The Fokker-Planck formalism for closed bosonic strings

Talk at SFT@Cloud

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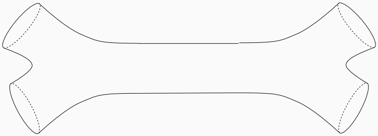
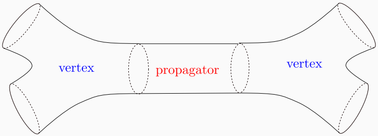
March 9, 2023

String Field Theory (SFT)





- The amplitudes in string theory are expressed by Feynman diagrams = worldsheets~Riemann surfaces
- In order to construct an SFT, we should define a rule to cut all the worldsheets into propagators and vertices systematically.
  - In general, we need infinitely many vertices to do so.

$$S = \phi K \phi + \phi^3 + \phi^4 + \cdots + \hbar \phi + \cdots$$

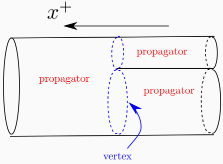
  - Such a theory can be studied by using the homotopy algebra methods. (Zwiebach, ...)

We would like to find out a way to construct an SFT as simple as

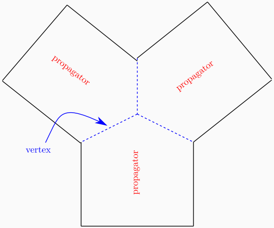
$$S = \phi K \phi + \phi^3$$

- So far, there exist essentially two known rules for which the theory looks like that.

LC type

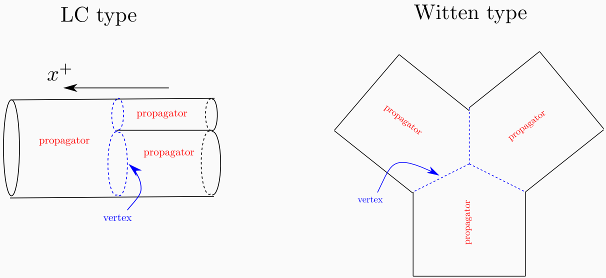


Witten type



- We would like to find out yet another rule.

SFT with only three string vertices



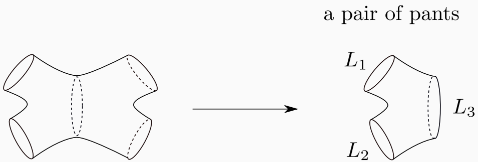
- SFT's for bosonic strings were constructed using these rules.

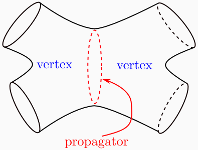
$$S = \phi K \phi + \phi^3$$

  - Light-cone gauge SFT(Kaku-Kikkawa), $\alpha = p^+$ HIKKO (Kugo-Zwiebach theory), covariantized light-cone
  - Witten's SFT

- These rules do not work for superstrings, because of the "spurious singularity" problem.

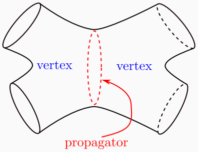
The pants decomposition

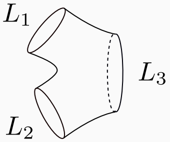




- A Riemann surface with $2g-2+n>0$ admits a hyperbolic metric such that the boundaries are geodesics. (cf. Moosavian-Pius, Costello-Zwiebach)
- It can be decomposed into pairs of pants whose boundaries are geodesics.

An SFT based on the pants decomposition?





- We may be able to construct an SFT based on the pants decomposition at least for closed bosonic strings

$$S = \phi K \phi + \phi^3$$

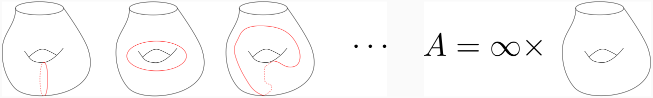
- The SFT will be quite different from the usual ones.
  - The string field $|\phi(L)\rangle$ depends on the length L of the string
    - We may consider $|\phi(L)\rangle$ as an operator from which we can derive various properties of the particles.
  - The kinetic term should be different from the conventional one

$$\langle \phi | Q c_0^+ | \phi \rangle$$

An SFT based on the pants decomposition?

$$S = \phi K \phi + \phi^3$$

- This action does not work. (D'Hoker-Gross)
    - One-loop one point amplitudes diverge because the pants decomposition is not unique.



    - The decompositions are related by modular transformations.
    - Most of the amplitudes diverge in the same way.

- We cannot construct the action.
  We should take an alternative approach.$\longrightarrow$the Fokker-Planck formalism

The Fokker-Planck formalism

- Euclidean field theory: action $S[\phi]$

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = \frac{\int [d\phi] e^{-S[\phi]}\phi(x_1)\cdots\phi(x_n)}{\int [d\phi] e^{-S[\phi]}}$$

- Fokker-Planck formalism

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = \lim_{\tau\to\infty}\langle 0|e^{-\tau \hat{H}_{\mathrm{FP}}}\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)|0\rangle$$

$$[\hat{\pi}(x), \hat{\phi}(y)] = \delta(x-y)\,, [\hat{\pi}, \hat{\pi}] = [\hat{\phi}, \hat{\phi}] = 0$$

$$\langle 0|\hat{\phi}(x) = \hat{\pi}(x)|0\rangle = 0$$

$$\hat{H}_{\mathrm{FP}} = -\int dx \left(\hat{\pi}(x) + \frac{\delta S}{\delta\phi(x)}[\hat{\phi}]\right)\hat{\pi}(x)$$

- path integral: action $I_{\mathrm{FP}}[\phi, \pi]$

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = \frac{\int [d\pi d\phi] e^{-I_{\mathrm{FP}}}\phi(0, x_1)\cdots\phi(0, x_n)}{\int [d\pi d\phi] e^{-I_{\mathrm{FP}}}}$$

$$I_{\mathrm{FP}} = \int_0^\infty d\tau \left[-\int dx \pi \partial_\tau \phi + H_{\mathrm{FP}}\right]$$

In this talk

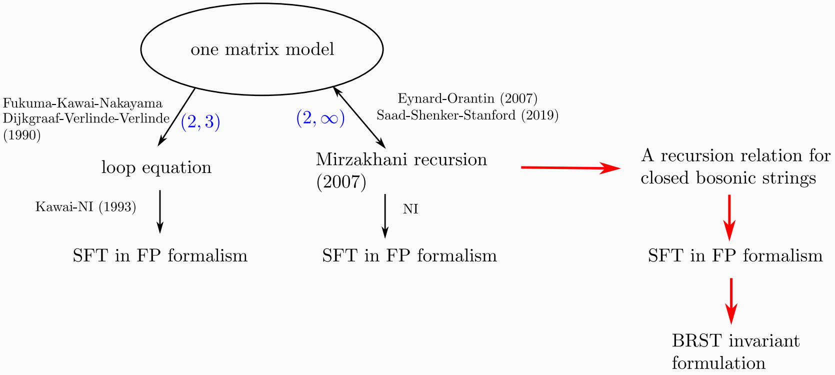
- I would like to show that it is possible to construct an SFT for closed bosonic strings based on the pants decomposition via the Fokker-Planck formalism.

$$
\begin{aligned}
&I_{\rm FP}[\phi,\pi,\lambda] \\
&\quad = \int_0^\infty d\tau \Bigg[- \int_0^\infty dL \langle R|\pi_\alpha(\tau,L)\rangle \frac{\partial}{\partial\tau}|\phi^\alpha(\tau,L)\rangle + H(\tau) \\
&\qquad\qquad + \int_0^\infty dL \Big(\langle R|\mathcal{Q}^\alpha(\tau,L)\rangle|\lambda_\alpha^{\mathcal{Q}}(\tau,L)\rangle + \langle R|\mathcal{T}^\alpha(\tau,L)\rangle|\lambda_\alpha^{\mathcal{T}}(\tau,L)\rangle \Big) \Bigg]
\end{aligned}
$$

  - $\lambda_\alpha^{\mathcal{Q}}, \lambda_\alpha^{\mathcal{T}}$: auxiliary fields
  - This action consists of kinetic terms and three string interaction terms.
  - $S[\phi]$ is not well-defined in our setup.

Based on PTEP **2023**,023B05 (2023)

This talk

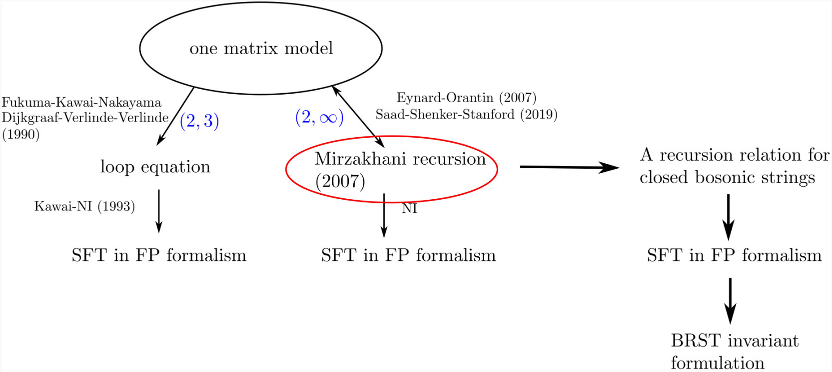

one matrix model
Fukuma-Kawai-Nakayama
Dijkgraaf-Verlinde-Verlinde
(1990)
$(2, 3)$
$(2, \infty)$
Eynard-Orantin (2007)
Saad-Shenker-Stanford (2019)
loop equation
Mirzakhani recursion
(2007)
A recursion relation for
closed bosonic strings
Kawai-NI (1993)
NI
SFT in FP formalism
SFT in FP formalism
SFT in FP formalism
BRST invariant
formulation

Plan of the talk

1. Mirzakhani recursion
2. A recursion relation for the off-shell amplitudes of closed bosonic strings
3. The Fokker-Planck formalism
4. BRST invariant formulation
5. Conclusions

1. Mirzakhani recursion

1. Mirzakhani recursion

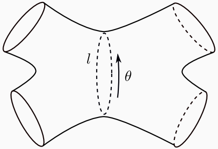


- Reviews: Moosavian-Pius, Do arXiv:1103.4674 [math], Huang arXiv:1509.06880 [math.GT]

The volume of the moduli space of Riemann surfaces with genus g and n boundaries ($2g-2+n>0$) whose lengths are $L_1,\cdots,L_n$ is given by

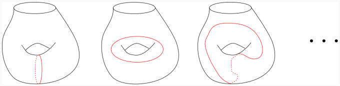
$$V_{g,n}(L_1,\cdots,L_n)=\int\prod_s\left[\frac{l_s dl_s d\theta_s}{2\pi}\right]$$

- The moduli space of Riemann surfaces (genus g, n boundaries) is parametrized by the coordinates $(l_s;\theta_s)$ $(s=1,\cdots,3g-3+n)$.
  - l_s denotes the length of a nonperipheral boundary and θ_s is the twist angle in a pants decomposition.



$$V_{g,n}(L_1,\cdots,L_n) = \int \prod_s \left[\frac{l_s dl_s d\theta_s}{2\pi}\right]$$

- Integrating over $0 < l_s < \infty$, the integral diverges.
  - The pants decomposition is not unique. There are infinitely many pants decomposition related by modular transformations.



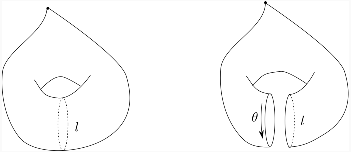
$\cdots$

$$\int_0^\infty \int_0^{2\pi} \frac{l dl d\theta}{2\pi} = \infty \times$$

- We should integrate over the fundamental domain $\mathcal{F}$, which is very complicated in general.

$$V_{g,n}(L_1,\cdots,L_n) = \int_{\mathcal{F}} \prod_s \left[\frac{l_s dl_s d\theta_s}{2\pi}\right]$$

McShane identity $(g = n = 1, L = 0)$



- McShane identity (1998): for $f(l) = \frac{2}{1+e^l}$

$$1 = \sum_{\gamma \in \textbf{modular group}} f(\gamma \cdot l)$$

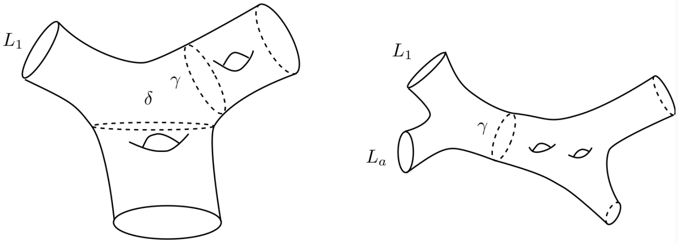
- $V_{1,1}$ can be calculated multiplying this by $\int_{\mathcal{F}} \frac{l dl d\theta}{2\pi}$ (Mirzakhani)

$$\begin{aligned} V_{1,1}(0) &= \int_{\mathcal{F}} \frac{l dl d\theta}{2\pi} = \int_{\mathcal{F}} \sum_{\gamma} f(\gamma \cdot l) \frac{l dl d\theta}{2\pi} \\ &= \int_{\mathcal{F}} \sum_{\gamma} f(\gamma \cdot l) \frac{\gamma \cdot l d(\gamma \cdot l) d(\gamma \cdot \theta)}{2\pi} = \sum_{\gamma} \int_{\gamma \mathcal{F}} f(l) \frac{l dl d\theta}{2\pi} \\ &= \int \frac{dl d\theta l}{2\pi} \frac{2}{1+e^l} = \frac{\pi^2}{6} \end{aligned}$$

Generalized McShane identity

- Mirzakhani obtained identities for general g, n with $2g - 2 + n > 0$.

$$L_1 = \sum_{\{\gamma,\delta\}\in\mathcal{C}_1} D_{L_1 l_\gamma l_\delta} + \sum_{a=2}^{n} \sum_{\gamma\in\mathcal{C}_a} (T_{L_1 L_a l_\gamma} + D_{L_1 L_a l_\gamma})$$



$$D_{LL'L''} = 2\left(\log(e^{\frac{L}{2}} + e^{\frac{L'+L''}{2}}) - \log(e^{-\frac{L}{2}} + e^{\frac{L'+L''}{2}})\right)$$

$$T_{LL'L''} = \log \frac{\cosh\frac{L''}{2} + \cosh\frac{L+L'}{2}}{\cosh\frac{L''}{2} + \cosh\frac{L-L'}{2}}$$

Mirzakhani recursion relation

Multiplying

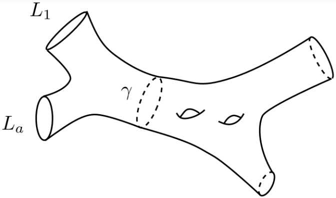
$$L_1 = \sum_{\{\gamma,\delta\}\in\mathscr{C}_1} \mathsf{D}_{L_1 l_\gamma l_\delta} + \sum_{a=2}^{n}\sum_{\gamma\in\mathscr{C}_a} (\mathsf{T}_{L_1 L_a l_\gamma} + \mathsf{D}_{L_1 L_a l_\gamma})$$

by $\int_{\mathcal{F}} \prod_s \left[\frac{l_s dl_s d\theta_s}{2\pi}\right]$, we get

$$\begin{aligned} LV_{g,n+1}(L,\mathbf{L}) &= \frac{1}{2}\int_0^\infty dL'L' \int_0^\infty dL''L'' D_{LL'L''} V_{g-1,n+2}(L',L'',\mathbf{L}) \\ &+\frac{1}{2}\int_0^\infty dL'L' \int_0^\infty dL''L'' D_{LL'L''} \sum_{\text{stable}} V_{g_1,n_1}(L',\mathbf{L}_1) V_{g_2,n_2}(L'',\mathbf{L}_2) \\ &+\sum_{a=1}^{n}\int_0^\infty dL'L' \left(T_{L_1 L_a L'} + D_{L_1 L_a L'}\right) V_{g,n}(L,\mathbf{L}\backslash L_a) \end{aligned}$$

- One can calculate $V_{g,n}(L_1,\cdots,L_n)$ by solving this equation.
  - The right hand side consists of quantities with less $2g-2+n$.

Mirzakhani recursion relation



$$\int_{\mathcal{F}} \prod_s \left[\frac{l_s dl_s d\theta_s}{2\pi}\right] \times$$

$$L_1 = \sum_{\{\gamma,\delta\}\in\mathcal{C}_1} D_{L_1 l_\gamma l_\delta} + \sum_{a=2}^{n} \sum_{\gamma\in\mathcal{C}_a} \left(T_{L_1 L_a l_\gamma} + D_{L_1 L_a l_\gamma}\right)$$

$$\begin{aligned}
&\int_{\mathcal{F}} \prod_s \left[\frac{l_s dl_s d\theta_s}{2\pi}\right] \sum_{\gamma\in\mathcal{C}_a} \left(T_{L_1 L_a l_\gamma} + D_{L_1 L_a l_\gamma}\right) \\
&= \sum_{\gamma\in\mathcal{C}_a} \int_{\mathcal{F}} \frac{l_\gamma dl_\gamma d\theta_\gamma}{2\pi} \left(T_{L_1 L_a l_\gamma} + D_{L_1 L_a l_\gamma}\right) \times \cdots \\
&= \int_0^\infty dl\, l \left(T_{L_1 L_a l} + D_{L_1 L_a l}\right) V_{g,n}(l, \mathbf{L} \backslash L_a)
\end{aligned}$$

2. A recursion relation for the off-shell amplitudes of closed bosonic strings

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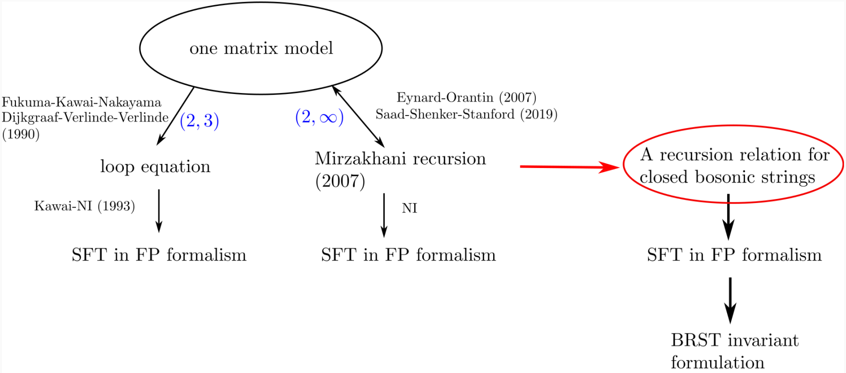
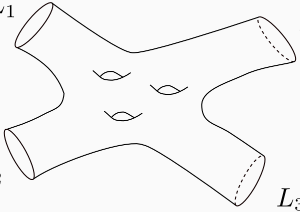


$$
\begin{aligned}
L_1 A^{I_1\cdots I_n}_{g,n} &= L_1 G^{I_1 I_2}\delta_{g,0}\delta_{n,2} \\
&\quad + \frac{1}{2} D^{I_1 J' J} G_{JI} G_{J'I'} \left[A^{II' I_2 \cdots I_n}_{g-1,n+1} + \sum{}' \frac{\varepsilon_{\mathcal{I}_1 \mathcal{I}_2}}{(n_1-1)!(n_2-1)!} A^{I\mathcal{I}_1}_{g_1,n_1} A^{I'\mathcal{I}_2}_{g_2,n_2} \right] \\
&\quad + \sum_{a=2}^{n} \varepsilon_a T^{I_1 I_a J} G_{JI} A^{I I_2 \cdots \hat{I}_a \cdots I_n}_{g,n-1}
\end{aligned}
$$



- In string theory, the amplitudes are given by integrals over the moduli space of Riemann surfaces

$$A_{g,n} = \int_{\mathcal{F}} \prod_s [dl_s d\theta_s] \langle \prod_s [b(\partial_{l_s}) b(\partial_{\theta_s})] V_{i_1} \cdots V_{i_n} \rangle$$

- It is conceivable that we can derive a recursion relation for these amplitudes in the same way as we did for the recursion relation for

$$V_{g,n}(L_1, \cdots, L_n) = \int_{\mathcal{F}} \prod_s \left[\frac{l_s dl_s d\theta_s}{2\pi} \right]$$

The recursion relation

generalized McShane identity

$$L_1 = \sum_{\{\gamma,\delta\}\in\mathcal{C}_1} D_{L_1 l_\gamma l_\delta} + \sum_{a=2}^{n}\sum_{\gamma\in\mathcal{C}_a}(T_{L_1 L_a l_\gamma} + D_{L_1 L_a l_\gamma})$$

$$\xrightarrow{\int_{\mathcal{F}}\prod_s\left[\frac{l_s dl_s d\theta_s}{2\pi}\right]\times}$$

recursion relation for

$$V_{g,n}(L_1,\cdots,L_n) = \int_{\mathcal{F}}\prod_s\left[\frac{l_s dl_s d\theta_s}{2\pi}\right]$$

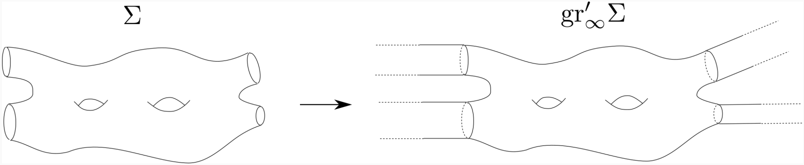
$$\Big\downarrow\ \int_{\mathcal{F}}\prod_s dl_s d\theta_s \left\langle \prod_s [b(\partial_{l_s})b(\partial_{\theta_s})]\, V_{i_1}\cdots V_{i_n}\right\rangle \times$$

recursion relation for

$$A^{i_1\cdots i_n}_{g,n} = \int_{\mathcal{F}}\prod_s dl_s d\theta_s \left\langle \prod_s [b(\partial_{l_s})b(\partial_{\theta_s})]\, V_{i_1}\cdots V_{i_n}\right\rangle$$

$$\begin{aligned} L_1 A^{I_1\cdots I_n}_{g,n} &= L_1 G^{I_1 I_2}\delta_{g,0}\delta_{n,2} \\ &+\frac{1}{2}D^{I_1 J' J}G_{JI}G_{J'I'}\left[A^{II'I_2\cdots I_n}_{g-1,n+1} + \sum{}' \frac{\varepsilon_{\mathcal{I}_1\mathcal{I}_2}}{(n_1-1)!(n_2-1)!}A^{I\mathcal{I}_1}_{g_1,n_1}A^{I'\mathcal{I}_2}_{g_2,n_2}\right] \\ &+\sum_{a=2}^{n}\varepsilon_a T^{I_1 I_a J}G_{JI}A^{II_2\cdots\hat{I}_a\cdots I_n}_{g,n-1} \end{aligned}$$

Details 1:The off-shell amplitudes



- The off-shell amplitudes on Σ can be defined using $\mathrm{gr}'_\infty \Sigma$. (Costello-Zwiebach)

$$A_{g,n} = \int_{\mathcal{F}} \prod_s [dl_s d\theta_s] \langle \prod_s [b(\partial_{l_s}) b(\partial_{\theta_s})] V_{i_1} \cdots V_{i_n} \rangle$$

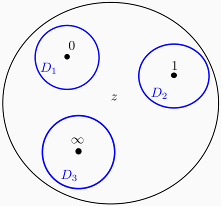
  - We can use the coordinates l_s, θ_s to parameterize the moduli space of the punctured Riemann surface. (Mondello)
  - For states $|\varphi^{i_a}\rangle = \mathcal{O}_{i_a}(0)|0\rangle$ in the state space of the bosonic string (in any background), satisfying

$$(b_0 - \bar{b}_0)|\varphi^{i_a}\rangle = (L_0 - \bar{L}_0)|\varphi^{i_a}\rangle = 0$$

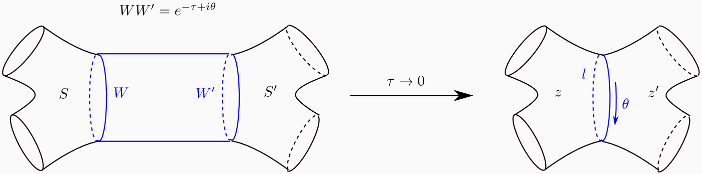
Details 2: b-ghost insertions

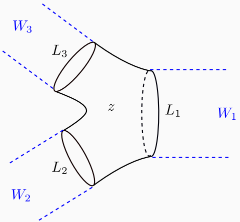
$$A_{g,n} = \int_{\mathcal{F}} \prod_s [dl_s d\theta_s] \langle \prod_s [b(\partial_{l_s}) b(\partial_{\theta_s})] V_{i_1} \cdots V_{i_n} \rangle$$

- $b(\partial_{l_s}), b(\partial_{\theta_s})$ are constructed following the standard prescription. (Sen 2015, Erbin's book, ...)
  - They are expressed by the variations of the transition functions between local patches.
  - In our case, we can take the patches to be the pairs of pants.
  - Since a pair of pants~ $\mathbb{C} - \cup_k D_k$, we take z on $\mathbb{C}$ as the local coordinate.



b-ghost insertions





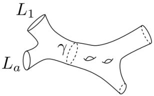
$$
\begin{aligned}
b(\partial_l) &= b_S(\partial_l) + b_{S'}(\partial_l) \\
& \quad b_S(\partial_l) = -\oint_{\partial S} \frac{dz}{2\pi i} \frac{\partial W_k}{\partial l} \left(\frac{\partial W_k}{\partial z}\right)^{-1} b(z) + \text{c.c.} \\
& \quad b_{S'}(\partial_l) = -\oint_{\partial S'} \frac{dz'}{2\pi i} \frac{\partial W_k}{\partial l} \left(\frac{\partial W_k}{\partial z'}\right)^{-1} b(z') + \text{c.c.} \\
b(\partial_\theta) &= b_0 - \bar{b}_0 \text{ on the cylinder}
\end{aligned}
$$

- The explicit forms of $W_k(z)$ are given in terms of the hypergeometric function (Fırat, Hadasz-Jaskolski)
- $b(\partial_l)$ has contributions from two adjacent pairs of pants.

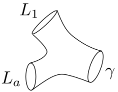
$$\int_{\mathcal{F}} \prod_s \left[\frac{l_s dl_s d\theta_s}{2\pi}\right] \langle \cdots \rangle \times$$

$$L_1 = \sum_{\{\gamma,\delta\}\in\mathcal{C}_1} D_{L_1 l_\gamma l_\delta} + \sum_{a=2}^{n} \sum_{\gamma\in\mathcal{C}_a} (T_{L_1 L_a l_\gamma} + D_{L_1 L_a l_\gamma})$$

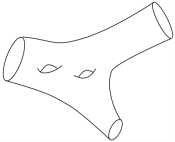
$$\int_{\mathcal{F}} \prod_s \left[\frac{l_s dl_s d\theta_s}{2\pi}\right] \sum_{\gamma\in\mathcal{C}_a} (T_{L_1 L_a l_\gamma} + D_{L_1 L_a l_\gamma}) \times$$

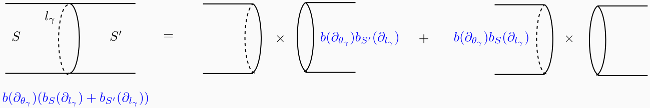


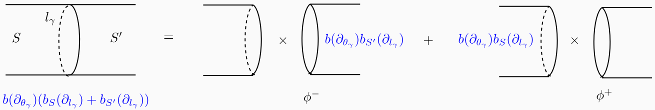
$$= \sum_{\gamma\in\mathcal{C}_a} \int \frac{l_\gamma dl_\gamma d\theta_\gamma}{2\pi} (T_{L_1 L_a l_\gamma} + D_{L_1 L_a l_\gamma})$$



$\times \int$







- The string field is labeled by $(i, L, \alpha) \equiv I$ $(\alpha = \pm)$

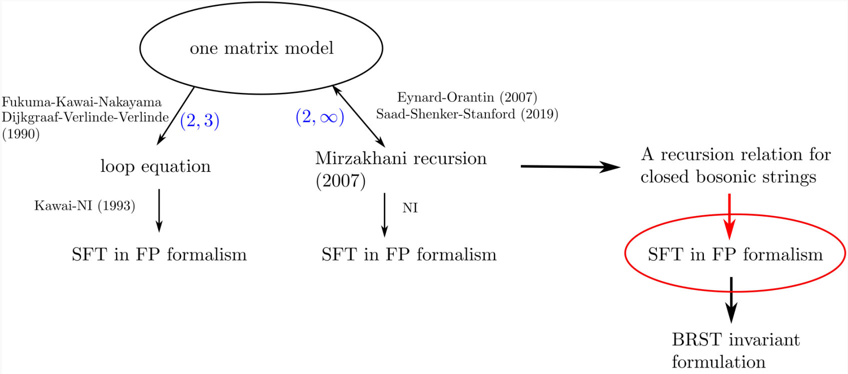
$$A_{g,n}^{I_1\cdots I_n} = \int_{\mathcal{F}} \prod_s [dl_s d\theta_s] \langle \prod_s [b(\partial_{l_s})b(\partial_{\theta_s})]\, B_{\alpha_1}\cdots B_{\alpha_n} V_{i_1}\cdots V_{i_n}\rangle$$

$$B_{\alpha_a} \equiv \begin{cases} 1 & \alpha_a = + \\ (b_0^{(a)} - \bar{b}_0^{(a)}) b_{S_a}(\partial_{L_a}) \int_0^{2\pi} \frac{d\theta_a}{2\pi} e^{i\theta_a(L_0^{(a)} - \bar{L}_0^{(a)})} & \alpha_a = - \end{cases}$$

$$\begin{aligned} L_1 A_{g,n}^{I_1\cdots I_n} &= L_1 G^{I_1 I_2}\delta_{g,0}\delta_{n,2} \\ &\quad + \frac{1}{2} D^{I_1 J' J} G_{JI} G_{J'I'} \left[A_{g-1,n+1}^{II' I_2\cdots I_n} + \sum{}' \frac{\varepsilon_{\mathcal{I}_1\mathcal{I}_2}}{(n_1-1)!(n_2-1)!} A_{g_1,n_1}^{I\mathcal{I}_1} A_{g_2,n_2}^{I'\mathcal{I}_2} \right] \\ &\quad + \sum_{a=2}^{n} \varepsilon_a T^{I_1 I_a J} G_{JI} A_{g,n-1}^{II_2\cdots\hat{I}_a\cdots I_n} \end{aligned}$$

$$\begin{aligned} T^{I_1 I_2 I_3} &\equiv T_{L_1 L_2 L_3} \langle B_{\alpha_1} B_{\alpha_2} B_{\alpha_3} V^{i_1} V^{i_2} V^{i_3}\rangle \\ D^{I_1 I_2 I_3} &\equiv D_{L_1 L_2 L_3} \langle B_{\alpha_1} B_{\alpha_2} B_{\alpha_3} V^{i_1} V^{i_2} V^{i_3}\rangle \\ G_{I_1 I_2} &\equiv \langle \varphi_{i_1}^c | \varphi_{i_2}^c \rangle (-1)^{n_{\varphi_{i_2}}} \delta(L_1 - L_2)\delta_{\alpha_1,-\alpha_2}\,, \end{aligned}$$

3. The Fokker-Planck formalism



$$
\begin{aligned}
\hat{H}_{\rm FP} \quad = \quad & -L\hat{\pi}_I\hat{\pi}_{I'}G^{I'I} + L\hat{\phi}^I\hat{\pi}_I \\
& -\frac{1}{2}g_{\rm s}D^{II'I''}G_{I''K''}G_{I'K'}\hat{\phi}^{K''}\hat{\phi}^{K'}\hat{\pi}_I \\
& -g_{\rm s}T^{II'I''}G_{I''K''}\hat{\phi}^{K''}\hat{\pi}_{I'}\hat{\pi}_I
\end{aligned}
$$

The Fokker-Planck formalism

$$
\begin{aligned}
L_1 A^{I_1\cdots I_n}_{g,n} \;=\;& L_1 G^{I_1 I_2}\delta_{g,0}\delta_{n,2} \\
&+\frac{1}{2}D^{I_1 J' J} G_{JI} G_{J'I'}\left[A^{II' I_2\cdots I_n}_{g-1,n+1} + \sum{}' \frac{\varepsilon_{\mathcal{I}_1\mathcal{I}_2}}{(n_1-1)!(n_2-1)!} A^{I\mathcal{I}_1}_{g_1,n_1} A^{I'\mathcal{I}_2}_{g_2,n_2}\right] \\
&+\sum_{a=2}^{n} \varepsilon_a T^{I_1 I_a J} G_{JI} A^{I I_2 \cdots \hat{I}_a \cdots I_n}_{g,n-1}
\end{aligned}
$$

- One can derive the amplitudes $A^{I_1\cdots I_n}_{g,n}$ perturbatively solving this equation.
- This equation can be regarded as the Schwinger-Dyson equation of string theory.
  - We may be able to construct an SFT from this equation.

- This equation can be turned into an SFT in the FP formalism via the method developed by Kawai-NI, Jevicki-Rodrigues, Fukuma-Kawai-Ninomiya-NI, Ikehara-Kawai-Mogami-Nakayama-Sasakura-NI , Ikehara, ...

The Fokker-Planck formalism for closed bosonic strings

- The off-shell amplitudes

$$\langle\!\langle \phi^{I_1}\cdots\phi^{I_n}\rangle\!\rangle \longleftrightarrow \lim_{\tau\to\infty}\langle\!\langle 0|e^{-\tau\hat{H}_{\rm FP}}\hat{\phi}^{I_1}\cdots\hat{\phi}^{I_n}|0\rangle\!\rangle$$

$$\langle\!\langle \phi^{I_1}\cdots\phi^{I_n}\rangle\!\rangle^{\rm c} = \sum_{g=0}^{\infty} g_{\rm s}^{2g-2+n} A_{g,n}^{I_1\cdots I_n}$$

- The Fokker-Planck formalism

$$[\hat{\pi}_I,\hat{\phi}^K] = \delta_I{}^K$$
$$[\hat{\pi}_I,\hat{\pi}_K] = [\hat{\phi}^I,\hat{\phi}^K] = 0$$
$$\langle\!\langle 0|\hat{\phi}^I = \hat{\pi}_I|0\rangle\!\rangle = 0$$

∗ The recursion relation

∗ SD equation

$$\mathcal{T}^I\langle\!\langle e^{J_I\phi^I}\rangle\!\rangle = 0 \longleftrightarrow H_{\rm FP}[\frac{\delta}{\delta J},J]\lim_{\tau\to\infty}\langle\!\langle 0|e^{-\tau\hat{H}_{\rm FP}}e^{J_I\hat{\phi}^I}|0\rangle\!\rangle = 0$$

$$H_{\rm FP}[\frac{\delta}{\delta J},J] = J_I\mathcal{T}^I$$

$$\lim_{\tau\to\infty}\partial_\tau\langle\!\langle 0|e^{-\tau\hat{H}_{\rm FP}[\hat{\phi},\hat{\pi}]}\hat{\phi}^{I_1}\cdots\hat{\phi}^{I_n}|0\rangle\!\rangle = 0$$

$$\begin{aligned}\mathcal{T}^I \;\equiv\; & -LG^{I'I}J_{I'} + L\frac{\delta}{\delta J_I}\\ & -\frac{1}{2}g_{\rm s}D^{II'I''}G_{I''K''}G_{I'K'}\frac{\delta^2}{\delta J_{K''}\delta J_{K'}}\\ & -g_{\rm s}T^{II'I''}G_{I''K''}J_{I'}\frac{\delta}{\delta J_{K''}}(-1)^{|I||I'|},\end{aligned}$$

$$\hat{H}_{\rm FP} = \hat{T}^I\hat{\pi}_I$$

The Fokker-Planck formalism for closed bosonic strings

$$
\begin{aligned}
\hat{H}_{\mathrm{FP}} &= \hat{T}^{I}\hat{\pi}_{I} \\
&= -L\hat{\pi}_{I}\hat{\pi}_{I'}G^{I'I} + L\hat{\phi}^{I}\hat{\pi}_{I} \\
&\quad -\frac{1}{2}g_{\mathrm{s}}D^{II'I''}G_{I''K''}G_{I'K'}\hat{\phi}^{K''}\hat{\phi}^{K'}\hat{\pi}_{I} \\
&\quad -g_{\mathrm{s}}T^{II'I''}G_{I''K''}\hat{\phi}^{K''}\hat{\pi}_{I'}\hat{\pi}_{I} \\
\hat{T}^{I} &= -L\hat{\pi}_{I'}G^{II'} + L\hat{\phi}^{I} \\
&\quad -\frac{1}{2}g_{\mathrm{s}}D^{II'I''}G_{I''K''}G_{I'K'}\hat{\phi}^{K''}\hat{\phi}^{K'} \\
&\quad -g_{\mathrm{s}}T^{II'I''}G_{I''K''}\hat{\phi}^{K''}\hat{\pi}_{I'}
\end{aligned}
$$

$$
\langle\!\langle \phi^{I_1}\cdots\phi^{I_n}\rangle\!\rangle = \lim_{\tau\to\infty}\langle\!\langle 0|e^{-\tau\hat{H}_{\mathrm{FP}}}\hat{\phi}^{I_1}\cdots\hat{\phi}^{I_n}|0\rangle\!\rangle
$$

- The Hamiltonian consists of the kinetic terms and the three string interaction terms.

The action $S[\phi]$

- It is possible to (formally) define the action $S[\phi]$.

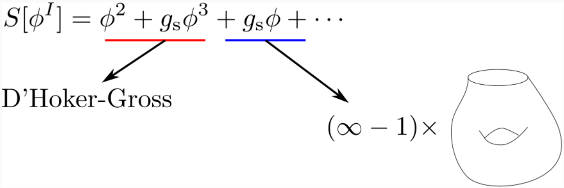
$$\frac{e^{-S[\phi]}}{\int [d\phi] e^{-S[\phi]}} = \lim_{\tau\to\infty} \langle\!\langle 0 | e^{-\tau \hat{H}} \delta(\phi - \hat{\phi}) | 0 \rangle\!\rangle$$

$$\begin{aligned} &\frac{\int [d\phi] e^{-S[\phi]} \phi^{I_1} \cdots \phi^{I_n}}{\int [d\phi] e^{-S[\phi]}} \\ &= \lim_{\tau\to\infty} \langle\!\langle 0 | e^{-\tau \hat{H}} \int [d\phi] \delta(\phi - \hat{\phi}) \phi^{I_1} \cdots \phi^{I_n} | 0 \rangle\!\rangle \\ &= \lim_{\tau\to\infty} \langle\!\langle 0 | e^{-\tau \hat{H}} \hat{\phi}^{I_1} \cdots \hat{\phi}^{I_n} | 0 \rangle\!\rangle \end{aligned}$$

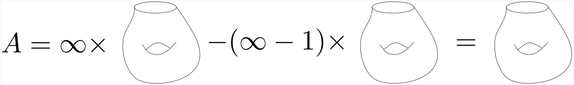
- One can calculate $S[\phi^I]$ perturbatively.

$$\begin{aligned} S[\phi^I] &= \frac{1}{2} G_{IJ} \phi^I \phi^J - \frac{g_s}{6} A_{0,3}^{II'I''} G_{IJ} G_{I'J'} G_{I''J''} \phi^{J''} \phi^{J'} \phi^J \\ &\quad + \frac{g_s}{L} T^{II'I''} G_{I'I''} G_{IJ} \phi^J + \mathcal{O}(g_s^2) \end{aligned}$$

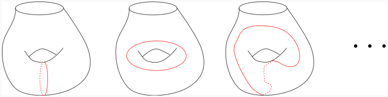
$$\begin{aligned} &[L G^{IJ} + g_s T^{IJI'} G_{I'J'} \phi^{J'}] \frac{\delta S}{\delta \phi^J} \\ &= L\phi^I - \frac{1}{2} g_s D^{II'I''} G_{I'J'} G_{I''J''} \phi^{J''} \phi^{J'} + g_s T^{II'I''} G_{I'I''} \end{aligned}$$



- $S[\phi^I]$ is divergent and ill defined.
  - The 1 loop 1 point amplitude



  - $S[\phi^I]$ includes infinitely many divergent counterterms.
- FP formalism breaks the modular invariance.



4. BRST invariant formulation

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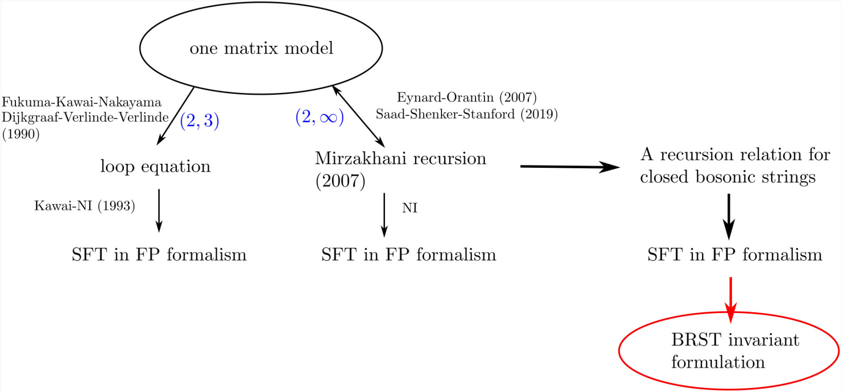

$$
\begin{aligned}
I_{\mathrm{FP}}[\phi,\pi,\lambda] \\
= \int_0^\infty d\tau \Big[- \int_0^\infty dL \langle R|\pi_\alpha(\tau,L)\rangle \frac{\partial}{\partial\tau}|\phi^\alpha(\tau,L)\rangle + H(\tau) \\
+ \int_0^\infty dL \left(\langle R|\mathcal{Q}^\alpha(\tau,L)\rangle |\lambda_\alpha^{\mathcal{Q}}(\tau,L)\rangle + \langle R|\mathcal{T}^\alpha(\tau,L)\rangle |\lambda_\alpha^{\mathcal{T}}(\tau,L)\rangle \right) \Big]
\end{aligned}
$$

BRST symmetry on the worldsheet

- We need the worldsheet BRST symmetry to define the physical states with positive norm.

$$
\begin{aligned}
Q|\text{phys.}\rangle &= 0 \\
|\ \rangle &\sim |\ \rangle + Q|\ \rangle'
\end{aligned}
$$

- In order to discuss this symmetry, we change the notation

$$
\begin{aligned}
|\phi^{\alpha}(L)\rangle &\equiv \sum_i \hat{\phi}^I |\varphi_i^c\rangle \\
|\pi_{\alpha}(L)\rangle &\equiv \sum_i |\varphi_i\rangle \hat{\pi}_I
\end{aligned}
$$

$$
\begin{aligned}
\hat{H}_{\mathrm{FP}} = & \int_0^{\infty} dL L \left[\langle R|\phi^{\alpha}(L)\rangle |\pi_{\alpha}(L)\rangle - \langle R|\pi_{\alpha}(L)\rangle |\pi_{-\alpha}(L)\rangle \right] \\
& - g_{\mathrm{s}} \int dL_1 dL_2 dL_3 \langle T_{L_2 L_3 L_1} | B^1_{-\alpha_1} B^2_{\alpha_2} B^3_{\alpha_3} |\phi^{\alpha 1}(L_1)\rangle_1 |\pi_{\alpha_2}(L_2)\rangle_2 |\pi_{\alpha_3}(L_3)\rangle_3 \\
& - \frac{1}{2} g_{\mathrm{s}} \int dL_1 dL_2 dL_3 \langle D_{L_3 L_1 L_2} | B^1_{-\alpha_1} B^2_{-\alpha_2} B^3_{\alpha_3} |\phi^{\alpha 1}(L_1)\rangle_1 |\phi^{\alpha 2}(L_2)\rangle_2 |\pi_{\alpha_3}(L_3)\rangle_3
\end{aligned}
$$

- The BRST transformation

$$
\begin{aligned}
&\delta_{\epsilon}|\phi^+(L)\rangle = \tfrac{1}{2}\epsilon c_0^- b_0^- P Q |\phi^+(L)\rangle && \delta_{\epsilon}|\pi_+(L)\rangle = \epsilon Q|\pi_+(L)\rangle - \epsilon b_0^- P \partial_L |\pi_-(L)\rangle \\
&\delta_{\epsilon}|\phi^-(L)\rangle = \epsilon Q|\phi^-(L)\rangle - \epsilon b_0^- P \partial_L |\phi^+(L)\rangle && \delta_{\epsilon}|\pi_-(L)\rangle = \tfrac{1}{2}\epsilon c_0^- b_0^- P Q |\pi_-(L)\rangle
\end{aligned}
$$

$\hat{H}$ is not BRST invariant

- $\hat{H}_{\mathrm{FP}}$ is not BRST invariant.
  - If it were, FP formalism would be modular invariant
  - Let $\hat{Q}$ be the generator of the BRST transformation

$$\delta \hat{H}_{\mathrm{FP}} = [\hat{Q}, \hat{H}_{\mathrm{FP}}] = \int_0^\infty dL \left(\langle R | \mathcal{Q}^\alpha(L) \rangle | \pi_\alpha(L) \rangle + \langle R | \mathcal{T}^\alpha(L) \rangle [\hat{Q}, |\pi_\alpha(L)\rangle] \right)$$

$$\hat{H}_{\mathrm{FP}} = \int_0^\infty dL \langle R | \mathcal{T}^\alpha(L) \rangle | \pi_\alpha(L) \rangle$$

$$|\mathcal{Q}^\alpha(L)\rangle \equiv [\hat{Q}, |\mathcal{T}^\alpha(L)\rangle]$$

- The amplitudes are invariant, because $|\mathcal{Q}^\alpha(L)\rangle, |\mathcal{T}^\alpha(L)\rangle$ are "null quantities" satisfying

$$\left[\lim_{\tau \to \infty} \langle\!\langle 0 | e^{-\tau \hat{H}_{\mathrm{FP}}} \right] |\mathcal{T}^\alpha(L)\rangle = 0$$

$$\left[\lim_{\tau \to \infty} \langle\!\langle 0 | e^{-\tau \hat{H}_{\mathrm{FP}}} \right] |\mathcal{Q}^\alpha(L)\rangle = 0$$

and do not contribute in $\lim_{\tau \to \infty} \langle\!\langle 0 | e^{-\tau \hat{H}_{\mathrm{FP}}} \hat{\phi}^{I_1} \cdots \hat{\phi}^{I_n} | 0 \rangle\!\rangle$.

BRST invariant formulation

- We can modify the Hamiltonian by introducing the auxiliary fields $|\lambda_\alpha^{\mathcal{Q}}(L)\rangle, |\lambda_\alpha^{\mathcal{T}}(L)\rangle$ so that it becomes BRST invariant and still yields the correct amplitudes.

$$\hat{H}_{\rm FP} \to \hat{H}_{\rm FP} + \int_0^\infty dL \Big(\langle R|\mathcal{Q}^\alpha(L)\rangle |\lambda_\alpha^{\mathcal{Q}}(L)\rangle + \langle R|\mathcal{T}^\alpha(L)\rangle |\lambda_\alpha^{\mathcal{T}}(L)\rangle \Big)$$

$$\delta \hat{H}_{\rm FP} = \int_0^\infty dL \left(\langle R|\mathcal{Q}^\alpha(L)\rangle |\pi_\alpha(L)\rangle + \langle R|\mathcal{T}^\alpha(L)\rangle [\hat{Q}, |\pi_\alpha(L)\rangle] \right)$$

- The action

$$\begin{aligned} I_{\rm FP} \quad = \quad & \int_0^\infty d\tau \Big[-\int_0^\infty dL \langle R|\pi_\alpha(\tau,L)\rangle \frac{\partial}{\partial\tau} |\phi^\alpha(\tau,L)\rangle + H(\tau) \\ & + \int_0^\infty dL \Big(\langle R|\mathcal{Q}^\alpha(\tau,L)\rangle |\lambda_\alpha^{\mathcal{Q}}(\tau,L)\rangle + \langle R|\mathcal{T}^\alpha(\tau,L)\rangle |\lambda_\alpha^{\mathcal{T}}(\tau,L)\rangle \Big) \Big] \end{aligned}$$

  - This action is invariant under the BRST transformation.
  - It consists of kinetic terms and three string interaction terms.

5. Conclusions

5. Conclusions

$$
\begin{aligned}
& I_{\mathrm{FP}}[\phi, \pi, \lambda] \\
& = \int_0^\infty d\tau \Bigg[- \int_0^\infty dL \langle R | \pi_\alpha(\tau, L) \rangle \frac{\partial}{\partial \tau} | \phi^\alpha(\tau, L) \rangle + H(\tau) \\
& \qquad + \int_0^\infty dL \Big(\langle R | \mathcal{Q}^\alpha(\tau, L) \rangle | \lambda^{\mathcal{Q}}_\alpha(\tau, L) \rangle + \langle R | \mathcal{T}^\alpha(\tau, L) \rangle | \lambda^{\mathcal{T}}_\alpha(\tau, L) \rangle \Big) \Bigg]
\end{aligned}
$$

- We have constructed an SFT for closed bosonic strings based on the pants decomposition via the Fokker-Planck formalism.
  - The action consists of kinetic terms and three string interaction terms.
  - It is manifestly invariant under a nilpotent BRST transformation and we can define the physical states using it.

- How can one interpret the procedure to select the physical states in terms of the 2nd quantized language?
- SFT for superstrings?