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CP Breaking in Lattice Chiral Gauge Theories

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1. Introduction

• Transparent understanding of chiral symmetry on the lattice has become available after the (re)discovery of

⋆ Ginsparg-Wilson relation **(Hasenfratz, Neuberger)**

$$\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D$$

Hermitian operator H

$$H = a\gamma_5 D = aD^\dagger \gamma_5 = H^\dagger$$

$$\gamma_5 H + H\gamma_5 = 2H^2$$

or more generally $\gamma_5 H + H\gamma_5 = 2H^2 f(H^2)$ with a regular $f(x)$ **(Fujikawa)**

• Important consequences:

⋆ Exact lattice chiral symmetry of the action **(Lüscher)**

⋆ and its "quantum" breaking (the axial anomaly) due to nontrivial Jacobian **(Kikukawa-Yamada, Adams, H.S., Fujikawa,. . .)**

⋆ Reproduces the desired breaking pattern of the Ward-Takahashi identity in QCD

Flavor non-singlet axial ⇒ Exact symmetry
Flavor singlet U(1) axial ⇒ Anomalous

⋆ (analytic) Index theorem

(Hasenfratz-Laliena-Niedermayer)

$$\Gamma_5 = \gamma_5 - H$$

$$\{\Gamma_5, H\} = 0 \quad \text{(but note } \Gamma_5^2 \neq 1\text{)}$$

then

$$H\varphi_n(x) = \lambda_n \varphi_n(x) \rightarrow H\Gamma_5\varphi_n(x) = -\lambda_n \Gamma_5 \varphi_n(x)$$

and thus

$$\begin{aligned}
\mathrm{Tr}\,\Gamma_5 &= \sum_n (\varphi_n, \Gamma_5 \varphi_n) \\
&= \sum_{\lambda_n = 0} (\varphi_n, \Gamma_5 \varphi_n) \quad [(\varphi_n, \Gamma_5 \varphi_n) = 0 \text{ for } \lambda_n \neq 0] \\
&= \sum_{\lambda_n = 0} (\varphi_n, \gamma_5 \varphi_n) \\
&= n_+ - n_-
\end{aligned}$$

⋆ Thus the integer $\mathrm{Tr}\,\Gamma_5 = n_+ - n_-$ provides a topological characteristic of a gauge field configuration

⋆ Since the space of gauge field configurations is simply connected, however, this shows that the the Dirac operator D cannot be a smooth function of the gauge field. . .

• For Neuberger's overlap Dirac operator D

$$2aD = 1 - \frac{A}{\sqrt{A^\dagger A}} \quad A = 1 - aD_W$$

where D_W is the massless Wilson-Dirac operator, the sufficient condition of the smoothness (and the locality) is

⋆ Admissibility **(Hernández-Jansen-Lüscher, Neuberger)**

$$\|1 - U(x, \mu, \nu)\| < \epsilon \quad \text{for all plaquettes}$$

$$\|\mathcal{O}\| = \sup_{v \neq 0} \frac{\|\mathcal{O}v\|}{\|v\|}$$

⋆ This condition divides (otherwise simply-connected) space of gauge field configurations into topological sectors (will be denoted by M)

⋆ Each topological sector has a definite value of index $\mathrm{Tr}\,\Gamma_5 = n_+ - n_-$

⋆ In this way, a topological structure of the gauge theory is naturally incorporated in the lattice framework (e.g., Fermion number violation, see below)

⋆ *Chiral* gauge theories on the lattice with GW **(Lüscher)** [= Overlap **(Narayanan-Neuberger)** + additional elaborate materials]

⋆ One-parameter family of modified chiral matrices

$$\gamma_5^{(t)} = \frac{\gamma_5 - tH}{\sqrt{1 + t(t-2)H^2}}$$

and its "conjugate"

$$\overline{\gamma}_5^{(t)} = \gamma_5 \gamma_5^{(2-t)} \gamma_5$$

then

$$\gamma_5^{(t)\dagger} = \gamma_5^{(t)} \quad \overline{\gamma}_5^{(t)\dagger} = \overline{\gamma}_5^{(t)} \quad (\gamma_5^{(t)})^2 = (\overline{\gamma}_5^{(t)})^2 = 1$$

$$D\gamma_5^{(t)} = -\overline{\gamma}_5^{(t)} D$$

Thus we impose the chirality of fermion fields by

$$P_-^{(t)}\psi = \psi \quad \overline{\psi P}_+^{(t)} = \overline{\psi}$$

with

$$P_\pm^{(t)} = \frac{1}{2}(1 \pm \gamma_5^{(t)}) \quad \overline{P}_\pm^{(t)} = \frac{1}{2}(1 \pm \overline{\gamma}_5^{(t)})$$

such that the kinetic term in the action

$$S_{\mathrm{F}} = a^4 \sum_x [\overline{\psi}(x) D\psi(x) - \overline{\psi}(x)\eta(x) - \overline{\eta}(x)\psi(x)]$$

is consistently decomposed.

⋆ Expectation value of an operator $\mathcal{O}$

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \sum_M \int_M \mathsf{D}[U]\, e^{-S_G} \mathcal{N}_M e^{i\vartheta_M} \langle \mathcal{O} \rangle_{\mathsf{F}}^M$$

The average over the fermion fields is defined by

$$Z_{\mathsf{F}}[U, \eta, \overline{\eta}; t] = \int \mathsf{D}[\psi]\mathsf{D}[\overline{\psi}]\, e^{-S_{\mathsf{F}}}$$

⋆ The weight and phase for each topological sector M, $\mathcal{N}_M e^{i\vartheta_M}$, is included because Z_{F} is defined sector by sector (see below)

⋆ To define the fermion measure, we introduce an orthonormal complete set of vectors, s.t.,

$$P_-^{(t)} v_j(x) = v_j(x), \qquad \overline{v}_k \overline{P}_+^{(t)}(x) = \overline{v}_k(x)$$

and expand the fields as

$$\psi(x) = \sum_j v_j(x) c_j \quad \overline{\psi}(x) = \sum_k \overline{c}_k \overline{v}_k(x)$$

Then the measure is defined by

$$\mathsf{D}[\psi]\mathsf{D}[\overline{\psi}] = \prod_j \mathrm{d}c_j \prod_k \mathrm{d}\overline{c}_k$$

⋆ The dimension of the Grassman integral is

$$\mathrm{Tr}(P_-^{(t)} + \overline{P}_+^{(t)}) = \mathrm{Tr}\, 1 - \frac{t-1}{|t-1|}(n_+ - n_-)$$

and depends on the topological sector M

$\star$ However, the chirality constraint does not specify basis vectors, v_j and $\overline{v}_k$ uniquely (and thus the measure $\prod_j \mathrm{d}c_j \prod_k \mathrm{d}\bar{c}_k$)

• Under an infinitesimal variation of the gauge field

$$\delta_\eta U(x,\mu) = a\eta_\mu(x) U(x,\mu)$$

The chirality constraint reads

$$(1 - P_-^{(t)})\delta_\eta v_j = \delta_\eta P_-^{(t)} v_j$$

however, since

$$\delta_\eta v_j = P_-^{(t)}\delta_\eta v_j + (1 - P_-^{(t)})\delta_\eta v_j$$

the component of $\delta_\eta v_j$ residing in the constrained space, $P_-^{(t)}\delta_\eta v_j$, is not fixed

$\star$ For two different basis, $\{v, \overline{v}\}$ and $\{w, \overline{w}\}$, one has

$$Z_{\mathsf{F}}^{\{v,\overline{v}\}}[U,\eta,\overline{\eta};t] = e^{i\theta[U;t]} Z_{\mathsf{F}}^{\{w,\overline{w}\}}[U,\eta,\overline{\eta};t]$$

where the phase is given by the Jacobian for the change of basis

$$e^{i\theta[U;t]} = \det \mathcal{Q} \det \overline{\mathcal{Q}}$$
$$\mathcal{Q}_{jk} = (w_j, v_k) \quad \overline{\mathcal{Q}}_{jk} = (\overline{v}_j^\dagger, \overline{w}_k^\dagger)$$

$\star$ This phase may depend on the gauge field because so do $\{v, \overline{v}\}$ and $\{w, \overline{w}\}$

$\star$ Under an infinitesimal variation of the gauge field, the variation of the phase is given by

$$\delta_\eta \theta[U;t] = -\mathcal{L}_\eta^{\{v,\overline{v}\}}[U;t] + \mathcal{L}_\eta^{\{w,\overline{w}\}}[U;t]$$

where the "measure term" is defined by

$$\mathcal{L}_\eta^{\{v,\overline{v}\}}[U;t] = i\sum_j (v_j, \delta_\eta v_j) + i\sum_k (\delta_\eta \overline{v}_k^\dagger, \overline{v}_k^\dagger)$$

which parameterizes the above mentioned $P_-^{(t)}\delta_\eta v_j$

$\star$ Thus we need to fix the phase ambiguity in the fermion measure in some way

$\star$ This must be done in a consistent way with some physical requirements:

1. Locality. The effective action, $\ln Z_F$, has a desired locality property

2. Smoothness. Z_F should be a single-valued function of the gauge field

3. Gauge invariance, assuming that the fermion representation is anomaly-free in the continuum theory

How to find such an "ideal" measure?

$\star$ Reconstruction theorem **(Lüscher)**
If there exists a current $j_\mu^a(x)$ which depends smoothly and locally on the gauge field, such that

$$(\nabla_\mu^* j_\mu)^a(x) = \mathcal{A}^a(x)$$

where

$$\nabla_\mu^* j_\mu(x) = \frac{1}{a}[j_\mu(x) - U(x-a\hat{\mu},\mu)^{-1} j_\mu(x-a\hat{\mu}) U(x-a\hat{\mu},\mu)]$$

and

$$\mathcal{A}^a(x) = -\frac{i}{2}\,\mathrm{tr}\, R(T^a)(\gamma_5^{(t)} + \gamma_5^{(2-t)})(x,x)$$

and the combination $\mathcal{L}_\eta = a^4 \sum_x \eta_\mu^a(x) j_\mu^a(x)$ satisfies

$$\begin{aligned} W &= \exp(i\int_0^1 \mathrm{d}\tau\, \mathcal{L}_\eta) \quad [a\eta_\mu(x) = \partial_\tau U_\tau(x,\mu) U_\tau(x,\mu)^{-1}] \\ &= \det(1 - P_0^{(t)} + P_0^{(t)} Q_1^{(t)}) \\ &\quad \times \det{}^{-1}(1 - \overline{P}_0^{(t)} + \overline{P}_0^{(t)} \gamma_5 Q_1^{(2-t)} \gamma_5) \end{aligned}$$

for any loop $U_\tau(x,\mu)$ $(0 \le \tau \le 1)$ and $U_1 = U_0$, then there exists a fermion measure which satisfies the above 3 requirements. Symbols:

$$P_\tau^{(t)} = P_-^{(t)}|_{U=U_\tau} \quad P_\tau^{(t)} = \overline{P}_+^{(t)}|_{U=U_\tau}$$

The unitary operator $Q_\tau^{(t)}$ in this expression is defined by

$$\partial_\tau Q_\tau^{(t)} = \frac{1}{4}[\partial_\tau \gamma_\tau^{(t)}, \gamma_\tau^{(t)}] Q_\tau^{(t)}, \qquad Q_0^{(t)} = 1$$

where $\gamma_\tau^{(t)} = \gamma_5^{(t)}|_{U=U_\tau}$

$\star$ In fact, taking basis vectors as

$$v_j = \begin{cases} Q_1^{(t)} u_1 W^{-1}, & \text{for } j = 1 \\ Q_1^{(t)} u_j & \text{otherwise} \end{cases} \qquad \overline{v}_k = \overline{u}_k (\gamma_5 Q_1^{(2-t)} \gamma_5)^\dagger$$

where u_j and $\overline{u}_k$ are fixed bases defined for the reference configuration U_0. The operators $Q_\tau^{(t)}$ and $Q_\tau^{(2-t)}$ and the Wilson line W is defined along a certain curve connecting U_0 and $U = U_1$, it can be seen that all the 3 requirements are fulfilled.

$\star$ For example, the gauge variation of the expectation value is given by

$$\delta_\eta \langle \mathcal{O} \rangle_{\mathsf{F}} = \langle \delta_\eta \mathcal{O} \rangle_{\mathsf{F}} + i a^4 \sum_x \omega^a(x) [\mathcal{A}^a(x) - (\nabla_\mu^* j_\mu)^a(x)] \langle \mathcal{O} \rangle_{\mathsf{F}}$$

$\star$ Present status of our knowledge on the "ideal" measure current $j_\mu^a(x)$

1. For anomaly-free reps. of $G = \mathrm{U}(1)$, $j_\mu^a(x)$ is known in a non-perturbative level on a finite volume lattice **(Lüscher)**

2. For pseudo-real reps. of SU(2), $j_\mu^a(x)$ does not exist **(Bär-Campos)** ($\leftrightarrow$ Witten anomaly)

3. For anomaly-free reps. of a general gauge group G, $j_\mu^a(x)$ is known to all orders of the perturbation series **(H.S., Lüscher)**

4. For electroweak $\mathrm{SU}(2) \times \mathrm{U}(1)$, $j_\mu^a(x)$ is known in a non-perturbative level at least on an infinite volume lattice **(Kikukawa-Nakayama)**

5. For real reps. of a general gauge group G, $j_\mu^a(x)$ is known in a non-perturbative level on a finite volume lattice **(H.S.)**

$\star$ To show the existence of the ideal $j_\mu^a(x)$ for non-abelian G in a non-perturbative level is remaining a challenging problem. . .

• In this work, we proceed by simply assuming the existence of the ideal current (or the ideal basis or the ideal measure)

2. The Problem and Our Results

⋆ The problem: The above formulation is not manifestly invariant under CP (= basic discrete symmetry in chiral gauge theories)

⋆ CP transformation (conventional one)

$$U(x,\mu) \to U^{\mathrm{CP}}(x,\mu) = \begin{cases} U(\bar{x} - a\hat{i}, i)^{-1*} & \text{for } \mu = i \\ U(\bar{x}, 4)^{*} & \text{for } \mu = 4 \end{cases}$$

where

$$\bar{x} = (-x_i, x_4) \text{ for } x = (x_i, x_4) \ (i = 1,\ 2,\ 3)$$

Under this, one has

$$D(U^{\mathrm{CP}})(x,y) = W^{-1} D(U)(\bar{x},\bar{y})^T W$$

On the fermion, we define

$$\psi(x) \to -W^{-1}\overline{\psi}^T(\bar{x}) \quad \overline{\psi}(x) \to \psi^T(\bar{x}) W$$

where $W = \gamma_2$ $(W^\dagger W = 1)$

• The CP exchanges $\psi \leftrightarrow \overline{\psi}$. But the chirality constraint is *not* symmetric for ψ and for $\overline{\psi}$.

⋆ The action changes under CP as

$$S_{\mathsf{F}} = a^4 \sum_x \overline{\psi}(x) \overline{P}_+ D P_- \psi(x)$$
$$\to S_{\mathsf{F}} = a^4 \sum_x \overline{\psi}(x) \gamma_5 P_+ \gamma_5 D \overline{P}_- \psi(x)$$

and consequently the propagator changes as

$$\frac{\langle \psi(x)\overline{\psi}(y)\rangle_{\mathsf{F}}}{\langle 1\rangle_{\mathsf{F}}} = P_- \frac{1}{D} \overline{P}_+(x,y)$$
$$\to \frac{\langle \psi(x)\overline{\psi}(y)\rangle_{\mathsf{F}}}{\langle 1\rangle_{\mathsf{F}}} = \overline{P}_- \frac{1}{D} \gamma_5 P_+ \gamma_5 (x,y)$$
$$= P_- \frac{1}{D} \overline{P}_+(x,y) - a\gamma_5 a^{-4} \delta_{x,y} \quad \text{(when } t = 2\text{)}$$

The last term is a constant term

⋆ However, it is not obvious how the breaking caused by the chirality constraint (which specifies the fermion measure) influences various places of the formulation (CP anomaly? for example) **cf. (Hasenfratz)**

⋆ We want to clarify this issue. . .

$\star$ Our results

It is always possible to choose the fermion measure such that an effect of the above CP breaking appears only at

1. Constant coefficient of Z_F. This constant depends only on the topological sector concerned, through the index $n_+ - n_-$.

2. Constant phase of Z_F. This phase depends only on the topological sector concerned.

These two breaking effects can be absorbed into the topological factor $\mathcal{N}_M e^{i\vartheta_M}$. There is no CP anomaly in this sense.

3. Propagators connected to fermion external lines or to Yukawa vertices, when $t \neq 1$.

3-1. When Higgs field has no VEV, this breaking in the propagator is local (almost contact term)
3-2. When Higgs field acquires VEV, the breaking in the propagator becomes intrinsically non-local

[3-3. Internal fermion lines respect CP. For example, vacuum polarization is CP invariant]

In another work (to appear in PLB), we have shown

4. The choice $t = 1$ cannot be made, because Γ_5 in

$$\gamma_5^{(t=1)} = \frac{\Gamma_5}{\sqrt{\Gamma_5^2}}$$

has always zeros at (would-be) species doublers' momentum

5. Assuming the conventional CP transformation, the chirality constraint being invariant under CP always involves $\Gamma_5/\sqrt{\Gamma_5^2}$

Thus this CP breaking in the propagator is a generic feature of this formulation based on (generalized) GW relation.

3. Fermion Generating Functional

$\star$ We can determine a general structure of Z_F by utilizing the auxiliary basis $\{w, \overline{w}\}$. We take eigenfunctions of $H^2 = a^2 D^\dagger D$:

$$H^2 u_j(x) = \lambda_j^2 u_j(x) \quad \lambda_j \geq 0$$

and its projection

$$w_j(x) = P_-^{(t)} u_j(x) \quad (w_j, w_k) = \delta_{jk}$$

$\star$ We know many about this eigenvalue problem. Defining

$$H\varphi_n(x) = \lambda_n \varphi_n(x) \quad (\varphi_n, \varphi_m) = \delta_{nm}$$

obviously

$$u_j \leftrightarrow \varphi_n \quad \lambda_j \leftrightarrow |\lambda_n|$$

φ_n is classified into 3 categories [Λ is a solution of $\Lambda f(\Lambda^2) = 1$; in our present case $\Lambda = 1$]
(i) $\lambda_n = 0$:

$$\gamma_5^{(t)} \varphi_0^\pm = \gamma_5 \varphi_0^\pm = \pm \varphi_0^\pm$$

Number of φ_0^+ (φ_0^-) is n_+ (n_-).
(ii) $\lambda_n \neq 0$ and $\lambda_n \neq \pm\Lambda$:

$$\gamma_5^{(t)} \varphi_n^\pm(x) = \pm \varphi_n^\pm(x)$$

both has identical $|\lambda_n|$ and numbers of φ_n^+ and φ_n^- are the same.
(iii) $\lambda_n = \pm\Lambda$:

$$\gamma_5^{(t)}\Psi_\pm(x) = \begin{cases} \mp\Psi_\pm(x) & \text{for } t > 1 \\ \pm\Psi_\pm(x) & \text{for } t < 1 \end{cases}$$

Number of Ψ_+ (Ψ_-) is N_+ (N_-).

$\star$ Moreover one can show the chirality sum-rule **(Chiu, Fujikawa)**

$$\mathrm{Tr}\,\gamma_5 = n_+ - n_- + N_+ - N_- = 0$$

In terms of φ_n, our auxiliary basis w_j is given by

$$\begin{array}{ll} \varphi_0^-(x) & \lambda_j = 0 \\ \varphi_j^-(x) & \lambda_j > 0 \quad \lambda_j \neq \Lambda \\ \begin{cases} \Psi_+(x) & \text{for } t > 1 \\ \Psi_-(x) & \text{for } t < 1 \end{cases} & \lambda_j = \Lambda \end{array}$$

$\star$ For the basis for the anti-fermion, we may use the left eigenfunctions of the hermitian operator $DD^\dagger$:

$$\overline{w}_j(x) = \frac{a}{\lambda_j} w_j^\dagger D^\dagger(x)$$

and for zero-modes of $DD^\dagger$

$$\varphi_0^{+\dagger}(x)$$

⋆ Having specified the basis vector $\{w, \overline{w}\}$, we have

$$Z_{\mathsf{F}}^{\{w,\overline{w}\}}[U,\eta,\overline{\eta};t]$$
$$= \left(\frac{\Lambda}{a}\right)^{N} \prod^{n_-} \left[a^4 \sum_x \overline{\eta}(x)\varphi_0^-(x)\right] \prod^{n_+} \left[a^4 \sum_x \varphi_0^{+\dagger}(x)\eta(x)\right]$$
$$\times \prod_{\substack{\lambda_j>0 \\ \lambda_j\neq\Lambda}} \left(\frac{\lambda_j}{a}\right) \exp\left[a^8 \sum_{x,y} \overline{\eta}(x) G^{(t)}(x,y)\eta(y)\right]$$

where

$$DG^{(t)}(x,y) = \overline{P}_+^{(t)}(x,y) - \sum^{n_+} \varphi_0^+(x)\varphi_0^{+\dagger}(y),$$

and

$$N = \begin{cases} N_+ & \text{for } t > 1 \\ N_- & \text{for } t < 1 \end{cases}$$

⋆ The general relation thus implies

$$Z_{\mathsf{F}}^{\{v,\overline{v}\}}[U,\eta,\overline{\eta};t]$$
$$= e^{i\theta[U;t]} \left(\frac{\Lambda}{a}\right)^{N} \prod^{n_-} \left[a^4 \sum_x \overline{\eta}(x)\varphi_0^-(x)\right]$$
$$\times \prod^{n_+} \left[a^4 \sum_x \varphi_0^{+\dagger}(x)\eta(x)\right]$$
$$\times \prod_{\substack{\lambda_j>0 \\ \lambda_j\neq\Lambda}} \left(\frac{\lambda_j}{a}\right) \exp\left[a^8 \sum_{x,y} \overline{\eta}(x) G^{(t)}(x,y)\eta(y)\right]$$

$\star$ The phase $\theta[U;t]$ obeys the general rule

$$\delta_\eta \theta[U;t] = -\mathcal{L}_\eta^{\{v,\overline{v}\}}[U;t] + \mathcal{L}_\eta^{\{w,\overline{w}\}}[U;t]$$

and the measure term for $\{w,\overline{w}\}$ is given by

$$\begin{aligned}\mathcal{L}_\eta^{\{w,\overline{w}\}}[U;t] &= i\sum^{n_-}(\varphi_0^-,\delta_\eta\varphi_0^-) + i\sum^{n_+}(\delta_\eta\varphi_0^+,\varphi_0^+) \\ &\quad + \frac{i}{4}\,\mathrm{Tr}'\,\delta_\eta D(\gamma_5^{(t)} + \gamma_5^{(2-t)})D^{-1}\end{aligned}$$

• Incidentally, this auxiliary basis cannot be "ideal", because the associated measure current is not local, as the last expression involves the propagator D^{-1}. This basis is simply a convenient tool.

4. CP Transformed Z_F

We first note CP preserves the admissibility, because

$$\|1 - U^{\mathrm{CP}}(x,i,j)\| = \|1 - U(\bar{x} - a\hat{i} - a\hat{j}, i, j)\|$$
$$\|1 - U^{\mathrm{CP}}(x,i,4)\| = \|1 - U(\bar{x} - a\hat{i}, i, 4)\|$$

thus it is meaningful to discuss CP in this framework. Note, however, that CP in general maps a gauge field configuration to that in another topological sector, e.g.,

$$\mathrm{Tr}\,\Gamma_5(U) = n_+ - n_- = -\,\mathrm{Tr}\,\Gamma_5(U^{\mathrm{CP}})$$

$\star$ Now let us examine CP transformed Z_F:

$$Z_{\mathsf{F}}[U^{\mathrm{CP}}, -W^{-1}\overline{\eta}^T, \eta^T W; t] = \int \mathsf{D}[\psi]\mathsf{D}[\overline{\psi}]\, e^{-S_{\mathsf{F}}}$$

The point is the basis vectors are subject of the constraints

$$P_-^{(t)}(U^{\mathrm{CP}})v_j(x) = v_j(x) \quad \overline{v}_k(x)\overline{P}_+^{(t)}(U^{\mathrm{CP}}) = \overline{v}_k(x)$$

• We may repeat the calculation anew by using an auxiliary basis. This repetition may be avoided by the following argument:
We introduce new variables

$$\psi'(x) = [\overline{\psi}(\bar{x})W^{-1}]^T = \sum_k \bar{c}_k [\overline{v}_k(\bar{x})W^{-1}]^T$$
$$\overline{\psi}'(x) = [-W\psi(\bar{x})]^T = \sum_j [-Wv_j(\bar{x})]^T c_j$$

then

$$\begin{aligned} & Z_{\mathsf{F}}[U^{\mathsf{CP}}, -W^{-1}\overline{\eta}^T, \eta^T W; t] \\ &= \int \prod_j \mathrm{d}c_j \prod_k \mathrm{d}\bar{c}_k \\ &\quad \times \exp\{-a^4 \sum_x [\overline{\psi}'(x) D(U) \psi'(x) \\ &\qquad\qquad -\overline{\eta}(x)\psi'(x) - \overline{\psi}'(x)\eta(x)]\} \end{aligned}$$

Since the basis vectors for new variables

$$v'_k = (\overline{v}_k W^{-1})^T \quad \overline{v}'_j = (-Wv_j)^T$$

satisfy the constraints

$$P_-^{(2-t)}(U) v'_k = v'_k \quad \overline{v}'_j \overline{P}_+^{(2-t)}(U) = \overline{v}'_j$$

This shows

$$Z_{\mathsf{F}}[U^{\mathsf{CP}}, -W^{-1}\overline{\eta}^T, \eta^T W; t] = Z_{\mathsf{F}}[U, \eta, \overline{\eta}; 2-t]$$

$\star$ The sole effect of the CP transformation is given by the change of the parameter, $t \to 2 - t$

⋆ Applying this fact to $Z_F^{\{w,\overline{w}\}}$, we have

$$
\begin{aligned}
& Z_{\mathsf{F}}^{\{v,\overline{v}\}}[U^{\mathsf{CP}}, -W^{-1}\overline{\eta}^T, \eta^T W; t] \\
& = e^{i\theta[U^{\mathsf{CP}};t]} Z_{\mathsf{F}}^{\{w,\overline{w}\}}[U, \eta, \overline{\eta}; 2-t] \\
& = e^{i\theta[U^{\mathsf{CP}};t]} \left(\frac{\Lambda}{a}\right)^{\overline{N}} \prod^{n_-} \left[a^4 \sum_x \overline{\eta}(x)\varphi_0^-(x)\right] \\
& \qquad \times \prod^{n_+} \left[a^4 \sum_x \varphi_0^{+\dagger}(x)\eta(x)\right] \\
& \qquad \times \prod_{\substack{\lambda_j>0 \\ \lambda_j\neq\Lambda}} \left(\frac{\lambda_j}{a}\right) \exp\left[a^8 \sum_{x,y} \overline{\eta}(x) G^{(2-t)}(x,y)\eta(y)\right]
\end{aligned}
$$

where

$$
\overline{N} = \begin{cases} N_- & \text{for } t > 1 \\ N_+ & \text{for } t < 1 \end{cases}
$$

⋆ The phase $\theta[U^{\mathsf{CP}}; t]$ obeys the general rule

$$
\delta_\eta \theta[U^{\mathsf{CP}}; t] = -\mathcal{L}_\eta^{\{v,\overline{v}\}}[U^{\mathsf{CP}}; t] + \mathcal{L}_\eta^{\{w,\overline{w}\}}[U^{\mathsf{CP}}; t]
$$

and the measure term for $\{w, \overline{w}\}$ is shown to be CP invariant

$$
\mathcal{L}_\eta^{\{w,\overline{w}\}}[U^{\mathsf{CP}}; t] = \mathcal{L}_\eta^{\{w,\overline{w}\}}[U; t]
$$

if one takes (which we assume)

$$
\begin{aligned}
\varphi_0^-(U^{\mathsf{CP}}) &= (\varphi_0^{+\dagger}(U) W^{-1})^T \\
\varphi_0^{+\dagger}(U^{\mathsf{CP}}) &= (-W\varphi_0^-(U))^T
\end{aligned}
$$

$\star$ Now we come to the crucial step: We claim that the ideal basis $\{v, \overline{v}\}$ should be taken such that

$$\mathcal{L}_\eta^{\{v,\overline{v}\}}[U^{\mathsf{CP}}; t] = \mathcal{L}_\eta^{\{v,\overline{v}\}}[U; t]$$

We will show that this is always possible. With this choice, one has

$$\begin{aligned} \delta_\eta \theta[U^{\mathsf{CP}}; t] &= -\mathcal{L}_\eta^{\{v,\overline{v}\}}[U^{\mathsf{CP}}; t] + \mathcal{L}_\eta^{\{w,\overline{w}\}}[U^{\mathsf{CP}}; t] \\ &= -\mathcal{L}_\eta^{\{v,\overline{v}\}}[U; t] + \mathcal{L}_\eta^{\{w,\overline{w}\}}[U; t] \\ &= \delta_\eta \theta[U; t] \end{aligned}$$

Namely, the difference of phases, it it exists, is a constant (which may depend on the topological sector M)

$$\theta[U^{\mathsf{CP}}; t] = \theta[U; t] + \theta_M$$

Thus we finally conclude

$$\begin{aligned} & Z_{\mathsf{F}}[U^{\mathsf{CP}}, -W^{-1}\overline{\eta}^T, \eta^T W; t] \\ &= e^{i\theta_M} \left(\frac{\Lambda}{a}\right)^{\overline{N}-N} \frac{\exp\left[a^8 \sum_{x,y} \overline{\eta}(x) G^{(2-t)}(x,y) \eta(y)\right]}{\exp\left[a^8 \sum_{x,y} \overline{\eta}(x) G^{(t)}(x,y) \eta(y)\right]} \\ &\qquad \times Z_{\mathsf{F}}[U, \eta, \overline{\eta}; t] \end{aligned}$$

which summarizes our result.

(I) Constant phase θ_M may be absorbed into the "topological phase" ϑ_M as

$$\vartheta_M \to \vartheta_M + \frac{1}{2}\theta_M \quad \vartheta_{M^{\mathrm{CP}}} \to \vartheta_{M^{\mathrm{CP}}} - \frac{1}{2}\theta_M$$

There is no CP anomaly. The problem is then reduced to how one should choose the topological phase ϑ_M. This is a problem analogous to strong CP problem.

(II) Constant coefficient $(\Lambda/a)^{\overline{N}-N}$ may also be absorbed into the "topological weight" $\mathcal{N}_M$ as

$$\mathcal{N}_M \to \mathcal{N}_M \left(\frac{\Lambda}{a}\right)^{(-N+\overline{N})/2} \qquad \mathcal{N}_{M^{\mathrm{CP}}} \to \mathcal{N}_{M^{\mathrm{CP}}} \left(\frac{\Lambda}{a}\right)^{(N-\overline{N})/2}$$

Then if one takes $\mathcal{N}_M = 1$, the CP breaking from this part disappears. **cf. Majonara formulation (H.S.)**

(III) The breaking in the propagator $G^{(2-t)} \neq G^{(t)}$ cannot be avoided. The symmetric choice $t = 1$ cannot be made. The breaking, however, is rather modest:

$$G^{(2-t)} = G^{(t)} + \frac{a(1-t)}{\sqrt{1 + t(t-2)H^2}}\gamma_5$$

The breaking term is local.

$\star$ The above analysis shows if one takes basis vectors such that

$$\mathcal{L}_\eta^{\{v,\overline{v}\}}[U^{\mathrm{CP}};t] = \mathcal{L}_\eta^{\{v,\overline{v}\}}[U;t]$$

then there is no CP anomaly. Conversely, if one does not respect this condition, one will have an unnecessary CP violation which results from a wrong choice of the fermion measure ("fake CP anomaly")

$\star$ Fortunately, it is always possible to take the measure term such that the above is true: It is given by the average over CP

$$\begin{aligned}
&\mathcal{L}_\eta^{\{v,\overline{v}\}}[U;t] = a^4 \sum_x \eta_\mu^a(x) j_\mu^a(x)[U;t] \\
&\to \frac{1}{2}[a^4 \sum_x \eta_\mu^a(x) j_\mu^a(x)[U;t] \\
&\qquad + a^4 \sum_x \eta_\mu^{\mathrm{CP}a}(x) j_\mu^a(x)[U^{\mathrm{CP}};t]].
\end{aligned}$$

This average is possible even if U and U^{CP} belong to different topological sectors M and M^{CP}, because the CP operation defines a differentiable one-to-one onto-mapping from M to M^{CP}.

$\star$ Moreover, conditions in the reconstruction theorem is invariant under $U \to U^{\mathrm{CP}}$: Namely,

$$\det(1-P_0^{(t)}+P_0^{(t)}Q_1^{(t)})\det{}^{-1}(1-\overline{P}_0^{(t)}+\overline{P}_0^{(t)}\gamma_5 Q_1^{(2-t)}\gamma_5)$$

is invariant under $U_\tau \to U_\tau^{\mathrm{CP}}$, and

$$ia^4\sum_x \omega^a(x)\mathcal{A}^a(x)$$

is invariant under $U \to U^{\mathrm{CP}}$ (under which $\omega(x) \to \omega(\bar{x})^*$).

These arguments justify the "CP invariant choice"

$$\mathcal{L}_\eta^{\{v,\overline{v}\}}[U^{\mathrm{CP}};t] = \mathcal{L}_\eta^{\{v,\overline{v}\}}[U;t]$$

5. With Yukawa Couplings

⋆ The above analysis can be repeated straightforwardly even with Yukawa couplings

$$\begin{aligned} S_{\mathsf{F}} \;=\; a^4 \sum_x [& \overline{\psi}_L(x) D \psi_L(x) + \overline{\psi}_R(x) D' \psi_R(x) \\ & + \overline{\psi}_L(x) \phi(x) \psi_R(x) + \overline{\psi}_R(x) \phi^\dagger(x) \psi_L(x) \\ & - \overline{\psi}_L(x) \eta_L(x) - \overline{\eta}_L(x) \psi_L(x) \\ & - \overline{\psi}_R(x) \eta_R(x) - \overline{\eta}_R(x) \psi_R(x)] \end{aligned}$$

where

$$\begin{aligned} P_-^{(t)} \psi_L = \psi_L \quad & \overline{\psi}_L \overline{P}_+^{(t)} = \overline{\psi}_L \\ P_+^{\prime(t)} \psi_R = \psi_R \quad & \overline{\psi}_R \overline{P}_-^{\prime(t)} = \overline{\psi}_R \end{aligned}$$

$$\begin{aligned} \psi_L(x) &\in R_L \qquad D \\ \psi_R(x) &\in R_R \qquad D' \\ \phi(x) &\in R_L \otimes (R_R)^* \end{aligned}$$

because the fermion measure does not refer to the Higgs field ϕ

⋆ We examine the change under CP

$$Z_{\mathsf{F}}[U, \phi, \eta_L, \overline{\eta}_L, \eta_R, \overline{\eta}_R; t] = \int \mathsf{D}[\psi] \mathsf{D}[\overline{\psi}] \, e^{-S_{\mathsf{F}}}$$

$$\to Z_{\mathsf{F}}[U^{\mathsf{CP}}, \phi^\dagger, -W^{-1} \overline{\eta}_L^T, \eta_L^T W, -W^{-1} \overline{\eta}_R^T, \eta_R^T W; t]$$

We have

$$
\begin{aligned}
&Z_{\mathsf{F}}[U^{\mathsf{CP}}, \phi^\dagger, -W^{-1}\overline{\eta}_L^T, \eta_L^T W, -W^{-1}\overline{\eta}_R^T, \eta_R^T W; t] \\
&= e^{i\theta_M} \left(\frac{\Lambda}{a}\right)^{\overline{N}-N-\overline{N}'+N'} \\
&\quad \times \exp[\frac{\partial}{\partial \eta_L} \overline{P}_+^{(2-t)} \phi P_+'^{(2-t)} \frac{\partial}{\partial \overline{\eta}_R} \\
&\qquad\qquad + \frac{\partial}{\partial \eta_R} \overline{P}_-'^{(2-t)} \phi^\dagger P_-^{(2-t)} \frac{\partial}{\partial \overline{\eta}_L}] \\
&\quad \times \frac{\exp\{[\overline{\eta}_L G^{(2-t)} \eta_L + \overline{\eta}_R G'^{(2-t)}) \eta_R]\}}{\exp\{[\overline{\eta}_L G^{(t)} \eta_L + \overline{\eta}_R G'^{(t)} \eta_R]\}} \\
&\qquad\qquad \times Z_{\mathsf{F}}[U, 0, \eta_L, \overline{\eta}_L, \eta_R, \overline{\eta}_R; t]
\end{aligned}
$$

$\star$ The breaking appears precisely in the same places as before, except for the terms consisting of Yukawa couplings connected by propagators.

$\star$ Thus when Higgs has no VEV, the situation is the same as before.

$\star$ However, when Higgs acquires VEV

$$\phi(x) = v \quad v_+ = \overline{P}_+^{(t)} v P_+^{\prime(t)} \quad v_- = \overline{P}_-^{\prime(t)} v^\dagger P_-^{(t)}$$

fermion propagator reads, for example,

$$\frac{\langle \psi_L(x) \overline{\psi}_R(y) \rangle_{\mathsf{F}}}{\langle 1 \rangle_{\mathsf{F}}} = P_-^{(t)} \frac{1}{v_- - D' v_+^{-1} D} \overline{P}_-^{\prime(t)}(x,y)$$

The difference under $t \to 2 - t$

$$\overline{P}_-^{\prime(2-t)} - \overline{P}_-^{\prime(t)} \propto a(t-1) D' \gamma_5$$

cannot cancel the denominator $v_- - D' v_+^{-1} D$. Since the exponential decaying rate of the propagator

$$\frac{1}{v_- - D' v_+^{-1} D}(x,y)$$

is

$$\sim \frac{1}{\sqrt{v^\dagger v} a}$$

in the lattice unit, the difference in the propagator is a non-local function on the lattice.

6. Outlook

1. CP breaking in propagators connected to

External fermion lines

Yukawa vertices

vanishes $a \to 0$. However their influences have to be carefully examined.

2. CP breaking possibly arising from composite operators has to be analyzed. Implication to weak matrix elements?

3. It is possible to obtain the same result without relying on the auxiliary basis (Kikukawa, private communication)

4. The explicit CP breaking in this formulation might trigger a spontaneous CP breaking. Has to be examined.