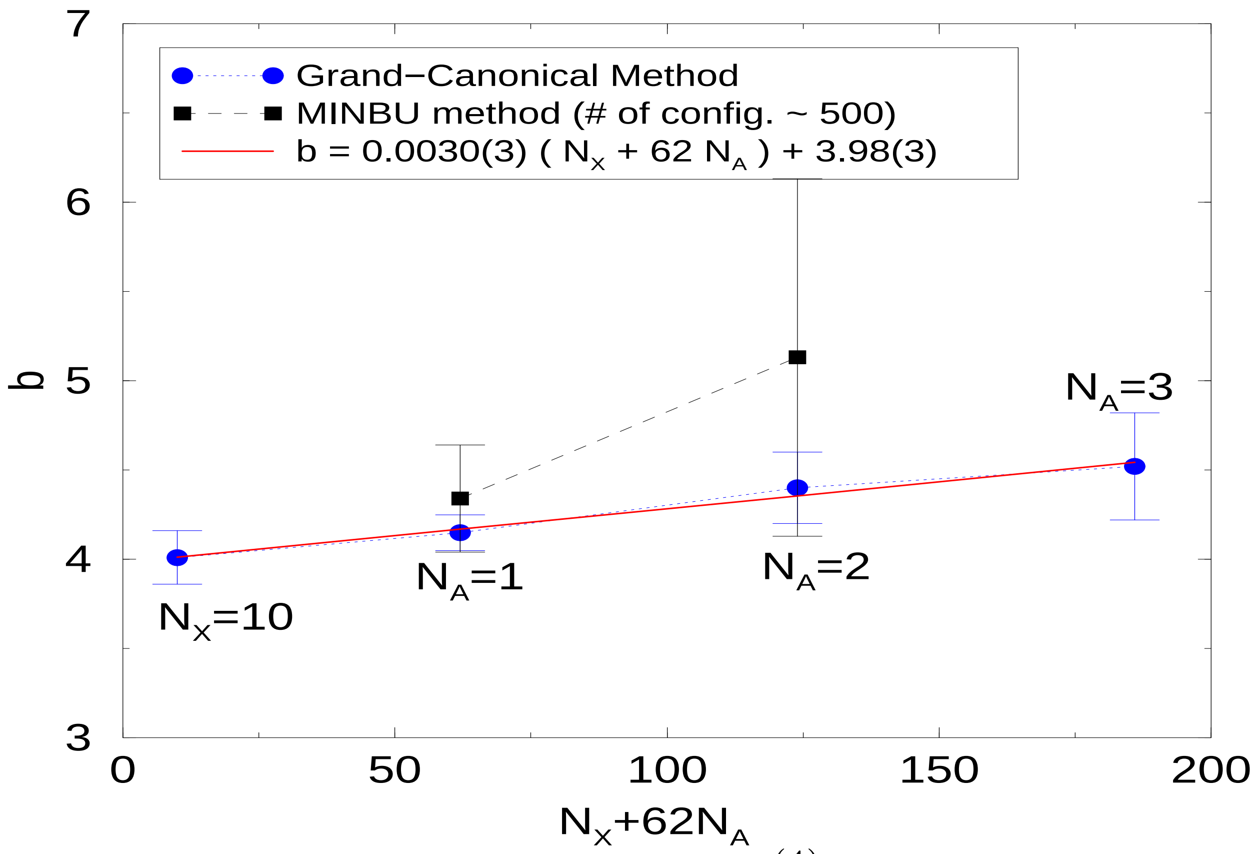


Figure 22: Plot the b estimated from $\gamma^{(4)}$ versus $N_X + 62N_A$.

However, it is hoped that $\gamma^{(4)}$ can be calculated with perturbation of traceless mode in 4D Background Metric Independent formalism (BMI).

K.Hamada, Prog.Theor.Phys. 105 (2001) 673-690; hep-th/0105069,0203250.

5. Conclusion and Future Developments

We carry out the Monte-Calro simulation of the 4D SQG coupled to the scalar and vector fields.

Numerical Results

- Three phases appear (Crumpled, Smooth, B.P.)

Crumpled Phase	...	$\gamma^{(4)} \to -\infty$,
Smooth Phase	...	$\gamma^{(4)} < 0$,
Branched Polymer	...	$\gamma^{(4)} > 0$.

- Second- or Higher-order phase transition (Crumpled $\leftrightarrow$ Smooth)

  It indicates the existence of non-trivial fixed-point.

- KPZ type partition function appears.

$$Z = \sum_{N_4} N_4^{\gamma^{(4)}(\kappa_2)-3} e^{(\kappa_4-\kappa_4^c(\kappa_2))N_4}.$$

- $\gamma^{(4)}$ can be determined.
  - It is expected that Grand-Canonical ensemble method has smaller finite size effect than MINBU method.
  - The central charge in 4D is estimated as,

    $$b = 0.0030\,(N_X + 62N_A) + 3.98(3).$$

    From BMI formalism, b_0 is computed as,

    $$b_0 = 0.0028\,(N_X + 62N_A) + 4.72.$$

    The difference from the result of the conformal limit of 4D BMI suggest that not only conformal mode but also dynamical graviton mode should be contained.

- What is continuum theory of quantum gravity?

  Although it is known that the perturbation from the flat space is failed, the numerical results suggest that the perturbation from the conformal field theory in 4D may be success. Furthermore, it is indicated that, for 4D Background Metric Independent formalism, the traceless 2-tensor field should be treated dynamically.

In order to look further into the physics of continuum theory of quantum gravity, we need to continue to perform simulations using Grand-Canonical ensemble method and proceed with analysis of the field theory of the gravity at the same time.

Future developments

- Function form of the Grand-Canonical Partition Function
  - κ_2 dependence of the string susceptibility exponent
- Geometries and Correlation on simplicial manifold
  - Geodesic Distance and Coordinates
  - Scalings and Geometries

I would like to thank you for listening.