Derivation of the Intriligator-Leigh-Seiberg linearity principle from Dijkgraaf-Vafa

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1. Introduction to DV

$d=4, \mathcal{N}=1$ SYM theories	DUALITY $\Longleftrightarrow$	Old matrix models

- Gauge Theory : N_c Finite and Fixed

Φ: adjoint, $N_c \times N_c$
S: gaugino condensate $\mathrm{tr}\, W^{\alpha} W_{\alpha}$
W_{α}: (λ_{α} (gaugino), ${F_{\mu\nu}}^{+}$ (field strength))

- Matrix Model : Planar Limit

Φ: $M \times M$ matrix,
S: 't Hooft coupling $M \to \infty$ with S fixed

	Gauge	Matrix
Bare	superpotential $W_{\mathrm{tree}}(\Phi)$	Action $W_{\mathrm{tree}}(\Phi)$
	$\Downarrow$	$\Downarrow$
Eff.	superpotential $W_{\mathrm{eff}}(S)$	Free energy $\mathcal{F}(S)$

- The Correspondence :

$$W_{\mathrm{eff}}(S) = N_c \frac{\partial \mathcal{F}}{\partial S}$$

conjectured to be EXACT

First Indication — Factorization

⋆ Matrix Model Side

Example:

$$\int d\Phi \exp\left[-\frac{M}{S}\left(\frac{1}{2}m\,\mathrm{tr}\,\Phi^2+\frac{1}{3}g\,\mathrm{tr}\,\Phi^3\right)\right]$$

- $(\text{Propagators})^{-1}$ and couplings $\propto M/S$
- $\underbrace{V}_{\text{vertices}} - \underbrace{E}_{\text{propagators}} + \underbrace{h}_{\text{index loops}} = \underbrace{\chi}_{\text{Euler\#}}$

$$\Downarrow$$

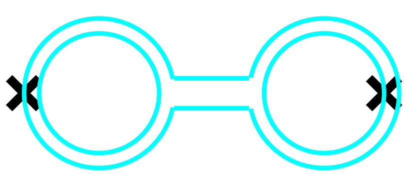
$$\text{diagram} \propto M^h\left(\frac{M}{S}\right)^{V-E} = M^{\chi}S^{V-E}$$

$$\rightsquigarrow$$

$M\to\infty$, planars dominate,
correlation functions factorize:

in $\langle AB\rangle$,

$\langle A\rangle\langle B\rangle$ $\qquad h=5 \rightsquigarrow M^4S^2$



$\langle AB\rangle_{\text{conn.}}$ $\qquad h=3 \rightsquigarrow M^2S^2$

$(E=5,\ V=4)$

⋆Gauge theory side

Chiral VEVs factorize:

$$\langle \Phi_1(x)\Phi_2(y)\rangle = \langle \Phi_1(x)\rangle\langle \Phi_2(y)\rangle$$

Because

- $\partial_\mu \Phi_1(x) = \{\bar{D}, [D, \Phi_1(x)]\}$ **is** $\bar{D}$**-exact.**
- **Hence,**

$$\begin{aligned} \partial_\mu \langle \Phi_1(x)\Phi_2(y)\rangle &= \langle \{\bar{D}, [D, \Phi_1(x)]\}\Phi_2(y)\rangle \\ &= \langle [D, \Phi_1(x)]\{\bar{D}, \Phi_2(y)\}\rangle \\ &= 0 \end{aligned}$$

- **Next, Take** $|x-y| \to \infty$
- **Finally use the Cluster Property.**

Stringy ideas behind the proposal

DV arrived this proposal by chasing the following dualities:

$$\begin{array}{ccc}
\mathcal{N}=1 \text{ SYM} & \overset{!!!}{\longleftrightarrow} & \text{matrix model} \\
\cap & & \parallel \\
\text{D5 wrapped on} & & \text{B brane on} \\
S^2 \text{ in CY} & \overset{\text{twist}}{\Longrightarrow} & S^2 \text{ in CY} \\
\text{large}N \text{ dual} \updownarrow & & \updownarrow \text{large}N \text{ dual} \\
\text{IIB on CY} & & \text{B model on CY} \\
\text{with flux in } S^3 & \overset{\text{twist}}{\Longrightarrow} & \text{with flux in } S^3
\end{array}$$

Checking the proposal

$W_{\text{tree}} = 0$

Known Exact Results
(ADS superpotential,
SW curve...)

$\Downarrow$

Use the (conjectured)
ILS linearity!
(more on this in section 2.)

$\Downarrow$

$W_{\text{tree}} \neq 0$

$W_{\text{eff}}(S)$

$\Uparrow$

Use Matrix Model!

$\Uparrow$

$W_{\text{tree}}(\Phi)$

Field-theoretic derivations

There are two:

- Dijkgraaf-Grisaru-Lam-Vafa-Zanon

Perturbation in an external gaugino

(more on this in section 3.)

- Cachazo-Douglas-Seiberg-Witten ,

Seiberg

Factorization and generalized anomaly

$\Downarrow$

We can now use DV to study SYM!

An example:

the ILS linearity principle from DV

2. Brief Review of ILS

From the bare lagrangian

$$\mathcal{L}_{\text{bare}} = \mathcal{L}_0 + \int d^2\theta W_{\text{tree}}(\Phi) + c.c.,$$

$$\text{where } W_{\text{tree}} = \sum g_i \mathcal{O}_i$$

$$\Downarrow$$

To the low energy superpotential

$$\int d^2\theta \ W_{\text{eff}}(\Phi)$$

holomorphy

Coupling constants g_i as the vevs of chiral superfields

$$\rightsquigarrow$$

$W_{\text{eff}}(\Phi)$ depends only on g_i , not on $\overline{g_i}$.

symmetry

- Many classical symmetries on chiral superfields and coupling constants.

- Even anomalous symmetries are useful once one assigns charges to the dynamical scale Λ appropriately.

classical and perturbative limit

• In the classical limit $\Lambda \to 0$, $W_{\text{eff}}(\Phi)$ must approach $W_{\text{tree}}(\Phi)$.

• If one takes some coupling $g_i \to 0$, W_{eff} should smoothly become W_{tree}.

$$\Downarrow$$

W_{eff} does not contain negative powers of Λ and g_i.

Strong constraint for in $W_{\text{eff}}(\Phi)$,
fixing completely in some simple cases .

Example.

$SU(N_c)$ SYM with $N_f < N_c$ pairs of quarks Q_i and $\tilde{Q}_i$.

• Gauge invariants are

$$T_{ij} = Q_i \tilde{Q}_j.$$

• $W_{\text{tree}} = m_{ij} T_{ij}$ as the bare superpotential

• $SU(N_f) \times SU(N_f)$ flavor symmetry on quarks and antiquarks.

- R-charges:

	θ	Q	$\tilde{Q}$	m	W_α
R-charge	1	0	0	2	1

- The anomaly from

$$q \to e^{-i\phi} q, \qquad \lambda \to e^{+i\phi} \lambda,$$

leads to:

$$\theta \to \theta + 2(N_c - N_f)\phi.$$

Because

$$\Lambda^{3N_c - N_f} \sim \Lambda_0^{3N_c - N_f} \exp\left(-\frac{8\pi^2}{g_0^2} + i\theta\right),$$

the R-charge of $\Lambda^{3N_c - N_f}$ is $2(N_c - N_f)$.

$$\Downarrow$$

$$W_{\text{eff}} = c \left(\frac{\Lambda^{3N_c - N_f}}{\det T_{ij}}\right)^{1/(N_c - N_f)} + \underbrace{m_{ij} T_{ij}},$$

the coeff. is fixed by $m \to 0$ limit.

the linearity principle

• The result can be summarized as

$$W_{\text{eff}} = \underbrace{W_{\text{n.p.}}}_{\substack{\text{non-perturbative,}\\ \text{independent of}\\ \text{coupling constants}}} + W_{\text{tree}}$$

• Various other models have this form. ILS proposed this linearity as an organizing principle.

• However, symmetry arguments are not always strong enough.

Another Example

• Consider $\mathcal{N}=1$ $SU(N_c)$ SYM with one adjoint Φ.

• The R-charge of Λ is zero .

• Introduce the bare superpotential

$$m\,\mathrm{tr}\,\Phi^2/2.$$

• Symmetry alone cannot exclude the corrections of the form $m\Lambda^2$.

3. Brief Review of DGLVZ

How to derive Dijkgraaf-Vafa?

$$\Downarrow$$

Directly equate Feynman rules in the gauge theory and in the matrix model .

Strategy

1. Introduce an external gauge superfield. In particular, introduce a spacetime constant gaugino.
2. Perturbatively integrate out the matter superfields. $W_{\text{inst}}(S)$
3. Make the gauge superfield dynamical.

$$W_{\text{eff}}(S) = \underbrace{W_{\text{VY}}(S)}_{\substack{\text{dynamical effect} \\ \text{from the gauge superfield}}} + W_{\text{inst}}(S)$$

Feynman rules for step 2
=Feynman rules for Matrix Model

the propagator

$$\langle \Phi\Phi \rangle = \frac{\bar{m}}{p^2 + m\bar{m} + W^\alpha \pi_\alpha}$$

where

$$\begin{array}{rcl} p & : & \text{bosonic momentum,} \\ W^\alpha & : & \text{field strength superfield,} \\ \pi_\alpha & : & \text{fermionic momentum.} \end{array}$$

Integration of loop momenta:

$$\prod_{a=1}^{l} \left(\int \frac{d^4 p_a}{(2\pi)^4} \int d^2 \pi_a \right)$$

$\Longrightarrow$ Every l-loop diagram carries a factor of $W_\alpha{}^{2l}$.

$\Longrightarrow$ if $W_\alpha = 0$, no correction to F terms.

This is the Perturbative Non-Renormalization Theorem!

What if $W_\alpha \neq 0$?

Origin of Planarity

• tr W_α^m is $\bar{D}$-exact for $m \geq 3$, i.e. $\sim$ zero in the F terms. (CDSW)

• Therefore, $W_{\rm eff}$ is a function of

$$32\pi S = \text{tr}\, W^\alpha W_\alpha.$$

To give W_α^{2l} requires at least l index loops.

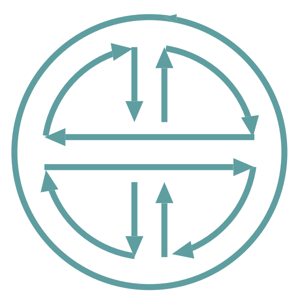
• # of index loops h satisfies

$$h = l + 1 - 2 \times \text{genus}.$$

$l = 3$

$\wedge$

$h = 4$



$l = 3$

$\vee$

$h = 2$

Only **GENUS 0** diagrams contribute!

• Further tricks reveal the resulting Feynman rules are the same as those of the matrix models in the planar limit.

Making the vector superfield dynamical

This introduces the Veneziano-Yankielowicz term

$$W_{\text{VY}} = N_c S(1 - \log \frac{S}{\Lambda^3}).$$

(Λ: the dynamical scale of SQCD)

There are arguments against further corrections.

The total effective superpotential:

$$W_{\text{eff}}(S) = W_{\text{VY}}(S) + W_{\text{inst}}(S).$$

• $\langle S \rangle$ is determined by the extremalization

$$\partial W_{\text{eff}} / \partial S = 0.$$

• For $W_{\text{tree}} = \sum g_i \mathcal{O}_i$, the vevs are

$$\begin{aligned} \langle \mathcal{O}_i \rangle &= \frac{\partial}{\partial g_i} W_{\text{eff}}(\langle S \rangle, g_i) \\ &= \left. \frac{\partial W_{\text{eff}}}{\partial g_i} \right|_S + \frac{\partial \langle S \rangle}{\partial g_i} \left. \frac{\partial W_{\text{eff}}}{\partial S} \right|_{g_i} = \left. \frac{\partial W_{\text{eff}}}{\partial g_i} \right|_S . \end{aligned}$$

4. ILS from DGLVZ

- Introduce $W_{\text{tree}} = \sum g_i \mathcal{O}_i$.
- From DV, calculate W_{eff} and $\langle W_{\text{tree}} \rangle$ in $W_{\text{eff}} = W_{\text{n.p.}} + \langle W_{\text{tree}} \rangle$.
- Is $W_{\text{n.p.}}$ independent of g_i as a function of Λ , g_i and $\langle \mathcal{O}_i \rangle$? This is the linearity principle.

Note:

$$
\begin{array}{lcll}
 & W_{\text{eff}} & = W_{\text{VY}} & + W_{\text{inst}} \\
\frac{\partial W}{\partial S} = 0 & \Downarrow & & \\
 & \langle W_{\text{eff}} \rangle & & \\
 & | & & \\
 & \langle W_{\text{tree}} \rangle & \text{Calculate separately} & \\
 & \| & & \\
 & W_{\text{n.p.}} & &
\end{array}
$$

Restriction on the matter rep.

We impose:

- **Vector-like** , i.e. gauge-invariant mass terms can be given to all fields.

- No $U(1)$ factor left in low energy.

- Each gauge invariants $\mathcal{O}_i$ is a polynomial of F_a's. e.g.

$F_i = \mathrm{tr}\,\Phi^i$, $(i = 1, \ldots, N_c)$

$\mathcal{O} = F_i F_j F_k \cdots = (\mathrm{tr}\,\Phi^i)(\mathrm{tr}\,\Phi^j)$, etc.

- Either the R-charge of Λ vanishes, or satisfies the Restriction (♣) :

no dynamical constraints among $\langle F_a \rangle$

$$P(\langle F_1 \rangle, \langle F_2 \rangle, \cdots) = 0.$$

on constraints...

(♣) may sounds a serious restriction, but it is **NOT!**

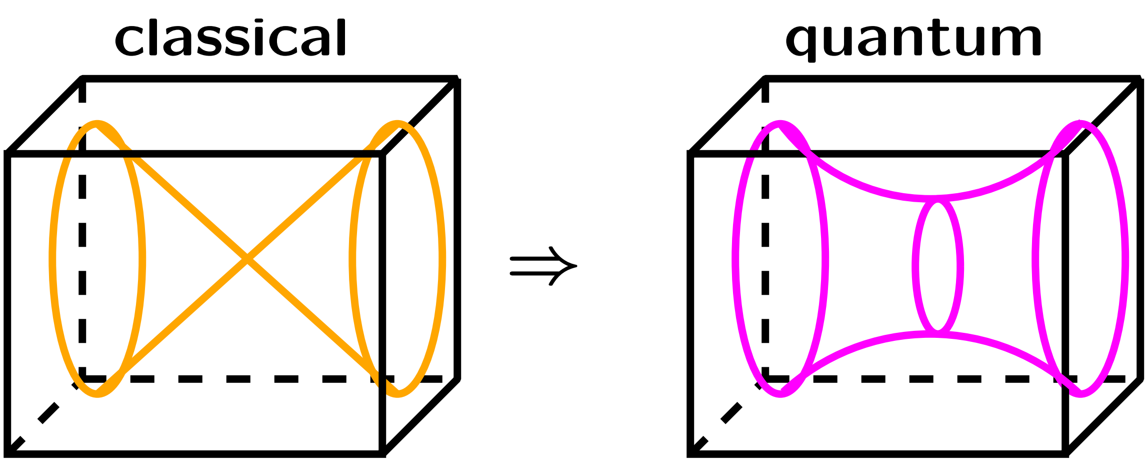
Example: $SU(Nc)$ SYM with N_f pairs of fundamentals Q_i, $\tilde{Q}_i$.

Gauge invariants:

$$T_{ij} = Q_i^a \tilde{Q}_{ja},$$
$$B_{ijkl\ldots} = \epsilon_{abc\ldots} Q_i^a Q_j^b Q_k^c \cdots$$
$$\tilde{B}_{ijkl\ldots} = \epsilon_{abc\ldots} \tilde{Q}_i^a \tilde{Q}_j^b \tilde{Q}_k^c \cdots$$

$N_c > N_f$: No constraints.

$N_c = N_f$: One constraint:

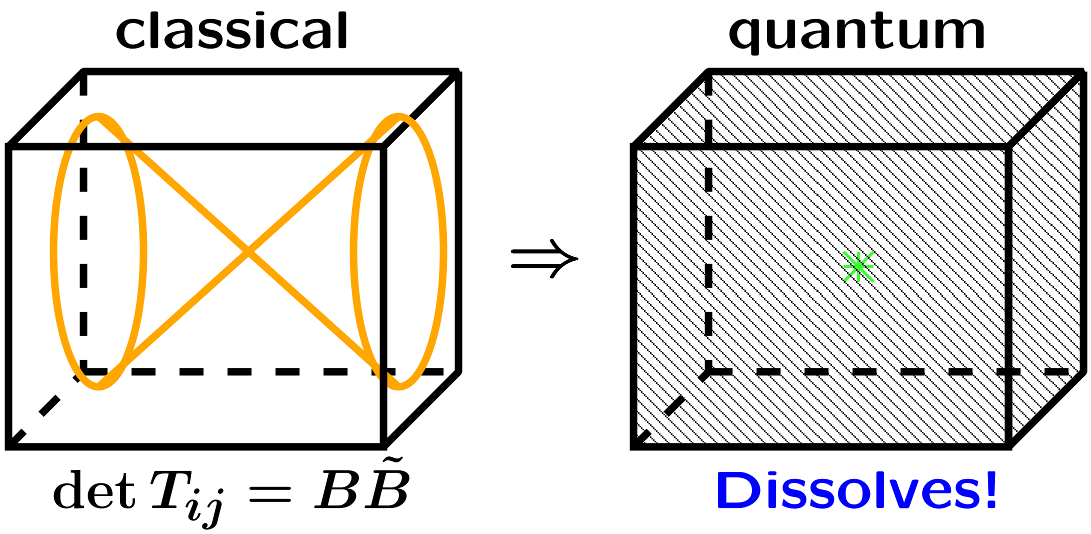


$\det T_{ij} = B\tilde{B}$ $\Rightarrow$ $\det T_{ij} - B\tilde{B} = \Lambda^{2N_f}$

However, R-charge of Λ is **zero.**

$\rightsquigarrow$ need not impose (♣)!

$N_c < N_f$:



Satisfies (♣)!

• The SO, Sp cases are quite similar.

• (Non-)existence of constraints can be checked by DV. (more about this later...)

How to prove the ILS from the DV?

Strategy

- Step 1. Shows $W_{\text{n.p.}} = (N_c - N_f)\langle S\rangle$.
- Step 2. Shows, if there's no dynamical constraint ($\clubsuit$),

$$\langle S\rangle = f(\langle F_i\rangle, \Lambda, \cancel{g_i})$$

Do not depend explicitly on g_i.

$$\rightsquigarrow$$

- If the R-charge of Λ vanishes, $W_{\text{n.p.}} = 0$ from Step 1.

- If it does not vanish, by the restriction ($\clubsuit$) and Step 2, $W_{\text{n.p.}} \propto S$ is independent of the coupling constants.

This is the linearity principle. Q.E.D.

Step 1.

We show $W_{\text{n.p.}} = (N_c - N_f)S$.

- $$\underbrace{V}_{\#(\text{vertices})} - \underbrace{E}_{\#(\text{propagators})} + \underbrace{L}_{\#(\text{loops})} = 1$$

- V, E and L can be counted by

$$g_j\partial/\partial g_j, \qquad -m_i\partial/\partial m_i, \qquad S\partial/\partial S$$

$$\Downarrow$$

$$\left(1 - S\frac{\partial}{\partial S}\right) D = \underbrace{\left(\sum m_i \frac{\partial}{\partial m_i} + \sum g_j \frac{\partial}{\partial g_j}\right) D}_{\text{the contrib. of } D \text{ to } \langle W_{\text{tree}}\rangle}.$$

- One loop diagrams $\propto N_f S \log m$ $\Rightarrow$
Another contrib. $N_f S$ to $\langle W_{\text{tree}}\rangle$ $\Rightarrow$

$$\langle W_{\text{tree}}\rangle = N_f S + \left(1 - S\frac{\partial}{\partial S}\right)\sum D$$

- Because $\dfrac{\partial W_{\text{eff}}}{\partial S} = \dfrac{\partial}{\partial S}(W_{\text{VY}} + \sum D) = 0$,

$$\langle W_{\text{tree}}\rangle = \sum D + S\frac{\partial W_{\text{VY}}}{\partial S}.$$

- Hence $W_{\text{n.p.}} = (N_c - N_f)S$.

Note $N_c - N_f$ is half the R-charge of Λ^β

Step 2.

We show S is **independent of g_i** when expressed as a function of Λ and $\langle O_i \rangle$.

$$\Uparrow$$

The same as inquiring whether S does **not change** when g_i's are varied as long as $\underbrace{\langle O_i \rangle \text{ remain fixed}}_{\substack{\text{by the factorization,} \\ \text{the same as fixing } \langle F_a \rangle}}$.

Take $O_1 = F_1$, $O_2 = F_2$, . . . , $O_r = F_r$, and O_i $(i > r)$ be polnomials

<DERIVATION>

- From $\delta(\partial W_{\text{eff}}/\partial S) = 0$,

$$-\frac{\partial^2 W_{\text{eff}}}{\partial S \partial S} \delta \langle S \rangle = \sum_i \delta g_i \frac{\partial^2 W_{\text{eff}}}{\partial g_i \partial S}.$$

- Hence $\delta \langle F_j \rangle = \sum_i \delta g_i G_{ij}$ where

$$G_{ij} = \frac{\partial^2 W_{\text{eff}}}{\partial g_i \partial g_j} - \frac{\partial^2 W_{\text{eff}}}{\partial g_i \partial S} \frac{\partial^2 W_{\text{eff}}}{\partial S \partial g_j} \Big/ \frac{\partial^2 W_{\text{eff}}}{\partial S \partial S}$$

$$= \frac{\partial \langle O_i \rangle}{\partial g_j} - \frac{\partial \langle O_i \rangle}{\partial S} \frac{\partial^2 W_{\text{eff}}}{\partial S \partial g_j} \Big/ \frac{\partial^2 W_{\text{eff}}}{\partial S \partial S}.$$

• From factorization,

$$G_{ij} = \langle \frac{\partial O_i}{\partial F_a} \rangle G_{aj}.$$

Hence

$$\delta\langle F_j\rangle = \underbrace{\sum_a}_{\text{This sum only runs over } 1,2,\ldots,r} \left(\delta g_i \langle \frac{\partial O_i}{\partial F_a}\rangle\right) G_{aj}$$

• Similarly, one obtains

$$-\frac{\partial^2 W_{\text{eff}}}{\partial S \partial S}\delta\langle S\rangle = \sum_a \left(\delta g_i \langle \frac{\partial O_i}{\partial F_a}\rangle\right) \frac{\partial \langle F_a\rangle}{\partial S}$$

• The restriction (♣) ensures that the rank of G_{ij} is maximal.

Thus, $\delta\langle F_i\rangle = 0$ implies $\delta\langle S\rangle = 0$.

N.B.

When the rank of G_{ij} is not maximal, G_{ij} contains the info of the dynamical constraints. Further, G_{ij} can be computed perturbatively!

$$\Downarrow$$

The dynamical constraints, if any, can be readily studied in this approach.

5. Conclusions

Summary

• We derived the ILS linearity principle in the framework of Dijkgraaf-Vafa

Future work

Directly related to this work:

• Lifting the restriction (♣)

More generally, interesting directions around the DV framework are:

• Direct topological twisting of $\mathcal{N} = 1$ SYM theories

• Extending to matter superfields not in a vector-like representation.